Polynomial Solutions of the Heun Equation

B. Shapiro, M. Tater

Abstract

We review properties of certain types of polynomial solutions of the Heun equation. Two aspects are particularly concerned, the interlacing property of spectral and Stieltjes polynomials in the case of real roots of these polynomials and asymptotic root distribution when complex roots are present.

Keywords: Heun equation, Van Vleck and Stieltjes polynomials, asymptotic root distribution, logarithmic potential.

1 Introduction

We study polynomial solutions of the Heun equation

$$\left\{Q(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}+P(z)\frac{\mathrm{d}}{\mathrm{d}z}+V(z)\right\}S(z)=0, \quad (1)$$

where Q, P, and V are given polynomials. Q is a polynomial of degree k, P is at most of degree $k-1$, and V is at most of degree $k-2$. E. Heine and T. Stieltjes posed the following problem:

Problem. *Given a pair of polynomials $\{Q, P\}$ and a positive integer n find all polynomials V such that (1) has a polynomial solution S of degree n.*

Polynomials V are referred to as Van Vleck polynomials and polynomials S as Stieltjes polynomials. For a generic pair $\{Q, P\}$ there exist $\binom{n+k-2}{n}$ distinct Van Vleck polynomials.

The simplest case is $k=2$, when equation (1) is an equation of hypergeometric type: Q is quadratic, P is at most linear and V reduces to a (spectral) parameter. This situation was thoroughly studied in the past and all polynomial solutions are brought to six types of either finite or infinite systems of orthogonal polynomials *e.g.* [4]. Asymptotic distribution of zeros of orthogonal polynomials has been studied for quite a long time and many important results are known [13].

2 $k = 3$ case

Next natural step is $k=3$. Even this problem has a long history, going back to G. Lamé. Already Heine and Stieltjes knew that for a fixed n the above mentioned problem has $n+1$ solutions, *i.e.* that there exist $n+1$ distinct Van Vleck polynomials. Moreover, in the case of the Lamé equation ($P = Q'/2$) and if we additionally assume that Q has three real and distinct roots $a_1 < a_2 < a_3$ then each root of each V and each S is real and simple, the roots of V and S lie between a_1 and a_3, none of the roots of S coincides with any a_i ($i = 1, 2, 3$), and $n+1$ polynomials S can be distinguished by the number of roots lying in the interval (a_1, a_2) (the remaining roots lie in (a_2, a_3)) [14]. Besides this, there is no zero of S between a_2 and the zero of the corresponding Van Vleck polynomial [1], *cf.* Figure 1.

Some additional results are known for fixed n. Each Van Vleck (linear) polynomial has a single zero ν_i, $i = 1, \ldots, n+1$. We can form a so-called *spectral* polynomial made of these zeros

$$Sp_n(\lambda) = \prod_{i=1}^{n+1}(\lambda - \nu_i).$$

Zeros of two successive spectral polynomials, *i.e.* Sp_n and Sp_{n+1} interlace: between any two roots of Sp_n lies a root of Sp_{n+1}, and vice versa [2]. On the other hand, in spite of the fact that these polynomials have simple zeros that interlace, the system $\{Sp_n\}_{n=1}^{\infty}$ is not orthogonal with respect to any measure. The proof in [2] is based on the finding that the asymptotic zero distribution of Sp_n [3] is different from that of orthogonal polynomials, showing also that Sp_n do not obey any three-term recurrence relation.

As already mentioned above, the roots of Van Vleck's ν_i lie between a_1 and a_3, and are mutually different, making it thus possible to order Stieltjes polynomials accordingly. So, for a fixed n, we have a sequence of $n+1$ Stieltjes polynomials $S_i^{(n)}$ of degree n, $i = 1, \ldots, n+1$. Two interesting results are proved in [1]. The n zeros of $S_i^{(n)}$ and the n zeros of $S_{i+1}^{(n)}$ interlace. In addition, the smallest zero of $S_{i+1}^{(n)}$ is smaller than the smallest zero of $S_i^{(n)}$. Besides this, the zeros of $S_i^{(n)}$ and $S_j^{(n+1)}$ interlace if and only if $i = j$ or $i = j+1$, otherwise they do not interlace. There is no definitive answer to the question of orthogonality of $S_i^{(n)}$.

If complex roots of Q are admitted, G. Pólya proved [9] that all roots of both V and S belong to the convex hull $Conv_Q$ of a_1, a_2, a_3 provided that all residues of P/Q are positive.

Investigations of the root asymptotics of both Van Vleck and Stieltjes polynomials have a considerably shorter history. We summarize here some salient results [10–12].

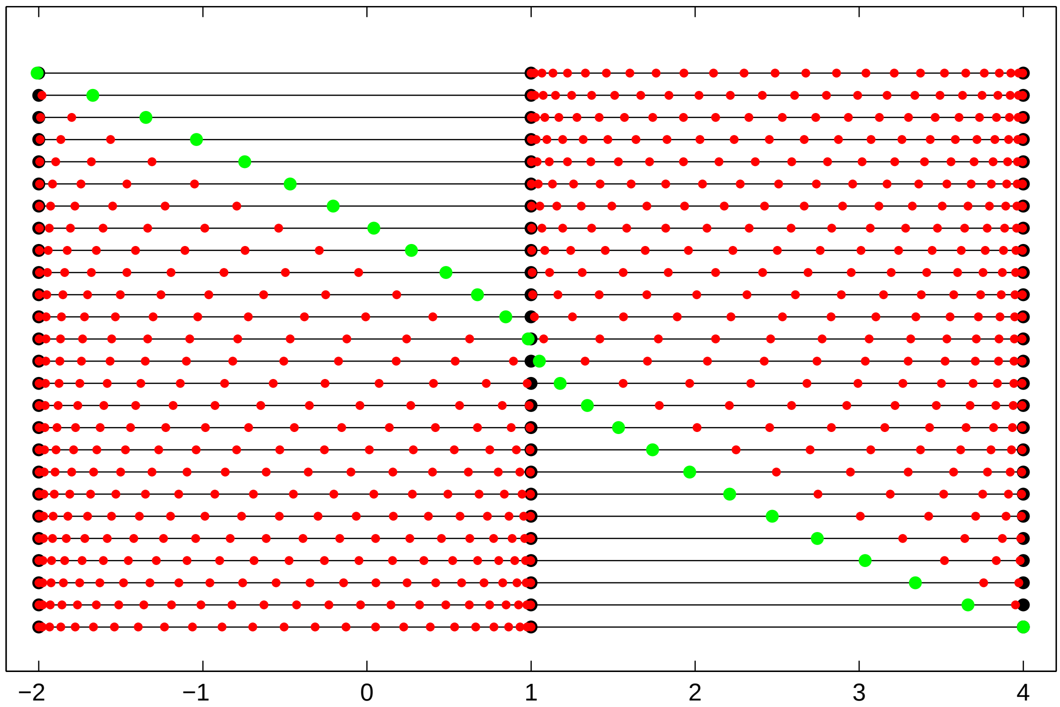

Fig. 1: The situation for $P = 0$ and $n = 25$. The thick black dots mark the roots of $Q(x) = (x + 2)(x - 1)(x - 4)$, the thick green dots mark the roots of $n+1$ Van Vleck polynomials, and the small red dots mark n roots of the corresponding Stieltjes polynomials

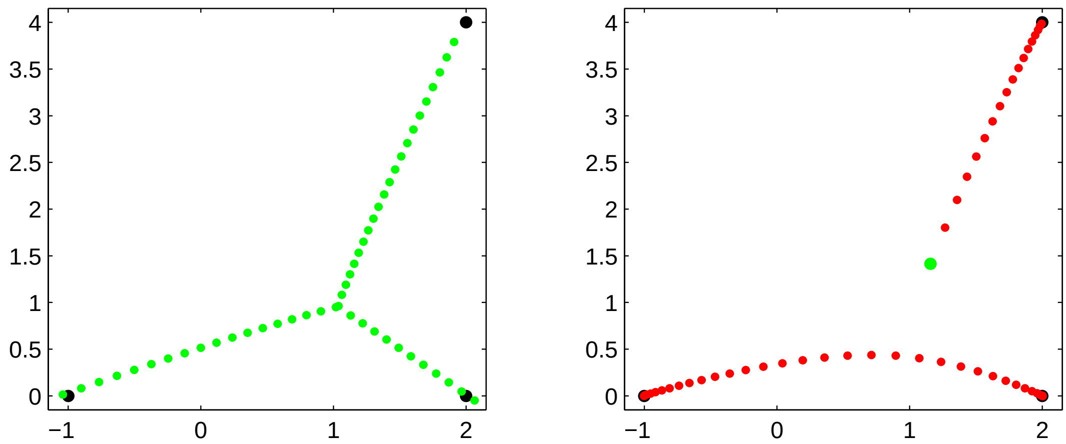

Fig. 2: The left part: The roots of the spectral polynomial $Sp_{51}(\lambda)$ for $Q(z) = (z + 1)(z - 2)(z - 2 - 4i)$ and $P(z) = (z+2+2i)(z-1+3i)$. The thick black dots mark the roots of Q, the green dots mark the roots of Van Vleck polynomials. The right part: The thick green dot marks one of the 51 Van Vleck polynomials and the small red dots mark 50 roots of the corresponding Stieltjes polynomial

The roots can be asymptotically localized. For any $\epsilon > 0$ there exist N_ϵ such that for any $n \geq N_\epsilon$ any root of any V as well as any root of the corresponding S lie in the ϵ-neighbourhood (in the usual Euclidean distance on $\mathbb{C}$) of the convex hull of a_1, a_2, a_3. This result shows that the asymptotic behaviour of roots is determined by Q, *i.e.* it is not influenced by P for sufficiently large n.

For a more detailed description of asymptotic distribution we associate to each polynomial p_n a finite real measure

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta(z - z_j),$$

where $\delta(z - z_j)$ is the Dirac measure supported at the root z_j. This probability measure is referred to as the *root-counting measure* of the polynomial p_n.

Now, two questions are to be answered. Does the sequence $\{\mu_n\}$ converge (in the weak sense) to a limiting measure μ and if so what does μ look like? We may ask these questions when $p_n = Sp_n$. The first question is answered positively [11, 12]. The sequence $\{\mu_n\}$ of the root-counting measures of its spectral polynomials converges to a probability measure μ supported on the union of three curves located inside $Conv_Q$ and connecting the three roots of Q with a certain interior point, *cf.* Figure 2. Moreover, μ depends only on Q.

The support of μ is a union of three curve segments $\gamma_i, i \in \{1, 2, 3\}$. They may be described as the set of all $b \in Conv_Q$ satisfying

$$\int_{a_j}^{a_k} \sqrt{\frac{b-t}{(t-a_1)(t-a_2)(t-a_3)}}\, \mathrm{d}t \in \mathbb{R},$$

here j and k are the remaining two indices in $\{1, 2, 3\}$ in any order and the integration is taken over the straight interval connecting a_j and a_k. We can see that a_i belong to γ_i and that these three curves connect the corresponding a_i with a common point within $Conv_Q$. Take a segment of γ_i connecting a_i with the common intersection point of all γ's. Let us denote the union of these three segments by Γ_Q. Then the support of the limiting root-counting measure μ coincides with Γ_Q.

Knowing the support of μ it is also possible to define its density along the support using the linear differential equation satisfied by its Cauchy transform [11]

$$Q(z)\mathcal{C}''_\nu(z) + Q'(z)\mathcal{C}'_\nu(z) + \frac{Q''(z)}{8}\mathcal{C}_\nu(z) + \frac{Q'''(z)}{24} = 0.$$

In the case when Q(z) has all real zeros, the density is explicitly given in [3].

The Cauchy transform $\mathcal{C}_\nu(z)$ and the logarithmic potential $pot_\nu(z)$ of a (complex-valued) measure ν supported in $\mathbb{C}$ are given by:

$$\mathcal{C}_\nu(z) = \int_{\mathbb{C}} \frac{\mathrm{d}\nu(\xi)}{z-\xi}$$

and

$$pot_\nu(z) = \int_{\mathbb{C}} \log|z-\xi|\, \mathrm{d}\nu(\xi).$$

$\mathcal{C}_\nu(z)$ is analytic outside the support of ν [5].

In [11] we were able to find an additional probability measure ν which is easily described and from which the measure μ is obtained by the inverse balayage, i.e. the support of μ will be contained in the support of the measure ν and they have the same logarithmic potential outside the support of the latter one. This measure is uniquely determined by the choice of a root of $Q(z)$, and thus we in fact have constructed three different measures ν_i having the same measure μ as their inverse balayage.

Let us try to formulate similar results for the asymptotic root behaviour of Stieltjes polynomials. To this end we must formulate in more detail which sequence of polynomials we are studying. Take a sequence of monic (the leading coefficient is 1) Van Vleck polynomials $\{\widetilde{V}_n\}$ converging to some monic linear polynomial $\widetilde{V}$. The existence of a linear polynomial $\widetilde{V}$ is ensured by the existence of the limit of the sequence of (unique) roots ν_{n,i_n} of $\{\widetilde{V}_n\}$. The above mentioned results guarantee the existence of plenty of such converging sequences in $Conv_Q$ and the limit $\widetilde{\nu}$ of these roots must necessarily belong to Γ_Q.

Having chosen $\{\widetilde{V}_n\}$ we take any sequence of the corresponding $\{S_{n,i_n}\}$, $deg\, S_{n,i_n} = n$ whose corresponding sequence $\{\widetilde{V}_n\}$ has a limit. If we denote by μ_{n,i_n} the root-counting measure of the corresponding Stieltjes polynomial, we have proved that the sequence $\{\mu_{n,i_n}\}$ converges weakly to the unique probability measure $\mu_{\widetilde{V}}$ whose Cauchy transform $\mathcal{C}_{\widetilde{V}}(z)$ satisfies the equation

$$\mathcal{C}^2_{\widetilde{V}}(z) = \frac{\widetilde{V}(z)}{Q(z)}$$

almost everywhere in $\mathbb{C}$.

In order to formulate further results we used [12] the notion of the quadratic differential (*cf.* also [7,8]). We avoid this way of formulating the results, because it would necessarily exceed the scope if this paper. Instead, we limit ourselves to presenting a typical example, *cf.* the right part of Figure 2. The support of the limit measure consists of singular trajectories of the quadratic differential. They run close to the roots shown in red. In this particular case, one trajectory joins two zeros of Q and the other one joins the third zero of Q with the root of the limiting Van Vleck polynomial.

3 Bispectral problems

Concerning the situation when $k = 4$ certain general statements have already been published (*e.g.* in [6, 7]). In the case when the roots of Van Vleck and Stieltjes polynomials are real we can still rely on the result of Stieltjes mentioned above, which make ordering of Stieltjes polynomials possible. The situation is shown in Figure 3.

When complex roots come into play, the picture is less clear. Figure 3 suggests that the asymptotic root distribution of Van Vleck polynomials has a more complicated structure than before. On the other hand, the structure of the asymptotic root distribution of Stieltjes polynomials bears some resemblance to the $k = 3$ case.

There are still several questions open. In addition, many other unsolved problems can be found for higher linear differential equations with polynomial coefficients.

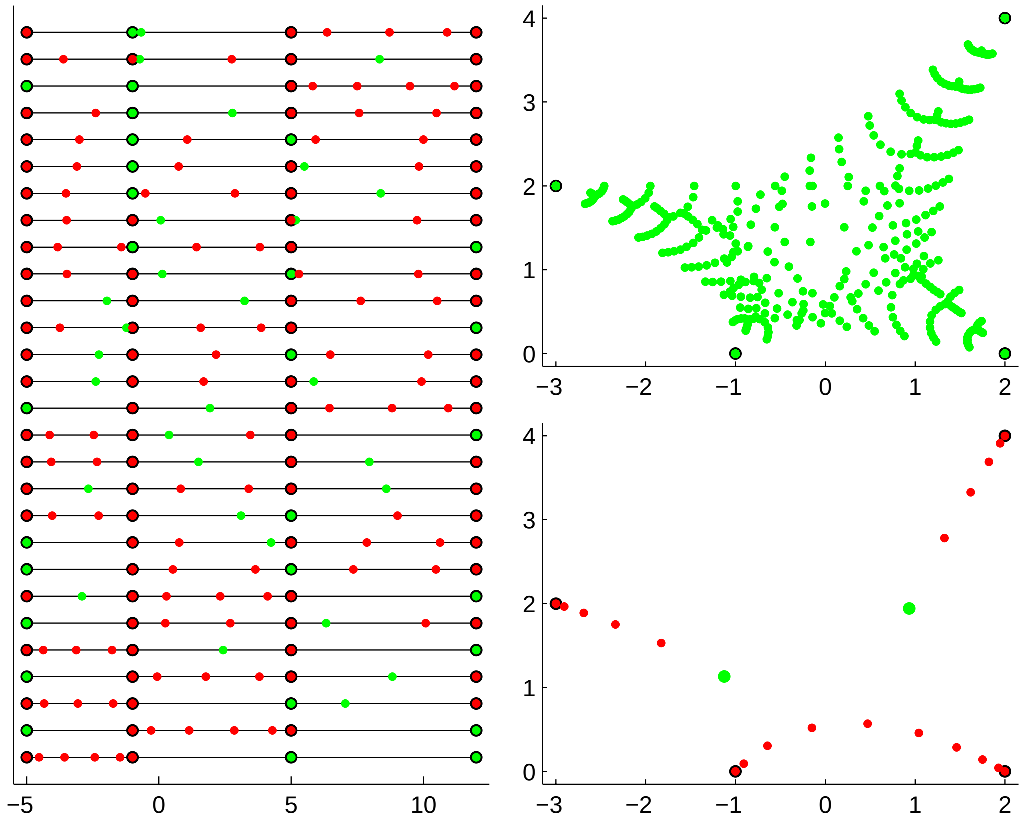

Fig. 3: The left part: The location of roots for $Q(x) = (x+5)(x+1)(x-5)(x-12)$, $P = 0$, and $n = 6$. The dots have the same meaning as in Fig. 1. The right upper part: The union of roots of (quadratic) Van Vleck polynomials for $Q(z) = (z+1)(z-2)(z-2-4i)(z+3-2i)$, $P = 0$, and $n = 20$. The lower part: roots of a particular Stieltjes polynomial (in red) and the roots of the corresponding Van Vleck polynomial (in green)

Acknowledgement

This work has been supported by the Czech Ministry of Education, Youth and Sports within the project LC06002 and by GACR grant P203/11/0701.

References

[1] Bourget, A., McMillen, T.: On the distribution and interlacing of the zeros of Stieltjes polynomials, *Proc. AMS* **138** (2010), 3 267–3 275.

[2] Bourget, A., McMillen, T., Vargas, A.: Interlacing and nonorthogonality of spectral polynomials for the Lamé operator, *Proc. AMS* **137** (2009), 1 699-1 710.

[3] Borcea, J., Shapiro, B.: Root asymptotics of spectral polynomials for the Lamé operator, *Commun. Math. Phys.* **282** (2008), 323–337.

[4] Cotfas, N.: Systems of orthogonal polynomials defined by hypergeometric type equations with application to quantum mechanics, *Cent. Eur. J. Phys.* **2** (2004), 456–466.

[5] Garnett, J.: Analytic capacity and measure. *Lecture Notes in Mathematics* **297**, Springer-Verlag, Berlin-New York, 1972.

[6] Holst, T., Shapiro, B.: On higher Heine-Stieltjes polynomials, to appear in *Isr. J. Math.* **183** (2011), 321–347.

[7] Martínez-Finkelshtein, A., Rakhmanov, E. A.: On asymptotic behavior of Heine-Stieltjes and Van Vleck polynomials, *Contemporary Mathematics* **507** (2010), 209–232.

[8] Martínez-Finkelshtein, A., Rakhmanov, E. A.: Critical measures, quadratic differentials, and weak limits of zeros of Stieltjes polynomials, *Comm. Math. Physics* **302** (2011), 53–111.

[9] Pólya:, G. Sur un théorème de Stieltjes, *C. R. Acad. Sci. Paris* **155** (1912), 76–769.

[10] Shapiro, B.: Algebro-geometric aspects of Heine-Stieltjes theory, *J. London Math. Soc.* **83** (2011), 36–56.

[11] Shapiro, B., Tater, M.: On spectral polynomials of the Heun equation. I, *J. Approx. Theory* **162** (2010), 766–781.

[12] Shapiro, B., Takemura, K., Tater, M.: On spectral polynomials of the Heun equation. II, arXiv:0904.0650.

[13] Szegő, G.: *Orthogonal Polynomials.* 1975, AMS, Pronidence, R.I.

[14] Whittaker, E. T., Watson, G.: *A course of modern analysis.* reprint of the 4th edition, 1996, Cambridge Univ. Press, UK.

Boris Shapiro
E-mail: shapiro@math.su.se
Department of Mathematics
Stockholm University,
SE-106 91 Stockholm, Sweden

Miloš Tater
E-mail: tater@ujf.cas.cz
Department of Theoretical Physics
Nuclear Physics Institute
AS CR v.v.i.
CZ-250 68 Řež, Czech Republic