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ITINERARIES INDUCED BY EXCHANGE OF TWO INTERVALS

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Abstract. We focus on the exchange T of two intervals with an irrational slope α. For a general subinterval I of the domain of T, the first return time to I takes three values. We describe the structure of the set of return itineraries to I. In particular, we show that it is equal to $\{R_1, R_2, R_1R_2, Q\}$ where Q is amicable with R_1, R_2 or R_1R_2.

Keywords: interval exchange, first return map, return time.

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1. Introduction

We study the symbolic dynamical system given by the transformation T of the unit interval, $T : [0,1) \to [0,1)$,

$$T(x) = \begin{cases} x - \alpha & \text{for } x \in [\alpha, 1), \\ x + 1 - \alpha & \text{for } x \in [0, \alpha), \end{cases} \tag{1}$$

where α is a fixed number in $[0,1)$. Transformation T has only one discontinuity point, such a dynamical system is the simplest dynamical system with discontinuous transformation. Dynamical systems defined by continuous transformations $F : J \to J$ have a number of nice properties, for example, there exists a fixed point $\rho \in J$, $F(\rho) = \rho$. The famous theorem of Sharkovskii [1] describes the structure of periodic points, i.e., fixed points of F^k for some $k \in \mathbb{N}$.

If one chooses the parameter α in (1) irrational, the map T has no periodic point, in other words, the orbit $\{\rho, T(\rho), T^2(\rho), \dots\}$ is infinite for every $\rho \in [0,1)$. Nevertheless, T has a weaker property, namely that although $T^k(\rho) \neq \rho$ for any $k \in \mathbb{N}$, one can get arbitrarily close to a point ρ with some of its iterations. More precisely,

$$\forall \varepsilon > 0 \ \exists n \in \mathbb{N},\ n \geq 1 : \left|T^n(\rho) - \rho\right| < \varepsilon. \tag{2}$$

Moreover, property (2) holds for every $\rho \in [0,1)$.

It is well known that every point $\rho \in [0,1)$ can be uniquely represented using the infinite string of 0 and 1, which constitutes the binary expansion of the number ρ. The mapping T of (1) allows another type of representation of ρ, namely by the coding of the orbit of ρ under T. Denote $J_0 = [0, \alpha)$, $J_1 = [\alpha, 1)$ and set

$$u_n = \begin{cases} 0 & \text{if } T^n(\rho) \in J_0, \\ 1 & \text{if } T^n(\rho) \in J_1. \end{cases}$$

Knowledge of the infinite word $\mathbf{u}_\rho := (u_n)_{n=0}^{\infty}$ allows one to determine the number ρ, i.e., the mapping $\rho \mapsto \mathbf{u}_\rho$ is one-to-one. The above defined infinite words $\mathbf{u}_\rho$ appear naturally in diverse mathematical problems; they were discovered and re-discovered several times and given different names. We will call the infinite word $\mathbf{u}_\rho$ a Sturmian word with slope α and intercept ρ.

Let us point out one important difference between binary expansion of numbers and their representation by Sturmian words with a fixed slope α. Every string of length n of letters 0 and 1 appears in the binary expansion of some real number $\rho \in [0,1)$. The number of such strings is obviously 2^n. By contrast, the list of all strings of length n appearing in the representation $\mathbf{u}_\rho$ of all $\rho \in [0,1)$ has exactly $n+1$ elements. Nevertheless, one can still represent a continuum of real numbers ρ. On the other hand, any type of representation using at most n strings of 0 and 1 of length n would allow representation of only countably many numbers. In that sense, Sturmian words represent real numbers in the most economical way.

Sturmian words have many other remarkable properties, for a review, see [2]. Generalizations of Sturmian words are treated in [3].

The property (2) expresses the fact that iterations $T^n(\rho)$ return arbitrarily close to ρ. This allows one to define, for a subinterval $I \subset [0,1)$ of positive length, the so-called return time $r : I \to \mathbb{N}$ by

$$r(\rho) := \min\{n \in \mathbb{N},\ n \geq 1, : T^n(\rho) \in I\}.$$

The return time represents the number of iterations needed for a point ρ to come back to the interval where it comes from. The movement of point ρ on its path from I back to I is recorded by the so-called I-itinerary of ρ, which we denote by $R(\rho)$. It is defined as the finite word $w_0 w_1 \cdots w_{n-1}$ in the alphabet $\mathcal{A} = \{0,1\}$ of length $n = r(\rho)$ such that

$$w_i = a, \quad \text{if } T^i(\rho) \in J_a,\ a \in \mathcal{A}.$$

Equivalently, the I-itinerary $R(\rho)$ of ρ is a prefix of the infinite word $\mathbf{u}_\rho$ of length $r(\rho)$. In our considerations, the interval I is fixed. Thus, for simplicity of notation, we avoid marking the dependence on I of the first return time and return itinerary, i.e., we write $r(x)$, $R(x)$ instead of $r_I(x)$, $R_I(x)$, respectively. The position of the point $\rho \in I$ after its return the interval I defines a new transformation $T_I : I \to I$ by

$$T_I(\rho) = T^{r(\rho)}(\rho), \tag{3}$$

which is usually called the first return map or induced map.

The I-itineraries for a special type of interval I were studied in diverse contexts:

- If the boundary points of the interval I are neighbouring elements of the set $\{\alpha, T^{-1}(\alpha), \dots, T^{-n}(\alpha)\}$ for some $n \in \mathbb{N}$, then the set of I-itineraries $R(\rho)$ for $\rho \in I$ consists of only two words. This reformulates the result of Vuillon [4] about the existence of exactly two return words to a fixed factor of a Sturmian word.
- If the Sturmian word $\mathbf{u}_\rho$ is invariant under a substitution $0 \mapsto \varphi(0)$, $1 \mapsto \varphi(1)$, then there exists an interval $I \subset [0,1)$, $\rho \in I$, such that the induced map T_I is homothetic to T, and the finite words $\varphi(0)$, $\varphi(1)$ are the I-itineraries. Invariance of Sturmian words under substitutions was studied by Yasutomi [5].
- An Abelian return word to a factor of a Sturmian word is an I-itinerary for $I = [0,\beta)$ or $I = [\beta,1)$ for some $\beta \in [0,1)$, see [6]. As follows from the result of [7], the intervals of the mentioned form have at most three itineraries R_1, R_2, R_3 and for their length one has $|R_3| = |R_1| + |R_2|$. In [8], we have shown that a stronger statement holds, namely that the word R_3 is a concatenation of words R_1 and R_2.

The aim of this paper is to describe the structure of the set of I-itineraries for a general position and length of the subinterval $I \subset [0,1)$. The set of all I-itineraries $R(x)$ for $x \in I$ is denoted by It_I. For the description, we will use the notion of word amicability. We say that two finite words w and v over the alphabet $\{0,1\}$ are amicable, if there exist words $p, q \in \{0,1\}^*$ such that $w = p01q$ and $v = p10q$ or $w = p10q$ and $v = p01q$. In other words, v is obtained from w by interchanging the order of letters 0 and 1 at two neighbouring positions $i-1, i$.

It follows from [9] that for every interval I there exist at most four I-itineraries, i.e., $\#It_I \le 4$. We will show the following theorem.

Theorem 1.1. *Let T be the transformation* (1) *for some irrational $\alpha \in (0,1)$ and let $I \subset [0,1)$ be an interval. Then there exist words $R_1, R_2 \in \{0,1\}^*$ such that for the set It_I of all I-itineraries one has*

$$It_I \subset \{R_1, R_2, R_1R_2, Q\},$$

where Q is amicable with R_1, R_2 or R_1R_2.

From the proof of Theorem 1.1 (at the end of Section 2) one can see that in the generic case,

$$It_I = \{R_1, R_2, R_1R_2, Q\}.$$

In Section 3 we discuss the possibilities for Q if $\#It_I = 4$ and determine the cases for which the set It_I has less than 4 elements.

2. Interval Exchange Transformations

First, let us recall the definition and certain properties of k-interval exchange maps, which we use for $k = 2$ and 3.

Definition 2.1. Let $J_0 \cup J_1 \cup \dots \cup J_{k-1}$ be a partition of the interval J, where J_i are intervals closed from the left and open from the right for every $i = 0, \dots, k-1$. The transformation $T : J \to J$ is called a k-interval exchange if there exist constants $c_0, c_1, \dots, c_{k-1} \in \mathbb{R}$ such that

$$T(x) = x + c_j, \quad x \in J_j,$$

and T is a bijection on J.

Since T is a bijection, intervals $T(J_i)$ for $j = 0, 1, \dots, k-1$ form a partition of J. The order of indices j which determines the ordering of intervals $T(J_i)$ in J is usually expressed by a permutation π. A trivial example of a k-interval exchange is the choice $c_j = 0$ for $j = 0, \dots, k-1$. Then T is the identity map and π is the identity permutation. The transformation T of (1) is a 2-interval exchange with permutation (21).

Example 2.2. Consider $a, b \in (0,1)$, $a < b$. Put

$$I_0 = [0,a), \quad I_1 = [a,b), \quad I_2 = [b,1).$$

Then the transformation $T : [0,1) \to [0,1)$ given by

$$T(x) = \begin{cases} x + 1 - a & \text{if } x \in [0,a), \\ x + 1 - a - b & \text{if } x \in [a,b), \\ x - b & \text{if } x \in [b,1), \end{cases} \tag{4}$$

is a 3-interval exchange with permutation $\pi = (321)$, see Figure 1.

From now on, we focus on the exchange T of two intervals given by the prescription (1) with an irrational slope α. We will study the first return map T_I defined by (3) to the subinterval $I \subset [0,1)$.

In [10] it is shown how T_I depends on the length of the interval I. For an irrational $\alpha \in (0,1)$ with the continued fraction $\alpha = [0, a_1, a_2, \dots]$ and convergents $\frac{p_n}{q_n}$ set

$$\delta_{k,s} := \big|(s-1)(p_k - \alpha q_k) + p_{k-1} - \alpha q_{k-1}\big|, \qquad \text{for } k \ge 0,\ 1 \le s \le a_{k+1}. \tag{5}$$

For the numbers $\delta_{k,s}$ one has $\delta_{k,s} > \delta_{k',s'}$ if and only if $k' > k$ or $k' = k$ and $s' > s$.

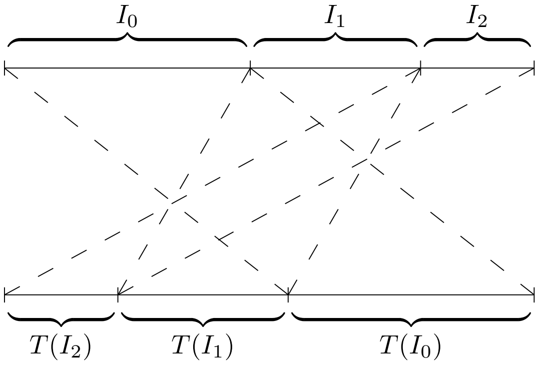

FIGURE 1. Exchange of three intervals.

In [10], we study infinite words associated to cut-and-project sequences which we show to be exactly codings of exchanges of two or three intervals. The following proposition is a reformulation of statements of Theorem 4.1 and Proposition 4.5 of [10] in the framework of interval exchanges.

Proposition 2.3. *Let $T : [0, 1) \to [0, 1)$ be an exchange of two intervals with irrational slope α and let $I = [c, d) \subset [0, 1)$. For the induced map T_I one has*

(1.) *If $d - c = \delta_{k,s}$ for some k, s, defined in (5), then T_I is an exchange of two intervals.*

(2.) *Otherwise, T_I is an exchange of three intervals with permutation (321). Moreover, the lengths of intervals I_0, I_1, I_2 forming the partition of I depend only on $d - c$ and for the return time $r(x_0)$, $r(x_1)$, $r(x_2)$ of points $x_0 \in I_0$, $x_1 \in I_1$, $x_2 \in I_2$, $x_0 < x_1 < x_2$, one has $r(x_1) = r(x_0) + r(x_2)$.*

Remark 2.4. Proposition 4.5 of [10] also allows to determine the exact two or three values of return time $r(x)$ to I. In fact, if $d - c = \delta_{k,s}$, then — keeping the notation of (5) — $r(x)$ takes two values

$$\{r(x) : x \in I\} = \{q_k, sq_k + q_{k-1}\}.$$

If $d - c$ is between $\delta_{k,s}$ and its successor in the decreasing sequence $(\delta_{k,s})$, then $r(x)$ takes three values

$$\{r(x) : x \in I\} = \{q_k, sq_k + q_{k-1}, (s + 1)q_k + q_{k-1}\}.$$

The values of return time are connected to the so-called three-distance theorem [11, 12]. Another point of view on return time in Sturmian words is presented in [13].

Although the return time $r(x)$ to a given interval I can take only three values, the set It_I of I-itineraries can have more than three elements. The following statement can be extracted from the proof of the Theorem in [9, §2]. It is convenient to provide the demonstration here.

Proposition 2.5. *Let $T : [0, 1) \to [0, 1)$ be an exchange of two intervals with irrational slope α and let $I = [c, d) \subset [0, 1)$. Then It_I has at most 4 elements.*

Proof. Choose $x \in I$. Denote $R(x)$ its I-itinerary and $r = r(x)$ its return time. Let $H \subset I$ be the maximal interval containing x such that for every $x' \in H$ one has $R(x) = R(x')$. For H, it holds that

(1.) $T^i(H) \subset [0, \alpha)$ or $T^i(H) \subset [\alpha, 1)$ for $i = 0, 1, \dots, r - 1$;

(2.) $T^i(H) \cap I = \emptyset$ for $i = 1, \dots, r - 1$;

(3.) $T^r(H) \subset I$.

The theorem will be established by showing that there are only four candidates for the left end-point of the interval $H = [\tilde{c}, \tilde{d})$. Obviously, one of them is $\tilde{c} = c$. If it is not the case, maximality of H and properties (1.), (2.), and (3.) imply that $c < \tilde{c} < \tilde{d} \le d$ and there exists

(a) $\tilde{l}$, $r - 1 \ge \tilde{l} \ge 1$ such that $T^{\tilde{l}}(\tilde{c}) = d$; or

(b) $\tilde{n}$, $r - 1 \ge \tilde{n} \ge 0$ such that $T^{\tilde{n}}(\tilde{c}) = \alpha$; or

(c) $\tilde{m}$, $r - 1 \ge \tilde{m} \ge 1$ such that $T^{\tilde{m}}(\tilde{c}) = c$.

Suppose that possibility (a) happened. Let us mention that it is possible only if $d < 1$. Denote

$$l = \min\{k \in \mathbb{Z},\ k \ge 1 : T^{-k}(d) \in I\}. \qquad (6)$$

Since $T^{-\tilde{l}}(d) = \tilde{c} \in H \subset I$, we have by definition of l that $\tilde{l} \ge l$. We will show by contradiction that $\tilde{l} = l$. If $\tilde{l} > l$, then $T^{\tilde{l}-l}(\tilde{c}) = T^{-l}\big(T^{\tilde{l}}(\tilde{c})\big) = T^{-l}(d) \in I$, and by definition of return time $r = r(\tilde{c}) \le \tilde{l} - l$. This contradicts the fact that $\tilde{l} \le r - 1$.

Similar discussion for possibilities (b) and (c) shows that the left end-point of the interval H is equal either to $T^{-l}(d)$ where l is defined by (6), or $T^{-n}(\alpha)$, where

$$n = \min\{k \in \mathbb{Z}, k \ge 0 : T^{-k}(\alpha) \in I\}, \qquad (7)$$

or $T^{-m}(c)$, where

$$m = \min\{k \in \mathbb{Z}, k \ge 1 : T^{-k}(c) \in I\}. \qquad (8)$$

This means that I is divided by the three (not necessarily distinct) points $T^{-l}(d)$, $T^{-n}(\alpha)$, $T^{-m}(c)$ into at most 4 subintervals H on which the I-itinerary is constant. ∎

Proposition 2.6. *Let It_I be the set of I-itineraries for the interval $I = [c, d) \subset [0, 1)$ under an exchange of two intervals with irrational slope α. There exist neighbourhoods H_c and H_d of c, d, respectively, such that for every $\tilde{c} \in H_c$ and $\tilde{d} \in H_d$, $0 \le \tilde{c} < \tilde{d} \le 1$ one has*

$$It_{\tilde{I}} \supseteq It_I, \quad \text{where } \tilde{I} = [\tilde{c}, \tilde{d}).$$

Proof. Let $It_I = \{R_1, \dots, R_p\}$. Proposition 2.5 implies that $p \le 4$ and for every $1 \le i \le p$ the elements x such that $R(x) = R_i$ form an interval, say I_i. Choose $x_i \in I_i$ such that for q with $0 \le q \le r(x_i) - 1 = |R_i| - 1$ one has $T^q(x_i) \notin \{c, d, \alpha\}$, (it suffices to choose $x_i \notin \mathbb{Z}[c, d, \alpha]$). Denote $M = \{c, d, \alpha\}$ and $N = \{T^q(x_i) : i = 1, \dots, p, 0 \le q \le r(x_i) - 1\}$. Put

$$\varepsilon := \min\{|a - b| : a \in M, b \in N\}.$$

Then for every $\tilde{c} \in (c-\varepsilon, c+\varepsilon)$ and $\tilde{d} \in (d-\varepsilon, d+\varepsilon)$, the I-itineraries $R(x_1), \dots, R(x_p)$ are also $\tilde{I}$-itineraries, where $\tilde{I} = [\tilde{c}, \tilde{d})$. ∎

Proof of Theorem 1.1. If $I = [c, d)$ where $c = 0$ or $d = 1$, then by Theorem 4.5 of [8], the set It_I of I-itineraries is of the form $It_I \subset \{R, R', RR'\}$. Without loss of generality, we can therefore assume that $c \neq 0$ and $d \neq 1$.

If c, d, or $d - c$ belongs to $\mathbb{Z}[\alpha]$ (which is dense in $\mathbb{R}$), we can always use Proposition 2.6 to find $\tilde{I} = [\tilde{c}, \tilde{d})$ such that $It_{\tilde{I}} \supseteq It_I$. Therefore, without loss of generality we assume $c, d, d - c \notin \mathbb{Z}[\alpha]$. In particular, $d - c \neq \delta_{k,s}$. From the proof of Proposition 2.5, the interval I is divided into at most four subintervals with constant I-itinerary by points

$$\begin{aligned} \lambda &= T^{-l}(d), & l &= \min\bigl\{k \geq 1 : T^{-k}(d) \in I\bigr\}, \\ \mu &= T^{-m}(c), & m &= \min\bigl\{k \geq 1 : T^{-k}(c) \in I\bigr\}, \\ \nu &= T^{-n}(\alpha), & n &= \min\bigl\{k \geq 0 : T^{-k}(\alpha) \in I\bigr\}. \end{aligned}$$

Moreover, λ and μ separate intervals with different return times. In particular, for sufficiently small ε, one has

$$\begin{aligned} l &= r(\lambda - \varepsilon) < r(\lambda + \varepsilon), \\ m &= r(\mu + \varepsilon) < r(\mu - \varepsilon). \end{aligned} \tag{9}$$

By Proposition 2.3, the induced map T_I is an exchange of three intervals with permutation (321). Let $I = I_0 \cup I_1 \cup I_2$ be the corresponding partition of I, where for every $x_0 \in I_0$, $x_1 \in I_1$, $x_2 \in I_2$ one has $x_0 < x_1 < x_2$. By the same proposition $r(x_1) = r(x_0) + r(x_2)$, which together with inequalities (9) implies that the right end-point of I_0 is equal to λ, the left end-point of I_2 is equal to μ, and $r(x_1) = l + m$.

Since $c, d, d - c \notin \mathbb{Z}[\alpha]$, we also have $\lambda \notin \mathbb{Z}[\alpha]$, and thus one can choose ε sufficiently small, so that the interval $[\lambda - \varepsilon, \lambda + \varepsilon]$ does not contain any of the points $T^{-j}(\alpha)$ for $0 \leq j \leq l + m$. This implies that $T^j\bigl([\lambda - \varepsilon, \lambda + \varepsilon]\bigr)$ is an interval not containing α for any $j = 0, 1, \dots, l + m - 1$, and consequently, the prefix of length $l + m$ of the infinite word $\mathbf{u}_\rho$ is the same for any $\rho \in [\lambda - \varepsilon, \lambda + \varepsilon]$. We have

$$\begin{aligned} T^l(\lambda - \varepsilon) &= d - \varepsilon \in I, \\ T^l(\lambda + \varepsilon) &= d + \varepsilon \notin I. \end{aligned}$$

For the corresponding I-itineraries, we thus have

$$R(\lambda + \varepsilon) = R(\lambda - \varepsilon) R(d - \varepsilon).$$

We can set $R_1 = R(\lambda - \varepsilon)$, $R_2 = R(d - \varepsilon)$, to have

$$It_I \supset \{R_1, R_2, R_1R_2\}.$$

By Proposition 2.5, the set It_I may have four elements. Let us determine the fourth element Q. Consider the point $\nu = T^{-n}(\alpha)$, $n = \min\{k \geq 0 : T^{-k}(\alpha) \in I\}$, which, by the proof of Proposition 2.5 splits one of the intervals I_0, I_1, I_2, into two, so that the I-itinerary on the new partition is constant. By the assumption that $c, d \notin \mathbb{Z}[\alpha]$, we have $\nu \neq \lambda$, $\nu \neq \mu$.

Consider the points $\nu - \varepsilon$, $\nu + \varepsilon$ for sufficiently small ε. Obviously, their return time coincides, $r(\nu - \varepsilon) = r(\nu + \varepsilon) = r(\nu)$, thus the I-itineraries $R(\nu - \varepsilon)$, $R(\nu + \varepsilon)$ are of the same length $r(\nu)$. Since $T^n(\nu) = \alpha$, we have $T^{n+1}(\nu) = 0 \notin I$, and thus $r(\nu) \geq n + 1$. We can see that

$$\begin{aligned} T^{n+1}(\nu + \varepsilon) &= \varepsilon, & T^{n+2}(\nu + \varepsilon) &= 1 - \alpha + \varepsilon, \\ T^{n+1}(\nu - \varepsilon) &= 1 - \varepsilon, & T^{n+2}(\nu - \varepsilon) &= 1 - \alpha - \varepsilon, \end{aligned}$$

which implies that

$$\begin{aligned} R(\nu - \varepsilon) &= u_0 \cdots u_{n-1} 01 u_{n+2} \cdots u_{r(\nu)-1}, \\ R(\nu + \varepsilon) &= u_0 \cdots u_{n-1} 10 u_{n+2} \cdots u_{r(\nu)-1}. \end{aligned}$$

Necessarily, $R(\nu - \varepsilon)$ and $R(\nu + \varepsilon)$ are amicable words. One of them is Q, the other one is equal to R_1, R_2 or R_1R_2, according to whether the point ν belongs to I_0, I_1 or I_2. ∎

3. Case study

Let us give several examples illustrating the possible outcomes for the set It_I of I-itineraries for general subinterval $I = [c, d) \subset [0, 1)$. According to our main Theorem 1.1, we have

$$It_I \subset \{R_1, R_2, R_1R_2, Q\},$$

where Q is a word amicable with one of R_1, R_2, R_1R_2. In fact, as we see in the following examples, we can have all possibilities.

For simplicity in the examples, we always keep $\alpha = \sigma$, where $\sigma = \frac{1}{2}(\sqrt{5} - 1)$ is the reciprocal of the golden ratio. In calculations, we use the relation $\sigma^2 = \sigma + 1$.

First, we choose the most generic cases, namely examples where $\#It_I = 4$. Let $I = [c, d)$ where $d - c = \sigma^3 + \sigma^6$. Since $d - c \neq \delta_{k,s}$ for any k, s, by Proposition 2.3, the induced map T_I is an exchange of three intervals with permutation (321), and, moreover, the lengths of exchanged intervals I_0, I_1, I_2 do not depend on the position of the interval I. In the notation introduced in the proof of Theorem 1.1,

$$\lambda = c + \sigma^4, \qquad \mu = c + \sigma^3.$$

Hence, in particular,

$$\begin{aligned} I_0 &= [c, c + \sigma^4), \\ I_1 &= [c + \sigma^4, c + \sigma^3), \\ I_2 &= [c + \sigma^3, c + \sigma^3 + \sigma^6). \end{aligned}$$

Independently on c, the return time $r(x)$ to the interval I satisfies

$$r(x) = \begin{cases} 3 & \text{if } x \in I_0, \\ 5 & \text{if } x \in I_1, \\ 2 & \text{if } x \in I_2. \end{cases}$$

(In fact, for any subinterval $I \subset [0,1)$ the return time takes two or three values, for $\alpha = \sigma$ always equal to two or three consecutive Fibonacci numbers.)

We consider several examples of positions of the interval I.

Example 3.1. Let $c = \sigma^4$. Then $\nu = T^{-1}(\alpha) = \sigma^3 \in I_0$ splits the interval I_0 into $I_0 = I_0^L \cup I_0^R$, where

$$I_0^L = [\sigma^4, \sigma^3), \qquad I_0^R = [\sigma^3, \sigma^3 + \sigma^6).$$

The I-itinerary satisfies

$$R(x) = \begin{cases} 001 & \text{if } x \in I_0^L, \\ 010 & \text{if } x \in I_0^R, \\ 01001 & \text{if } x \in I_1, \\ 01 & \text{if } x \in I_2. \end{cases}$$

We put

$$R_1 = 01, \quad R_2 = 001, \quad R_1R_2 = 01001, \quad Q = 010,$$

where Q is amicable with R_2. Note that we have another choice for notation,

$$R_1 = 010, \quad R_2 = 01, \quad R_1R_2 = 01001, \quad Q = 001,$$

where Q is amicable with R_1.

Example 3.2. Let $c = \sigma^6$. Then $\nu = T^{-1}(\alpha) = \sigma^3 \in I_1$ splits the interval I_1 into $I_1 = I_1^L \cup I_1^R$, where

$$I_1^L = [\sigma^4 + \sigma^6, \sigma^3), \qquad I_1^R = [\sigma^3, \sigma^3 + \sigma^6).$$

The I-itinerary satisfies

$$R(x) = \begin{cases} 001 & \text{if } x \in I_0, \\ 00101 & \text{if } x \in I_1^L, \\ 01001 & \text{if } x \in I_1^R, \\ 01 & \text{if } x \in I_2. \end{cases}$$

We put

$$R_1 = 001, \quad R_2 = 01, \quad R_1R_2 = 00101, \quad Q = 01001,$$

where $Q = R_2R_1$ is amicable with R_1R_2.

Example 3.3. Let $c = \sigma^3 + \sigma^5 + \sigma^7$. Then $\nu = T^0(\alpha) = \sigma \in I_2$ splits the interval I_2 into $I_2 = I_2^L \cup I_2^R$, where

$$I_2^L = [\sigma^2 + \sigma^4 + \sigma^6 + \sigma^9, \sigma), \quad I_2^R = [\sigma, \sigma + \sigma^7).$$

The I-itinerary satisfies

$$R(x) = \begin{cases} 010 & \text{if } x \in I_0, \\ 01010 & \text{if } x \in I_1, \\ 01 & \text{if } x \in I_2^L, \\ 10 & \text{if } x \in I_2^R. \end{cases}$$

We put

$$R_1 = 01, \quad R_2 = 010, \quad R_1R_2 = 01010, \quad Q = 10,$$

where Q is amicable with R_1, or

$$R_1 = 010, \quad R_2 = 10, \quad R_1R_2 = 01010, \quad Q = 01,$$

where Q is amicable with R_2.

Let us discuss the cases for which $\# It_I < 4$. This can happen if $d - c \neq \delta_{k,s}$, (i.e., T_I is still an exchange of three intervals), but $\nu \in \{c, \lambda, \mu\}$. It can be derived from the proof of Theorem 1.1, that, in this case, the set of I-itineraries is of the form

$$It_I = \{R_1, R_2, R_1R_2\}.$$

Note that $c = 0$ is a special case of such situation. For, we have $c = 0 = T(\alpha)$, whence $\mu = T^{-m}(0) = T^{-m+1}(\alpha) = \nu$. Similarly, the case $d = 1$ corresponds to $\lambda = \nu$.

Example 3.4. Let $c = \sigma^2$, $d - c = \sigma^3 + \sigma^6$. Then $\nu = T^0(\alpha) = \sigma = \mu$. The I-itinerary satisfies

$$R(x) = \begin{cases} 010 & \text{if } x \in I_0, \\ 01010 & \text{if } x \in I_1, \\ 10 & \text{if } x \in I_2. \end{cases}$$

With $R_1 = 010$, $R_2 = 10$, we have $It_I = \{R_1, R_2, R_1R_2\}$.

Consider the situation that $d - c = \delta_{k,s}$ for some k, s as defined in (5). By Proposition 2.3, the induced map T_I is an exchange of two intervals, since $\lambda = \mu$. The set of I-itineraries is then either $It_I = \{R_1, R_2\}$, which happens if $\nu \in \{c, \lambda\}$, or $It_I = \{R_1, R_2, Q\}$, where Q is amicable with R_1 or with R_2, according to the position of ν in the interval I.

4. Conclusions

Notions such as return time, return itinerary, first return map, etc. for the exchange of two intervals have been studied by many authors. For an overview, see for example [14]. This notion occurs in various contexts such as return words, Abelian return words, or substitution invariance of the corresponding codings, i.e., Sturmian words. The many equivalent definitions of Sturmian words allow one to combine different points of view which contributes substantially to the solution of such problems.

A detailed solution of analogous questions for exchanges of more than two intervals is still unknown. We believe that at least for exchanges of three intervals one can obtain an explicit description of return times and return itineraries, since the corresponding codings are geometrically representable by cut-and-project sequences, in a similar way that Sturmian words are identified with mechanical words.

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The results presented in this paper, as well as other results about exchange of intervals, have been obtained with the use of geometric representation of the associated codings in the frame of cut-and-project scheme, see [10]. We were lead to this topic from the study of mathematical models of quasicrystals, based on the rich collaboration of our institute with Jiří Patera from Centre de Recherches Mathématiques in Montréal. This collaboration was initiated, encouraged and is still maintained by prof. Miloslav

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