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MULTIDIMENSIONAL HYBRID BOUNDARY VALUE PROBLEM

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Abstract. The purpose of this paper is to discuss three types of boundary conditions for few families of special functions orthogonal on the fundamental region. Boundary value problems are considered on a simplex F in the real Euclidean space $\mathbb{R}^n$ of dimension $n > 2$.

Keywords: hybrid functions, Dirichlet boundary value problem, Neumann boundary value problem, mixed boundary value problem.

1. Introduction

The boundary value problems, considered in the paper, is a generalization of [24] in which the authors presented two-dimensional hybrids with mixed boundary value problems. Here we take a real Euclidean space $\mathbb{R}^n$ of dimension n on finite regions $F \subset \mathbb{R}^n$ that are polyhedral domains. The aim of this paper is to seek solutions of the Helmholtz equation with mixed boundary condition by analogy to two-dimensional cases. The solutions are presented as expansions into a series of special functions that satisfy required conditions at the $(n-1)$-dimensional boundaries of F. The recent discovery of special functions [5, 10, 11, 16, 19, 23] makes realization of this idea easy and straightforward in any dimension. The new functions, called 'multidimensional hybrids', satisfy the Dirichlet boundary condition on some parts of the boundary F and Neumann on the remaining ones. The methods used in the paper are the standard methods of separation of variables for differential equations (see for example [15, 18]) and the branching rule method for orbits of reflection groups (see for example [19, 24, 28]). The boundary value conditions play an important role in mathematics and physics. They are used, for example, in the theory of elasticity, electrostatics and fluid mechanics [4, 9, 26].

In §2, we present the well known Helmholtz equation and three types of boundary conditions. In § 3, we recall some facts about finite reflection groups. The next Section is devoted to special functions, projection matrices and branching rules. In § 5 we present 3D cases in details, namely $B_3, C_3, C_2\times A_1, G_2\times A_1, A_1\times A_1\times A_1$. In the Appendix we list tables containing the values of functions on the boundaries of fundamental region.

2. Helmholtz equation and boundary conditions

In this paper we consider the partial differential equation called the homogeneous Helmholtz equation [15, 18, 25, and references therein]:

$$\Delta\Psi(\mathbf{x}) = -\mathrm{w}^2\Psi(\mathbf{x}), \tag{1}$$

where w-positive real constant, $\mathbf{x} = (y_1, \dots, y_n)$ is given in Cartesian coordinates and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}$.

Using a standard method of separation of variables for (1) (see for example [15]) and searching for the solutions in the form $\Psi(\mathbf{x}) = X_1(y_1)\cdots X_n(y_n)$, we have the following differential equation

$$X_1''X_2\cdots X_n + X_1X_2''\cdots X_n + \cdots + X_1X_2\cdots X_n'' + \mathrm{w}^2X_1X_2\cdots X_n = 0. \tag{2}$$

By introducing $-\mathrm{k}_1^2, \dots, -\mathrm{k}_n^2$ so-called separation constants, we get the solution of (2) in the form

$$\begin{aligned} X_1^1(y_1) &= \cos(\mathrm{k}_1\, y_1),\\ &\vdots\\ X_{n-1}^1(y_{n-1}) &= \cos(\mathrm{k}_{n-1}\, y_{n-1}),\\ X_n^1(y_n) &= \cos(\mathrm{k}_n\, y_n),\\ X_1^2(y_1) &= \sin(\mathrm{k}_1\, y_1),\\ &\vdots\\ X_{n-1}^2(y_{n-1}) &= \sin(\mathrm{k}_{n-1}\, y_{n-1}),\\ X_n^2(y_n) &= \sin(\mathrm{k}_n\, y_n), \end{aligned} \tag{3}$$

where $k_n := \sqrt{\mathrm{w}^2 - \sum_{i=1}^{n-1} \mathrm{k}_i{}^2}$, $\mathrm{k}_i \neq 0$ for $i = 1, \dots, n$. The way of choosing separation constants is not unique. In this paper k_i, $i = 1, \dots, n-1$ are selected according to a branching rule method [19, 24, 28], see next sections.

Three types of boundary conditions.

D: A Dirichlet boundary condition defines the value of the function itself

$$\Psi(\mathbf{x}) = f(\mathbf{x}), \text{ for } \mathbf{x} \in \partial F,$$

where $f(\mathbf{x})$ is a given function defined on the boundary.

N: A Neumann boundary condition defines the value of the normal derivative of the function

$$\frac{\partial \Psi}{\partial \mathbf{n}}(\mathbf{x}) = f(\mathbf{x}), \text{ for } \mathbf{x} \in \partial F,$$

where $\mathbf{n}$ denotes normal vector to the boundary ∂F.

M: A mixed boundary condition defines the value of the function itself on one part of the boundary and the value of the normal derivative of the function on the other part of the boundary

$$\textbf{D:} \quad \Psi|_{\partial F_0} = f_0, \qquad \textbf{N:} \quad \frac{\partial \Psi}{\partial \mathbf{n}}\Big|_{\partial F_1} = f_1,$$

where $\partial F = \partial F_0 \cup \partial F_1$ and f_0, f_1 are given functions, defined on the appropriate boundary.

3. Finite reflection groups

Our method is general and can be presented for any crystallographic finite reflection groups G of any rank and any dimension which are associated with simple and semisimple Lie algebras/groups [1, 7, 10, 27]. There is a complete classification of finite reflection groups given by Dynkin diagrams [2, 3, 10]. These graphs provide the relative angles and relative length of the vectors of a set of simple roots of the root systems. There are two kinds of root systems according to the number of roots with different lengths: systems with one root length, and systems with two root lengths. A reflection r in a hyperplane orthogonal to the long/short root and passing through the origin of $\mathbb{R}^n$ be denoted by r_l/r_s respectively.

Working with finite reflection groups, it is convenient to use four bases in $\mathbb{R}^n$, namely natural e-, the simple root α-, co-root $\check{\alpha}$- and weight ω-bases [2, 7, 10]. The co-root basis $\check{\alpha}$ is defined by the formula

$$\check{\alpha}_i = \frac{2\alpha_i}{\langle \alpha_i | \alpha_i \rangle}.$$

The ω-basis is dual to simple root basis. The relationship between considered bases is standard for group theory and is expressed by

$$\langle \check{\alpha}_i | \omega_j \rangle = \delta_{ij}.$$

There are two types of fundamental region either simplex for simple Lie group G or prism for semisimple one. The simplex with $n+1$ vertices has the following coordinates

$$F = \Big\{0, \frac{\omega_1}{q_1}, \dots, \frac{\omega_n}{q_n}\Big\},$$

where $q_i, i = 1, \dots, n$, called co-marks, can be found in [6, 10] for any simple Lie group G of any rank and any dimension. The fundamental region for prisms can be given in the following sense. Let $G = G_1 \times G_2$, where G_1, G_2 are finite reflection groups. Let $\omega_1, \dots, \omega_k$ be a set of generating elements of G_1 and $\omega_{k+1}, \dots, \omega_n$ of G_2. Then the prism can be written as follows

$$F = \Big\{0, \frac{\omega_i}{q_i}, \frac{\omega_j}{q_j}, \frac{\omega_i}{q_i} + \frac{\omega_j}{q_j}\Big\},$$

where $i = 1, \dots, k$ and $j = k+1, \dots, n$ and q_i, q_j are co-marks [6, 10].

Let ∂F_i be contained in the hyperplane generated by a set of orthogonal reflections

$$r_0, r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n,$$

$i = \{0, \dots, n\}$, where r_0 is an affine reflection (it corresponds to long reflection). If r_i corresponds to the reflection orthogonal to the short/long root then we denote a part of the boundary by ∂F_s or ∂F_l respectively. In other words we can say that the boundary ∂F of the fundamental region F will be denoted by $\partial F_l/\partial F_s$ if its normal vector is perpendicular to the long/short root α respectively.

4. Special functions as a solution of Helmholtz equation

There are four kinds of special functions of interest to us whose orthogonality on lattice fragment F is known for any simple Lie group [5, 6, 10, 16, 17, 19, 20, 23, and references therein]. The general formula for special functions (called orbit functions) [10, 11] corresponding to the finite reflection group G is given by

$$\sum_{w \in G} \sigma(w) e^{2\pi i \langle w\lambda | \mathbf{x} \rangle}, \quad \lambda \in P^+, \, x \in F \tag{4}$$

where the summation extends over the whole group G, P^+ denotes the set of dominant weights [10] and $\sigma(w) = \pm 1$ depends on the type of the orbit function. The homomorphism $\sigma : G \to \{\pm 1\}$ is a product of $\sigma(r_l), \sigma(r_s) \in \{\pm 1\}$. There are four types of maps σ [16, 17]:

$$\begin{aligned} \sigma(r_l) = \sigma(r_s) = 1 \quad &\Longrightarrow \quad C, \\ \sigma(r_l) = \sigma(r_s) = -1 \quad &\Longrightarrow \quad S, \\ \sigma(r_l) = -1, \quad \sigma(r_s) = 1 \quad &\Longrightarrow \quad S^l, \\ \sigma(r_l) = 1, \quad \sigma(r_s) = -1 \quad &\Longrightarrow \quad S^s. \end{aligned} \tag{5}$$

All four families of functions defined above are formed as finite sums of exponential terms. The first two families, namely C- and S-functions are generalized cosine and sine functions. They are symmetric and skew-symmetric with respect to the finite reflection group [6, 10, 16, 19–21, 23]. The other two, S^s- and S^l-functions [11, 12, 16, 17, 23] have analogous properties as C- and S-functions. The main difference between them is their behaviour at the boundary of their domain of orthogonality in $\mathbb{R}^n$.

Every finite group G generated by reflections can be reduced to a subgroup $A_1 \times \cdots \times A_1$ using a branching rule method described in [13, 14, 19, 22, 24, 28]. This method allows us to do the separation of variables for special functions (5) corresponding to group G. As a result, we have all the functions written as a product of sine and cosine functions.

Remark 1. *All four families of functions* (5) *presented above are solutions of the Helmholtz equation* (1) *where* $\mathrm{w}^2 = 4\pi^2\langle\lambda|\lambda\rangle$ *with one of the three types of boundary conditions described in § 2.*

Projection matrix reduces any n-dimensional group G to a subgroup $A_1 \times \ldots \times A_1$ [13, 19]. The branching rule allows one to divide any orbit of group G into a union of orbits of group A_1. As an example see 3D cases described in § 5.

Remark 2. *The union of orbits which we get after reduction determine our choice of separating constants used in solution of Helmholtz equation* (1).

The behaviour of the functions C, S, S^s and S^l on the boundary ∂F can be summarize in the Tab. 1.

	D		**N**	
	∂F_s	∂F_l	∂F_s	∂F_l
$C_\lambda(\mathbf{x})$	$*$	$*$	0	0
$S_\lambda(\mathbf{x})$	0	0	$*$	$*$
$S^s_\lambda(\mathbf{x})$	0	$*$	$*$	0
$S^l_\lambda(\mathbf{x})$	$*$	0	0	$*$

Table 1. Behaviour of the functions C, S, S^s and S^l on the boundary ∂F for any finite refleciton group G where $*$ denotes any function non-equivalent to 0.

For any group G considered in the paper C-functions fulfil the Dirichlet condition with value non-equivalent to 0 and the Neumann condition with 0 value on the whole boundary. The S-functions behave inversely. The S^s-functions fulfil the Dirichlet condition with a value non-equivalent to 0 on the part of boundary denoted by ∂F_l and the Neumann condition with a value non-equivalent to 0 on the part of the boundary denoted by ∂F_s. The S^l functions behave inversely. In the case of C-functions we talk about Dirichlet boundary condition and S-functions - Neumann boundary condition. For S^s- and S^l-functions we talk about mixed boundary condition. In the next section we present 3D cases in details.

5. 3D finite reflection groups

The 3 dimensional groups which we considered here are $B_3, C_3, C_2 \times A_1, G_2 \times A_1, A_1 \times A_1 \times A_1$ [2, 7, 8, 10]. We use the following notation for coordinates:

$$\mathbb{R}^3 \ni \lambda = (a, b, c)_\omega = a\omega_1 + b\omega_2 + c\omega_3,$$
$$\mathbb{R}^3 \ni \mathbf{x} = (x_1, x_2, x_3)_{\check{\alpha}} = (y_1, y_2, y_3)_e,$$

where indexes e, ω, and $\check{\alpha}$ denote natural-, ω-, and $\check{\alpha}$-basis, respectively. The action of the Laplace operator ∇ on the functions given in different bases can be found in [10]. In the next subsections we describe each case in details.

For each case we present functions which are the solutions of Helmholtz equation (1). We give the exact forms of the projection matrices and branching rules which allow us to choose the separation constants used in (3). All functions described below fulfil one of the three types of boundary conditions described in §,2.

5.1. B_3 and C_3 groups

The α-basis vectors in Cartesian coordinates are

$$\begin{aligned}
&B_3: && C_3:\\
&\alpha_1 := (1,-1,0)_e, && \alpha_1 := \tfrac{1}{\sqrt{2}}(1,-1,0)_e,\\
&\alpha_2 := (0,1,-1)_e, && \alpha_2 := \tfrac{1}{\sqrt{2}}(0,1,-1)_e,\\
&\alpha_3 := (0,0,1)_e, && \alpha_3 := \tfrac{1}{\sqrt{2}}(0,0,2)_e.
\end{aligned}$$

As one can easily notice the short root for B_3 is α_3 and for C_3 are α_1, α_2. The fundamental regions F for B_3 and C_3 groups, written in ω-basis, have the vertices:

$$F_{B_3} = \{0, \omega_1, \tfrac{1}{2}\omega_2, \omega_3\},$$
$$F_{C_3} = \{0, \omega_1, \omega_2, \omega_3\},$$

and are shown in Fig. 1.

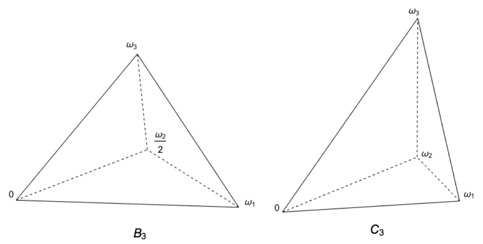

Figure 1. The fundamental region F for B_3 and C_3 group.

The reduction of B_3 and C_3 to a subgroup $A_1 \times A_1 \times A_1$ is given by the projection matrices

$$P_{B_3} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \qquad P_{C_3} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the branching rule is the following:

$$\begin{aligned}
O(a,b,c) \xrightarrow{P_{B_3}} & O(2a{+}2b{+}c)O(2b{+}c)O(c)\\
&\cup O(2b{+}c)O(2a{+}2b{+}c)O(c)\\
&\cup O(2a{+}2b{+}c)O(c)O(2b{+}c)\\
&\cup O(c)O(2a{+}2b{+}c)O(2b{+}c)\\
&\cup O(2b{+}c)O(c)O(2a{+}2b{+}c)\\
&\cup O(c)O(2b{+}c)O(2a{+}2b{+}c),
\end{aligned}$$

$$O(a,b,c) \xrightarrow{P_{C_3}} O(a{+}b{+}c)O(b{+}c)O(c) \cup O(b{+}c)O(a{+}b{+}c)O(c) \cup O(a{+}b{+}c)O(c)O(b{+}c) \cup O(b{+}c)O(c)O(a{+}b{+}c) \cup O(c)O(a{+}b{+}c)O(b{+}c) \cup O(c)O(b{+}c)O(a{+}b{+}c).$$

According to Remarks 1 and 2 the separation constants for B_3 and C_3 group we can choose as

$$\begin{aligned} -k_1^2 &= -\pi^2(2a+2b+c)^2, \\ -k_2^2 &= -\pi^2(2b+c)^2, \\ -k_3^2 &= -\pi^2 c^2, \end{aligned} \tag{6}$$

where $\mathrm{w}^2 = 4\pi^2(a^2+2ab+2b^2+ac+2bc+\frac{3}{4}c^2)$. The separation constants for C_3 group are

$$\begin{aligned} -k_1^2 &= -\pi^2(a+b+c)^2, \\ -k_2^2 &= -\pi^2(b+c)^2, \\ -k_3^2 &= -\pi^2 c^2, \end{aligned} \tag{7}$$

where $\mathrm{w}^2 = 4\pi^2(\frac{1}{2}a^2+ab+b^2+ac+2bc+\frac{3}{2}c^2)$.

The explicit forms of orbit functions for B_3 and C_3 group have form

$$\begin{aligned} B_3\colon\quad C_{a,b,c}(\mathbf{x}) &= C_{2a+2b+c}(\mathrm{x}_1)C_{2b+c}(\mathrm{x}_2)C_c(\mathrm{x}_3) + C_{2b+c}(\mathrm{x}_1)C_{2a+2b+c}(\mathrm{x}_2)C_c(\mathrm{x}_3) \\ &+ C_{2a+2b+c}(\mathrm{x}_1)C_c(\mathrm{x}_2)C_{2b+c}(\mathrm{x}_3) + C_c(\mathrm{x}_1)C_{2a+2b+c}(\mathrm{x}_2)C_{2b+c}(\mathrm{x}_3) \\ &+ C_{2b+c}(\mathrm{x}_1)C_c(\mathrm{x}_2)C_{2a+2b+c}(\mathrm{x}_3) + C_c(\mathrm{x}_1)C_{2b+c}(\mathrm{x}_2)C_{2a+2b+c}(\mathrm{x}_3), \end{aligned}$$

$$\begin{aligned} S^l_{a,b,c}(\mathbf{x}) &= C_{2a+2b+c}(\mathrm{x}_1)C_{2b+c}(\mathrm{x}_2)C_c(\mathrm{x}_3) - C_{2b+c}(\mathrm{x}_1)C_{2a+2b+c}(\mathrm{x}_2)C_c(\mathrm{x}_3) \\ &- C_{2a+2b+c}(\mathrm{x}_1)C_c(\mathrm{x}_2)C_{2b+c}(\mathrm{x}_3) + C_c(\mathrm{x}_1)C_{2a+2b+c}(\mathrm{x}_2)C_{2b+c}(\mathrm{x}_3) \\ &+ C_{2b+c}(\mathrm{x}_1)C_c(\mathrm{x}_2)C_{2a+2b+c}(\mathrm{x}_3) - C_c(\mathrm{x}_1)C_{2b+c}(\mathrm{x}_2)C_{2a+2b+c}(\mathrm{x}_3), \end{aligned}$$

$$\begin{aligned} S_{a,b,c}(\mathbf{x}) &= S_{2a+2b+c}(\mathrm{x}_1)S_{2b+c}(\mathrm{x}_2)S_c(\mathrm{x}_3) - S_{2b+c}(\mathrm{x}_1)S_{2a+2b+c}(\mathrm{x}_2)S_c(\mathrm{x}_3) \\ &- S_{2a+2b+c}(\mathrm{x}_1)S_c(\mathrm{x}_2)S_{2b+c}(\mathrm{x}_3) + S_c(\mathrm{x}_1)S_{2a+2b+c}(\mathrm{x}_2)S_{2b+c}(\mathrm{x}_3) \\ &+ S_{2b+c}(\mathrm{x}_1)S_c(\mathrm{x}_2)S_{2a+2b+c}(\mathrm{x}_3) - S_c(\mathrm{x}_1)S_{2b+c}(\mathrm{x}_2)S_{2a+2b+c}(\mathrm{x}_3), \end{aligned}$$

$$\begin{aligned} S^s_{a,b,c}(\mathbf{x}) &= S_{2a+2b+c}(\mathrm{x}_1)S_{2b+c}(\mathrm{x}_2)S_c(\mathrm{x}_3) + S_{2b+c}(\mathrm{x}_1)S_{2a+2b+c}(\mathrm{x}_2)S_c(\mathrm{x}_3) \\ &+ S_{2a+2b+c}(\mathrm{x}_1)S_c(\mathrm{x}_2)S_{2b+c}(\mathrm{x}_3) + S_c(\mathrm{x}_1)S_{2a+2b+c}(\mathrm{x}_2)S_{2b+c}(\mathrm{x}_3) \\ &+ S_{2b+c}(\mathrm{x}_1)S_c(\mathrm{x}_2)S_{2a+2b+c}(\mathrm{x}_3) + S_c(\mathrm{x}_1)S_{2b+c}(\mathrm{x}_2)S_{2a+2b+c}(\mathrm{x}_3); \end{aligned}$$

$$\begin{aligned} C_3\colon\quad C_{a,b,c}(\mathbf{x}) &= C_{a+b+c}(\mathrm{x}_1)C_{b+c}(\mathrm{x}_2)C_c(\mathrm{x}_3) + C_{b+c}(\mathrm{x}_1)C_{a+b+c}(\mathrm{x}_2)C_c(\mathrm{x}_3) \\ &+ C_{a+b+c}(\mathrm{x}_1)C_c(\mathrm{x}_2)C_{b+c}(\mathrm{x}_3) + C_c(\mathrm{x}_1)C_{a+b+c}(\mathrm{x}_2)C_{b+c}(\mathrm{x}_3) \\ &+ C_{b+c}(\mathrm{x}_1)C_c(\mathrm{x}_2)C_{a+b+c}(\mathrm{x}_3) + C_c(\mathrm{x}_1)C_{b+c}(\mathrm{x}_2)C_{a+b+c}(\mathrm{x}_3), \end{aligned}$$

$$\begin{aligned} S^s_{a,b,c}(\mathbf{x}) &= C_{a+b+c}(\mathrm{x}_1)C_{b+c}(\mathrm{x}_2)C_c(\mathrm{x}_3) - C_{b+c}(\mathrm{x}_1)C_{a+b+c}(\mathrm{x}_2)C_c(\mathrm{x}_3) \\ &- C_{a+b+c}(\mathrm{x}_1)C_c(\mathrm{x}_2)C_{b+c}(\mathrm{x}_3) + C_c(\mathrm{x}_1)C_{a+b+c}(\mathrm{x}_2)C_{b+c}(\mathrm{x}_3) \\ &+ C_{b+c}(\mathrm{x}_1)C_c(\mathrm{x}_2)C_{a+b+c}(\mathrm{x}_3) - C_c(\mathrm{x}_1)C_{b+c}(\mathrm{x}_2)C_{a+b+c}(\mathrm{x}_3), \end{aligned}$$

$$\begin{aligned} S_{a,b,c}(\mathbf{x}) &= S_{a+b+c}(\mathrm{x}_1)S_{b+c}(\mathrm{x}_2)S_c(\mathrm{x}_3) - S_{b+c}(\mathrm{x}_1)S_{a+b+c}(\mathrm{x}_2)S_c(\mathrm{x}_3) \\ &- S_{a+b+c}(\mathrm{x}_1)S_c(\mathrm{x}_2)S_{b+c}(\mathrm{x}_3) + S_c(\mathrm{x}_1)S_{a+b+c}(\mathrm{x}_2)S_{b+c}(\mathrm{x}_3) \\ &+ S_{b+c}(\mathrm{x}_1)S_c(\mathrm{x}_2)S_{a+b+c}(\mathrm{x}_3) - S_c(\mathrm{x}_1)S_{b+c}(\mathrm{x}_2)S_{a+b+c}(\mathrm{x}_3) \end{aligned}$$

$$\begin{aligned} S^l_{a,b,c}(\mathbf{x}) &= S_{a+b+c}(\mathrm{x}_1)S_{b+c}(\mathrm{x}_2)S_c(\mathrm{x}_3) + S_{b+c}(\mathrm{x}_1)S_{a+b+c}(\mathrm{x}_2)S_c(\mathrm{x}_3) \\ &+ S_{a+b+c}(\mathrm{x}_1)S_c(\mathrm{x}_2)S_{b+c}(\mathrm{x}_3) + S_c(\mathrm{x}_1)S_{a+b+c}(\mathrm{x}_2)S_{b+c}(\mathrm{x}_3) \\ &+ S_{b+c}(\mathrm{x}_1)S_c(\mathrm{x}_2)S_{a+b+c}(\mathrm{x}_3) + S_c(\mathrm{x}_1)S_{b+c}(\mathrm{x}_2)S_{a+b+c}(\mathrm{x}_3). \end{aligned}$$

The functions on the right side of the above equations are special functions corresponding to group A_1

$$\begin{aligned} C_\mu(\mathrm{x}_i) &= \sum_{\mu\in A_1} e^{2\pi i\langle\mu|\mathrm{x}_i\rangle} = 2\cos(2\pi\mu\mathrm{x}_i), \\ S_\mu(\mathrm{x}_i) &= \sum_{\mu\in A_1} \sigma(\mu)e^{2\pi i\langle\mu|\mathrm{x}_i\rangle} = 2i\sin(2\pi\mu\mathrm{x}_i), \end{aligned}$$

where $\mu \in P^+_{A_1}, \mathrm{x}_i \in F_{A_1}, i = 1,2,3$. The coordinate x_i respond to the i-th coordinate in $A_1 \times A_1 \times A_1$. The functions $C_\mu(\mathrm{x}_i)$ and $S_\mu(\mathrm{x}_i)$ are the solutions of Helmholtz equation (1) in the form (3) in 1D case.

For group B_3 the functions C- and S^l- are real valued and S- and S^s are purely imaginary. In the case of C_3, the functions C- and S^s- are real valued and S- and S^l are purely imaginary.

The normal vectors shown on Fig. 2 are

$$\begin{aligned} &B_3\colon && C_3\colon \\ &n_1 = \{-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\}, && n_1 = \{0,0,1\}, \\ &n_2 = \{0, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\}, && n_2 = \{0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\}, \\ &n_3 = \{0,0,-1\}, && n_3 = \{\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0\}, \\ &n_4 = \{\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\}, && n_4 = \{1,0,0\}. \end{aligned}$$

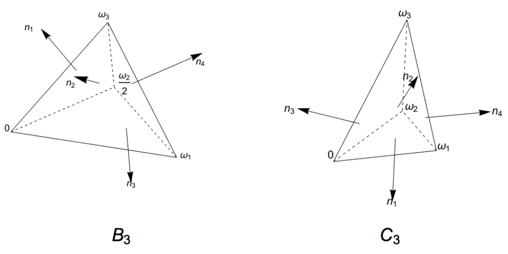

Figure 2. Normal vectors for B_3 and C_3.

In the case of B_3 group normal vector n_4 is perpendicular to the short simple root. The rest of them, namely n_1, n_2, n_3 are perpendicular to the long simple roots. So the boundaries that correspond to normal vectors are ∂F_s for n_1 and ∂F_l for the others. The values of the functions on the boundaries are summarized in the Appendix in Tab. 2. In the case of C_3 it is a little bit different, the normal vectors n_2, n_3 are perpendicular to the short simple roots and n_1, n_4 - to the long simple roots. The values of the functions on the boundaries are given in the Appendix in Tab. 3.

5.2. $C_2 \times A_1$ and $G_2 \times A_1$ groups

The α-basis vectors in Cartesian coordinates have the form

$$\begin{aligned}
&C_2 \times A_1: && G_2 \times A_1:\\
&\alpha_1 := \frac{1}{\sqrt{2}}(1,-1,0)_e, && \alpha_1 := (\sqrt{2},0,0)_e,\\
&\alpha_2 := \frac{2}{\sqrt{2}}(0,2,0)_e, && \alpha_2 := \Big(-\frac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{6}}, 0\Big)_e,\\
&\alpha_3 := \frac{1}{\sqrt{2}}(0,0,2)_e, && \alpha_3 := \frac{1}{\sqrt{2}}(0,0,2)_e.
\end{aligned}$$

The vertices of the fundamental regions F for $C_2 \times A_1, G_2 \times A_1$ groups, shown in Fig. 3, written in ω-basis are

$$F_{C_2\times A_1} = \{0, \omega_1, \omega_2, \omega_3, \omega_1+\omega_3, \omega_2+\omega_3\},$$
$$F_{G_2\times A_1} = \Big\{0, \frac{1}{2}\omega_1, \omega_2, \omega_3, \frac{1}{2}\omega_1+\omega_3, \omega_2+\omega_3\Big\}.$$

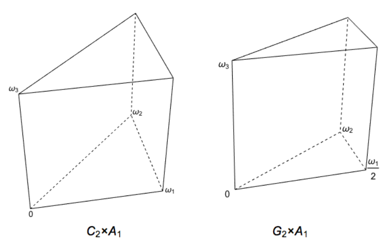

Figure 3. The fundamental region F for $C_2 \times A_1$ and $G_2 \times A_1$ group.

The groups $C_2 \times A_1, G_2 \times A_1$ can be reduced to a subgroup $A_1 \times A_1 \times A_1$ using a branching rule method described in [19, 24].

The projection matrices and the branching rules are

$$P_{C_2\times A_1} = \begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}, \quad P_{G_2\times A_1} = \begin{pmatrix}1&1&0\\3&1&0\\0&0&1\end{pmatrix},$$

$$O(a,b)O(c) \xrightarrow{P_{C_2\times A_1}} O(a+b)O(b)O(c) \cup O(b)O(a+b)O(c),$$

$$O(a,b)O(c) \xrightarrow{P_{G_2\times A_1}} O(a+b)O(3a+b)O(c) \cup O(2a+b)O(b)O(c) \cup O(a)O(3a+2b)O(c).$$

The separation constants for $C_2 \times A_1$ are

$$-k_1^2 = -\pi^2(a+b)^2, \quad -k_2^2 = -\pi^2 b^2, \quad -k_3^2 = -\pi^2 c^2, \tag{8}$$

where $\mathrm{w}^2 = 4\pi^2(a^2+ab+b^2+\frac{1}{2}c^2)$ and for $G_2 \times A_1$ equal

$$\begin{aligned}
-k_1^2 &= -2\pi^2(2a+b)^2, & -l_1^2 &= -2\pi^2(a+b)^2,\\
-k_2^2 &= -\tfrac{2}{3}\pi^2 b^2, & -l_2^2 &= -\tfrac{2}{3}\pi^2(3a+b)^2,\\
-k_3^2 &= -2\pi^2 c^2, & -l_3^2 &= -2\pi^2 c^2,
\end{aligned}$$

$$\begin{aligned}
-m_1^2 &= -2\pi^2 a^2,\\
-m_2^2 &= -\tfrac{2}{3}\pi^2(3a+2b)^2,\\
-m_3^2 &= -2\pi^2 c^2,
\end{aligned} \tag{9}$$

where $\mathrm{w}^2 = 4\pi^2(2a^2+2ab+\frac{2}{3}b^2+\frac{1}{2}c^2)$.

The explicit forms of orbit functions are

$C_2 \times A_1$:

$$\begin{aligned}
C_{a,b,c}(\mathbf{x}) &= C_{a+b}(\mathrm{x}_1)C_b(\mathrm{x}_2)C_c(\mathrm{x}_3) + C_b(\mathrm{x}_1)C_{a+b}(\mathrm{x}_2)C_c(\mathrm{x}_3),\\
S^s_{a,b,c}(\mathbf{x}) &= C_{a+b}(\mathrm{x}_1)C_b(\mathrm{x}_2)C_c(\mathrm{x}_3) - C_b(\mathrm{x}_1)C_{a+b}(\mathrm{x}_2)C_c(\mathrm{x}_3),\\
S_{a,b,c}(\mathbf{x}) &= S_{a+b}(\mathrm{x}_1)S_b(\mathrm{x}_2)S_c(\mathrm{x}_3) - S_b(\mathrm{x}_1)S_{a+b}(\mathrm{x}_2)S_c(\mathrm{x}_3),\\
S^l_{a,b,c}(\mathbf{x}) &= S_{a+b}(\mathrm{x}_1)S_b(\mathrm{x}_2)S_c(\mathrm{x}_3) + S_b(\mathrm{x}_1)S_{a+b}(\mathrm{x}_2)S_c(\mathrm{x}_3);
\end{aligned}$$

$G_2 \times A_1$:

$$\begin{aligned}
C_{a,b,c}(\mathbf{x}) &= C_a(\mathrm{x}_1)C_{3a+2b}(\mathrm{x}_2)C_c(\mathrm{x}_3) + C_{a+b}(\mathrm{x}_1)C_{3a+b}(\mathrm{x}_2)C_c(\mathrm{x}_3) + C_{2a+b}(\mathrm{x}_1)C_b(\mathrm{x}_2)C_c(\mathrm{x}_3),\\
S^l_{a,b,c}(\mathbf{x}) &= S_a(\mathrm{x}_1)C_{3a+2b}(\mathrm{x}_2)S_c(\mathrm{x}_3) - S_{a+b}(\mathrm{x}_1)C_{3a+b}(\mathrm{x}_2)S_c(\mathrm{x}_3) + S_{2a+b}(\mathrm{x}_1)C_b(\mathrm{x}_2)S_c(\mathrm{x}_3),\\
S_{a,b,c}(\mathbf{x}) &= S_a(\mathrm{x}_1)S_{3a+2b}(\mathrm{x}_2)S_c(\mathrm{x}_3) - S_{a+b}(\mathrm{x}_1)S_{3a+b}(\mathrm{x}_2)S_c(\mathrm{x}_3) + S_{2a+b}(\mathrm{x}_1)S_b(\mathrm{x}_2)S_c(\mathrm{x}_3),
\end{aligned}$$

$$S^s_{a,b,c}(\mathbf{x}) = C_a(\mathrm{x}_1)S_{3a+2b}(\mathrm{x}_2)C_c(\mathrm{x}_3) - C_{a+b}(\mathrm{x}_1)S_{3a+b}(\mathrm{x}_2)C_c(\mathrm{x}_3) - C_{2a+b}(\mathrm{x}_1)S_b(\mathrm{x}_2)C_c(\mathrm{x}_3),$$

where $C_\mu(\mathrm{x}_i), S_\mu(\mathrm{x}_i)$ for $i = 1, 2, 3$ are the same as in the previous cases.

For group $C_2 \times A_1$ the functions C- and S^s- are real valued and S- and S^l are purely imaginary. In the case of $G_2 \times A_1$, the functions C- and S^l- are real valued and S- and S^s are purely imaginary.

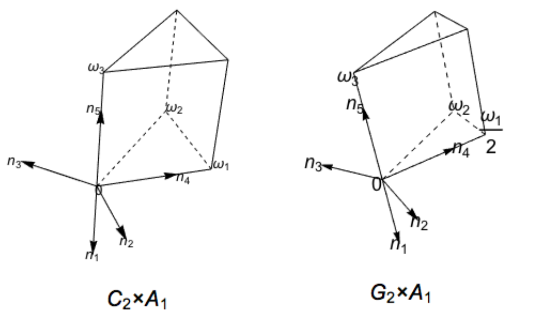

Figure 4. Normal vectors of F for $C_2 \times A_1, G_2 \times A_1$ groups.

The normal vectors shown in Fig. 4 are

$C_2 \times A_1$:	$G_2 \times A_1$:
$n_1 = \{0, 0, -1\}$,	$n_1 = \{0, 0, -1\}$,
$n_2 = \{0, -1, 0\}$,	$n_2 = \{\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\}$,
$n_3 = \{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\}$,	$n_3 = \{-1, 0, 0\}$,
$n_4 = \{1, 0, 0\}$,	$n_4 = \{\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\}$,
$n_5 = \{0, 0, 1\}$,	$n_5 = \{0, 0, 1\}$.

In the case of $C_2 \times A_1$ the group normal vector n_3 is perpendicular to the short simple root. The rest of them, namely n_1, n_2, n_4, n_5 are perpendicular to the long simple roots. So the boundaries that correspond to normal vectors are ∂F_s for n_3 and ∂F_l for the others. The values of the functions on the boundaries are summarized in Appendix in Tab. 4. In the case of $G_2 \times A_1$, the normal vector n_2 corresponds to the short simple root so to the boundary ∂F_s and the rest of normal vectors to the long simple roots i.e. to the boundaries ∂F_l. The values of the functions on the boundaries are given in Appendix in Tab. 5.

5.3. $A_1 \times A_1 \times A_1$ group

Although the root system of $A_1 \times A_1 \times A_1$ does not have two different lengths of roots, it is still an interesting case for us. The α-basis vectors in Cartesian coordinates have the form

$$\alpha_1 := (\sqrt{2}, 0, 0)_e, \quad \alpha_2 := (0, \sqrt{2}, 0)_e, \quad \alpha_3 := (0, 0, \sqrt{2})_e.$$

According to (4) and (5) there are two families of special functions C and S. By the analogy to homomorphism (5) we can define new families of functions.

$$\begin{aligned}
\sigma(r_1) = \sigma(r_2) = \sigma(r_3) = 1 &\implies CCC,\\
\sigma(r_1) = \sigma(r_2) = \sigma(r_3) = -1 &\implies SSS,\\
\sigma(r_1) = \sigma(r_2) = 1, \quad \sigma(r_3) = -1 &\implies CCS,\\
\sigma(r_1) = \sigma(r_2) = -1, \quad \sigma(r_3) = 1 &\implies SSC,\\
\sigma(r_1) = \sigma(r_3) = 1, \quad \sigma(r_2) = -1 &\implies CSC,\\
\sigma(r_1) = -1, \quad \sigma(r_2) = \sigma(r_3) = 1 &\implies SCC,\\
\sigma(r_1) = 1, \quad \sigma(r_2) = \sigma(r_3) = -1 &\implies CSS,\\
\sigma(r_2) = -1, \quad \sigma(r_1) = \sigma(r_3) = 1 &\implies SCS,
\end{aligned}$$

where CCC, SSS correspond to C and S-functions, respectively and the rest of them to S^l- and S^s-functions. All families of functions defined on the fundamental region

$$F_{A_1\times A_1\times A_1} = \{0, \omega_1, \omega_2, \omega_3, \omega_1+\omega_2, \omega_1+\omega_3, \omega_2+\omega_3, \omega_1+\omega_2+\omega_3\}.$$

fulfill mixed boundary condition (see Tab. 6).

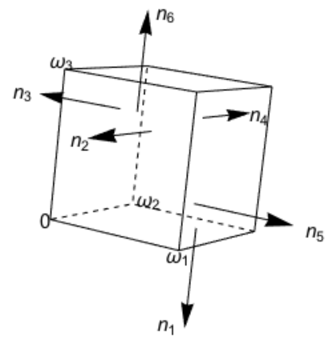

Figure 5. The fundamental region F with normal vectors of $A_1 \times A_1 \times A_1$ group.

The projection matrix is the identity matrix and then the choice of separation constants is trivial:

$$-k_1^2 = -\pi^2 a^2, \quad -k_2^2 = -\pi^2 b^2, \quad -k_3^2 = -\pi^2 c^2. \tag{10}$$

According to the branching rule

$$O(a, b, c) \xrightarrow{P_{A_1\times A_1\times A_1}} O(a)O(b)O(c)$$

we have

$$\begin{aligned}
CCC_{a,b,c}(\mathbf{x}) &:= C_a(\mathrm{x}_1)C_b(\mathrm{x}_2)C_c(\mathrm{x}_3),\\
SCS_{a,b,c}(\mathbf{x}) &:= S_a(\mathrm{x}_1)C_b(\mathrm{x}_2)S_c(\mathrm{x}_3),\\
CSS_{a,b,c}(\mathbf{x}) &:= C_a(\mathrm{x}_1)S_b(\mathrm{x}_2)S_c(\mathrm{x}_3),\\
SSC_{a,b,c}(\mathbf{x}) &:= S_a(\mathrm{x}_1)S_b(\mathrm{x}_2)C_c(\mathrm{x}_3),\\
SSS_{a,b,c}(\mathbf{x}) &:= S_a(\mathrm{x}_1)S_b(\mathrm{x}_2)S_c(\mathrm{x}_3),\\
CSC_{a,b,c}(\mathbf{x}) &:= C_a(\mathrm{x}_1)S_b(\mathrm{x}_2)C_c(\mathrm{x}_3),\\
CCS_{a,b,c}(\mathbf{x}) &:= C_a(\mathrm{x}_1)C_b(\mathrm{x}_2)S_c(\mathrm{x}_3),\\
SCC_{a,b,c}(\mathbf{x}) &:= S_a(\mathrm{x}_1)C_b(\mathrm{x}_2)C_c(\mathrm{x}_3),
\end{aligned}$$

where $C_\mu(\mathrm{x}_i), S_\mu(\mathrm{x}_i)$ for $i = 1, 2, 3$ are the same as in the previous cases. The first four families of functions are real valued and the rest of them are pure imaginary.
Normal vectors shown on Fig. 5 are

$$n_1 = \{0, 0, -1\}, \qquad n_2 = \{0, -1, 0\},$$
$$n_3 = \{-1, 0, 0\}, \qquad n_4 = \{0, 1, 0\},$$
$$n_5 = \{1, 0, 0\}, \qquad n_6 = \{0, 0, 1\}.$$

The values of the functions on the boundaries are shown in Appendix in Tab. 6.

6. Appendix

In Tables 2–6 we collect the values of special functions on the boundaries of the fundamental region F for each of 3D finite reflection groups presented in the paper.

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B_3	C		S	
	D	**N**	**D**	**N**
F_1	$2(C_{2b+c}(x)C_{2a+2b+c}(x)C_c(z)+C_c(x)C_{2b+c}(z)C_{2a+2b+c}(x)$ $+C_c(x)C_{2b+c}(x)C_{2a+2b+c}(z))$	0	0	$\sqrt{2}i(-k_1S_{2b+c}(x)C_{2a+2b+c}(x)S_c(z)+k_1S_c(x)C_{2a+2b+c}(x)S_{2b+c}(z)$ $-k_2S_c(x)C_{2b+c}(x)S_{2a+2b+c}(z)+k_2S_{2a+2b+c}(x)C_{2b+c}(x)S_c(z)$ $-k_3S_{2a+2b+c}(x)C_c(x)S_{2b+c}(z)+k_3S_{2b+c}(x)C_c(x)S_{2a+2b+c}(z))$
F_2	$2(C_c(y)C_{2b+c}(y)C_{2a+2b+c}(x)+C_c(x)C_{2b+c}(y)C_{2a+2b+c}(y)$ $+C_c(y)C_{2b+c}(x)C_{2a+2b+c}(y))$	0	0	$\sqrt{2}i(k_1S_{2b+c}(x)C_{2a+2b+c}(y)S_c(y)-k_1S_c(x)C_{2a+2b+c}(y)S_{2b+c}(y)$ $+k_2S_c(x)C_{2b+c}(y)S_{2a+2b+c}(y)-k_2S_{2a+2b+c}(x)C_{2b+c}(y)S_c(y)$ $+k_3S_{2a+2b+c}(x)S_{2b+c}(y)C_c(y)-k_3S_{2b+c}(x)S_{2a+2b+c}(y)C_c(y))$
F_3	$2(C_c(y)C_{2a+2b+c}(x)+C_{2b+c}(y)C_{2a+2b+c}(x)+C_c(x)C_{2a+2b+c}(y)$ $C_{2b+c}(x)C_{2a+2b+c}(y)+C_c(y)C_{2b+c}(x)+C_c(x)C_{2b+c}(y))$	0	0	$2i(-k_1S_{2b+c}(x)S_c(y)+k_1S_c(x)S_{2b+c}(y)$ $+k_2S_{2a+2b+c}(x)S_c(y)-k_2S_c(x)S_{2a+2b+c}(y)$ $-k_3S_{2a+2b+c}(x)S_{2b+c}(y)+k_3S_{2b+c}(x)S_{2a+2b+c}(y))$
F_4	$C_c(z)C_{2b+c}(1-x)C_{2a+2b+c}(x)+C_c(z)C_{2b+c}(x)C_{2a+2b+c}(1-x)$ $C_c(x)C_{2b+c}(1-x)C_{2a+2b+c}(x)+C_c(x)C_{2a+2b+c}(1-x)C_{2b+c}(z)$ $C_c(1-x)C_{2b+c}(z)C_{2a+2b+c}(x)+C_c(1-x)C_{2b+c}(x)C_{2a+2b+c}(z)$	0	0	$\frac{\sqrt{2}}{2}i(k_1C_{2a+2b+c}(x)S_{2b+c}(1-x)S_c(z)-k_2C_{2b+c}(x)S_{2a+2b+c}(1-x)S_c(z)$ $+k_3C_c(x)S_{2a+2b+c}(1-x)S_{2b+c}(z)-k_3C_c(x)S_{2b+c}(1-x)S_{2a+2b+c}(z)$ $-k_1C_{2a+2b+c}(x)S_c(1-x)S_{2b+c}(z)+k_2C_{2b+c}(x)S_c(1-x)S_{2a+2b+c}(z)$ $+k_2S_{2a+2b+c}(x)C_{2b+c}(1-x)S_c(z)-k_1S_{2b+c}(x)C_{2a+2b+c}(1-x)S_c(z)$ $+k_1S_c(x)C_{2a+2b+c}(1-x)S_{2b+c}(z)-k_2S_c(x)C_{2b+c}(1-x)S_{2a+2b+c}(z)$ $-k_3S_{2a+2b+c}(x)C_c(1-x)S_{2b+c}(z)+k_3S_{2b+c}(x)C_c(1-x)S_{2a+2b+c}(z))$

B_3	S^l	
	D	**N**
F_1	0	$\sqrt{2}i(k_2C_{2a+2b+c}(x)S_{2b+c}(x)C_c(z)-k_1C_{2b+c}(x)S_{2a+2b+c}(x)C_c(z)$ $+k_1C_c(x)S_{2a+2b+c}(x)C_{2b+c}(z)-k_2C_c(x)S_{2b+c}(x)C_{2a+2b+c}(z)$ $-k_3C_{2a+2b+c}(x)S_c(x)C_{2b+c}(z)+k_3C_{2b+c}(x)S_c(x)C_{2a+2b+c}(z))$
F_2	0	$\sqrt{2}i(-k_2C_{2a+2b+c}(x)S_{2b+c}(y)C_c(y)+k_1C_{2b+c}(x)S_{2a+2b+c}(y)C_c(y)$ $-k_1C_c(x)S_{2a+2b+c}(y)C_{2b+c}(y)+k_2C_c(x)S_{2b+c}(y)C_{2a+2b+c}(y)$ $+k_3C_{2a+2b+c}(x)S_c(y)C_{2b+c}(y)-k_3C_{2b+c}(x)S_c(y)C_{2a+2b+c}(y))$
F_3	$2(C_{2b+c}(y)C_{2a+2b+c}(x) - C_{2b+c}(x)C_{2a+2b+c}(x)+C_c(x)C_{2a+2b+c}(y)$ $-C_c(x)C_{2b+c}(y) - C_c(y)C_{2a+2b+c}(x)+C_c(y)C_{2b+c}(y))$	0
F_4	0	$\frac{\sqrt{2}}{2}i(k_1S_{2a+2b+c}(x)C_{2b+c}(1-x)C_c(z)-k_2S_{2b+c}(x)C_{2a+2b+c}(1-x)C_c(z)$ $+k_3S_c(x)C_{2a+2b+c}(1-x)C_{2b+c}(z)-k_3S_c(x)C_{2b+c}(1-x)C_{2a+2b+c}(z)$ $-k_1S_{2a+2b+c}(x)C_c(1-x)C_{2b+c}(z)+k_2S_{2b+c}(x)C_c(1-x)C_{2a+2b+c}(z)$ $+k_2C_{2a+2b+c}(x)S_{2b+c}(1-x)C_c(z)-k_1C_{2b+c}(x)S_{2a+2b+c}(1-x)C_c(z)$ $+k_1C_c(x)S_{2a+2b+c}(1-x)C_{2b+c}(z)-k_2C_c(x)S_{2b+c}(1-x)C_{2a+2b+c}(z)$ $-k_3C_{2a+2b+c}(x)S_c(1-x)C_{2b+c}(z)+k_3C_{2b+c}(x)S_c(1-x)C_{2a+2b+c}(z))$

B_3	S^s	
	D	**N**
F_1	$2(S_{2b+c}(x)S_{2a+2b+c}(x)S_c(z)+S_c(x)S_{2b+c}(z)S_{2a+2b+c}(x)$ $+S_c(x)S_{2b+c}(x)S_{2a+2b+c}(z))$	0
F_2	$2(S_c(z)S_{2b+c}(y)S_{2a+2b+c}(x)+S_c(x)S_{2b+c}(y)S_{2a+2b+c}(y)$ $+S_c(y)S_{2b+c}(x)S_{2a+2b+c}(y))$	0
F_3	0	$2i(-k_3S_{2a+2b+c}(x)S_{2b+c}(y)-k_3S_{2b+c}(x)S_{2a+2b+c}(y)$ $-k_2S_c(x)S_{2a+2b+c}(y)-k_1S_c(x)S_{2b+c}(y)-k_2S_{2a+2b+c}(x)S_c(y)-k_1S_{2b+c}(x)S_c(y))$
F_4	$S_c(z)S_{2b+c}(1-x)S_{2a+2b+c}(x)+S_c(z)S_{2b+c}(x)S_{2a+2b+c}(1-x)$ $S_c(x)S_{2b+c}(1-x)S_{2a+2b+c}(x)+S_c(x)S_{2b+c}(1-x)S_{2a+2b+c}(z)$ $S_c(1-x)S_{2b+c}(z)S_{2a+2b+c}(x)+S_c(z)S_{2b+c}(1-x)S_{2a+2b+c}(z)$	0

Table 2. The values of C-, S-, S^l- and S^s-functions on the boundaries of fundamental region F of B_3. The separation constants k_i, $i = 1, 2, 3$ are given by (6) in § 5.1.

C_3	C		S	
	D	**N**	**D**	**N**
F_1	$2(C_{b+c}(x)C_c(y)+C_{a+b+c}(x)C_c(y)$ $+C_{a+b+c}(x)C_{b+c}(y)+C_c(x)C_{b+c}(y)$ $+C_c(x)C_{a+b+c}(y)+C_{b+c}(x)C_{a+b+c}(y))$	0	0	$-2i(k_1S_{b+c}(x)S_c(y)+k_2S_{a+b+c}(x)S_c(y)$ $-k_3S_{a+b+c}(x)S_{b+c}(y)+k_1S_c(x)S_{b+c}(y)$ $-k_2S_c(x)S_{a+b+c}(y)+k_3S_{b+c}(x)S_{a+b+c}(y)$
F_2	$2(C_{a+b+c}(x)C_{b+c}(z)C_c(z)$ $+C_{b+c}(x)C_{a+b+c}(z)C_c(z)$ $+C_c(x)C_{a+b+c}(z)C_{b+c}(z))$	0	0	$i\sqrt{2}(k1C_c(z)S_{b+c}(x)C_{a+b+c}(z)-k1S_c(x)S_{b+c}(z)C_{a+b+c}(z)$ $-k2S_c(z)C_{b+c}(z)S_{a+b+c}(x)+k2S_c(x)C_{b+c}(z)S_{a+b+c}(z)$ $-k3C_c(z)S_{b+c}(x)S_{a+b+c}(z)+k3C_c(z)S_{b+c}(z)S_{a+b+c}(x))$
F_3	$2(C_c(z)C_{b+c}(y)C_{a+b+c}(y)$ $+C_c(y)C_{b+c}(z)C_{a+b+c}(y)$ $+C_c(y)C_{b+c}(y)C_{a+b+c}(z))$	0	0	$-i\sqrt{2}(k1S_c(z)S_{b+c}(y)C_{a+b+c}(y)-k1S_c(y)S_{b+c}(z)C_{a+b+c}(y)$ $-k2Sc(z)C_{b+c}(y)S_{a+b+c}(y) + k2S_c(y)C_{b+c}(y)S_{a+b+c}(z)$ $-k3C_c(y)S_{b+c}(y)S_{a+b+c}(z)+k3C_c(y)S_{b+c}(z)Sa + b + c(y))$
F_4	$C_{a+b+c}(\frac{1}{\sqrt{2}})C_{b+c}(y)C_c(z)$ $+C_{b+c}(\frac{1}{\sqrt{2}})C_{a+b+c}(y)C_c(z)$ $+C_c(\frac{1}{\sqrt{2}})C_{a+b+c}(y)C_{b+c}(z)$	0	0	$i(k1S_c(z)C_{a+b+c}(\frac{1}{\sqrt{2}})S_{b+c}(y)-k1S_c(y)C_{a+b+c}(\frac{1}{\sqrt{2}})S_{b+c}(z)$ $-k2C_{b+c}(\frac{1}{\sqrt{2}})S_c(z)S_{a+b+c}(y)+k2C_{b+c}(\frac{1}{\sqrt{2}})S_c(y)S_{a+b+c}(z)$ $-k3C_c(\frac{1}{\sqrt{2}})S_{b+c}(y)S_{a+b+c}(z)+k3C_c(\frac{1}{\sqrt{2}})S_{b+c}(z)S_{a+b+c}(y))$

C_3	S^l	
	D	**N**
F_1	0	$-2i(k_1S_{b+c}(x)C_c(y)+k_2S_{a+b+c}(x)S_c(y)$ $+k_3S_{a+b+c}(x)S_{b+c}(y)+k_1S_c(x)S_{b+c}(y)$ $+k_2S_c(x)S_{a+b+c}(y)+k_3S_{b+c}(x)S_{a+b+c}(y))$
F_2	$2(S_{a+b+c}(x)S_{b+c}(z)S_c(z)+S_{b+c}(x)S_{a+b+c}(z)S_c(z)$ $+S_c(x)S_{a+b+c}(z)S_{b+c}(z))$	0
F_3	$2(S_c(z)S_{b+c}(y)S_{a+b+c}(y) + S_c(y)S_{b+c}(z)S_{a+b+c}(y)$ $+S_c(y)S_{b+c}(y)S_{a+b+c}(z))$	0
F_4	0	$i(k1S_c(z)C_{a+b+c}(\frac{1}{\sqrt{2}})S_{b+c}(y)+k1S_c(y)C_{a+b+c}(\frac{1}{\sqrt{2}})S_{b+c}(z)$ $+k2C_{b+c}(\frac{1}{\sqrt{2}})S_c(z)S_{a+b+c}(y)+k2C_{b+c}(\frac{1}{\sqrt{2}})S_c(y)S_{a+b+c}(z)$ $+k3C_c(\frac{1}{\sqrt{2}})S_{b+c}(y)S_{a+b+c}(z)+k3C_c(\frac{1}{\sqrt{2}})S_{b+c}(z)S_{a+b+c}(y))$

C_3	S^s	
	D	**N**
F_1	$-2(C_{b+c}(x)C_{a+b+c}(y)-C_c(y)C_{a+b+c}(x)$ $+C_{b+c}(y)C_{a+b+c}(x)+C_c(x)C_{a+b+c}(y)$ $+C_c(y)C_{b+c}(x)-C_c(x)C_{b+c}(y))$	0
F_2	0	$i\sqrt{2}(k1C_c(z)C_{b+c}(x)S_{a+b+c}(z)-k1C_c(x)C_{b+c}(z)S_{a+b+c}(z)$ $-k2C_c(z)S_{b+c}(z)C_{a+b+c}(x)+k2C_c(x)S_{b+c}(z)C_{a+b+c}(z)$ $+k3S_c(z)C_{b+c}(z)C_{a+b+c}(x)-k3S_c(z)C_{b+c}(x)C_{a+b+c}(z))$
F_3	0	$-i\sqrt{2}(k1C_c(z)C_{b+c}(y)S_{a+b+c}(y)-k1C_c(y)C_{b+c}(z)S_{a+b+c}(y)$ $-k2C_c(z)S_{b+c}(y)C_{a+b+c}(y)+k2C_c(y)S_{b+c}(y)C_{a+b+c}(z)$ $+k3S_c(y)C_{b+c}(z)C_{a+b+c}(y)-k3S_c(y)C_{b+c}(y)C_{a+b+c}(z))$
F_4	$C_c(z)C_{a+b+c}(\frac{1}{\sqrt{2}})C_{b+c}(y)-C_c(y)C_{a+b+c}(\frac{1}{\sqrt{2}})C_{b+c}(z)$ $-C_{b+c}(\frac{1}{\sqrt{2}})C_c(z)C_{a+b+c}(y)+C_c(\frac{1}{\sqrt{2}})C_{b+c}(z)C_{a+b+c}(y)$ $+C_{b+c}(\frac{1}{\sqrt{2}})C_c(y)C_{a+b+c}(z)-C_c(\frac{1}{\sqrt{2}})C_{b+c}(y)C_{a+b+c}(z)$	0

TABLE 3. The values of C-, S-, S^l- and S^s-functions on the boundaries of fundamental region F of C_3. The separation constants k_i, $i = 1, 2, 3$ are given by (7) in § 5.1.

$C_2{\times}A_1$	C		S	
	D	**N**	**D**	**N**
F_1	$2(C_{a+b}(x)C_b(y){+}C_b(x)C_{a+b}(y))$	0	0	$-2ik_3(S_{a+b}(x)S_b(y){-}S_b(x)S_{a+b}(y))$
F_2	$2(C_{a+b}(x)C_c(z){+}C_b(x)C_c(z))$	0	0	$-2ik_2S_{a+b}(x)S_c(z){+}ik_1S_b(x)S_c(z)$
F_3	$2C_{a+b}(y)C_b(y)C_c(z)$	0	0	$-i\sqrt{2}(k_1C_{a+b}(y)S_b(y)S_c(z){-}k_2C_b(y)S_{a+b}(y)S_c(z))$
F_4	$C_{a+b}(\frac{\sqrt{2}}{2})C_b(y)C_c(z){+}C_b(\frac{\sqrt{2}}{2})C_{a+b}(y)C_c(z)$	0	0	$ik_1C_{a+b}(\frac{\sqrt{2}}{2})S_b(y)S_c(z){-}ik_2C_b(\frac{\sqrt{2}}{2})S_{a+b}(y)S_c(z)$
F_5	$C_{a+b}(x)C_b(y)C_c(\frac{\sqrt{2}}{2}){+}C_b(x)C_{a+b}(y)C_c(\frac{\sqrt{2}}{2})$	0	0	$ik_3(S_{a+b}(x)S_b(y)C_c(\frac{\sqrt{2}}{2}){-}S_b(x)S_{a+b}(y)C_c(\frac{\sqrt{2}}{2}))$

$C_2{\times}A_1$	S^l	
	D	**N**
F_1	0	$-2ik_3(S_{a+b}(x)S_b(y){+}S_b(x)S_{a+b}(y))$
F_2	0	$-2i(k_2S_{a+b}(x)S_c(z){+}k_1S_b(x)S_c(z))$
F_3	$2S_{a+b}(y)S_b(y)S_c(z)$	0
F_4	0	$i(k_1C_{a+b}(\frac{\sqrt{2}}{2})S_b(y)S_c(z){+}k_2C_b(\frac{\sqrt{2}}{2})S_{a+b}(y)S_c(z))$
F_5	0	$ik_3(S_{a+b}(x)S_b(y)C_c(\frac{\sqrt{2}}{2}){+}S_b(x)S_{a+b}(y)C_c(\frac{\sqrt{2}}{2}))$

$C_2{\times}A_1$	S^s	
	D	**N**
F_1	$2(C_{a+b}(x)C_b(y){-}C_b(x)C_{a+b}(y))$	0
F_2	$2(C_{a+b}(x)C_c(z){-}C_b(x)C_c(z))$	0
F_3	0	$-i\sqrt{2}(k_1S_{a+b}(y)C_b(y)C_c(z){-}k_2S_b(y)C_{a+b}(y)C_c(z))$
F_4	$C_{a+b}(\frac{\sqrt{2}}{2})C_b(y)C_c(z){-}C_b(\frac{\sqrt{2}}{2})C_{a+b}(y)C_c(z)$	0
F_5	$C_{a+b}(x)C_b(y)C_c(\frac{\sqrt{2}}{2}){-}C_b(x)C_{a+b}(y)C_c(\frac{\sqrt{2}}{2})$	0

Table 4. The values of C-, S-, S^l- and S^s-functions on the boundaries of fundamental region F of $C_2 \times A_1$. The separation constants k_i, $i = 1, 2, 3$ are given by (8) in § 5.2.

$G_2\times A_1$	C		S	
	D	**N**	**D**	**N**
F_1	$2(C_a(x)C_{3a+2b}(y)+C_{a+b}(x)C_{3a+b}(y)$ $+C_{2a+b}(x)C_b(y))$	0	0	$-i2k_3(S_a(x)S_{3a+2b}(y)-S_{a+b}(x)S_{3a+b}(y)+S_{2a+b}(x)S_b(y))$
F_2	$(C_a(x)C_{3a+2b}(\sqrt{3}x)+C_{a+b}(x)C_{3a+b}(\sqrt{3}x)$ $+C_{2a+b}(x)C_b(\sqrt{3}x))C_c(z)$	0	0	$2i(-m_2S_a(x)C_{3a+2b}(\sqrt{3}x)+l_2S_{a+b}(x)C_{3a+b}(\sqrt{3}x)$ $-k_2S_{2a+b}(x)C_b(\sqrt{3}x))S_c(z)$
F_3	$2(C_{3a+2b}(y)+C_{3a+b}(y)+C_b(y))C_c(z)$	0	0	$-2i(m_1S_{3a+2b}(y)-l_1S_{3a+b}(y)+k_1S_b(y))S_c(z)$
F_4	$C_a(x)C_{3a+2b}(-\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{3})C_c(z)$ $+C_{a+b}(x)C_{3a+b}(-\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{3})C_c(z)$ $+C_{2a+b}(x)C_b(-\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{3})C_c(z)$	0	0	$i(\frac{m_1}{2}C_a(x)S_{3a+2b}(-\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{3})S_c(z)-\frac{l_1}{2}C_{a+b}(x)S_{3a+b}(-\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{3})S_c(z)$ $+\frac{k_1}{2}C_{2a+b}(x)S_b(-\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{3})S_c(z)+\frac{m_2\sqrt{3}}{2}S_a(x)C_{3a+2b}(-\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{3})S_c(z)$ $-\frac{l_2\sqrt{3}}{2}S_{a+b}(x)C_{3a+b}(-\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{3})S_c(z)+\frac{k_2\sqrt{3}}{2}S_{2a+b}(x)C_b(-\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{3})S_c(z))$
F_5	$(C_a(x)C_{3a+2b}(y)+C_{a+b}(x)C_{3a+b}(y)+C_{2a+b}(x)C_b(y))C_c(\frac{\sqrt{2}}{2})$	0	0	$ik_3(S_a(x)S_{3a+2b}(y)-S_{a+b}(x)S_{3a+b}(y)+S_{2a+b}(x)S_b(y))C_c(\frac{\sqrt{2}}{2})$

$G_2\times A_1$	S^l	
	D	**N**
F_1	0	$2im_3(-S_a(x)C_{3a+2b}(y)+S_{a+b}(x)C_{3a+b}(y)-S_{2a+b}(x)C_b(y))$
F_2	$(S_a(x)C_{3a+2b}(\sqrt{3}x)+S_{a+b}(x)C_{3a+b}(\sqrt{3}x)-S_{2a+b}(x)C_b(\sqrt{3}x))S_c(z)$	0
F_3	0	$2i(m_1C_{3a+2b}(y)-l_1C_{3a+b}(y)+k_1C_b(y))S_c(z)$
F_4	0	$i(-\frac{k_1}{2}C_b(\frac{\sqrt{6}}{3}-\frac{\sqrt{3x}}{3})S_c(z)C_{a+b}(x)+\frac{l_1}{2}S_c(z)C_{a+b}(x)C_{3a+b}(\frac{\sqrt{6}}{3}-\frac{\sqrt{3x}}{3})$ $+\frac{m_1}{2}C_a(x)S_c(z)C_{3a+2b}(\frac{\sqrt{6}}{3}-\frac{\sqrt{3x}}{3})-\frac{1}{2}\sqrt{3}k_2S_b(\frac{\sqrt{6}}{3}-\frac{\sqrt{3x}}{3})S_c(z)S_{2a+b}(x)$ $+\frac{1}{2}\sqrt{3}l_2S_c(z)S_{a+b}(x)S_{3a+b}(\frac{\sqrt{6}}{3}-\frac{\sqrt{3x}}{3})+\frac{1}{2}\sqrt{3}m_2S_a(x)S_c(z)S_{3a+2b}(\frac{\sqrt{6}}{3}-\frac{\sqrt{3x}}{3}))$
F_5	0	$ik_3(S_a(x)S_{3a+2b}(y)-S_{a+b}(x)S_{3a+b}(y)+S_{2a+b}(x)S_b(y))C_c(\frac{\sqrt{2}}{2})$

$G_2\times A_1$	S^s	
	D	**N**
F_1	$2(C_a(x)S_{3a+2b}(y)-C_{a+b}(x)S_{3a+b}(y)-C_{2a+b}(x)C_b(y))$	0
F_2	0	$2iC_c(z)(m_1C_a(x)C_{3a+2b}(\sqrt{3}x)-l_2C_{a+b}(x)C_{3a+b}(\sqrt{3}x)$ $-k_2C_{2a+b}(x)C_b(\sqrt{3}x))-k_2C_{2a+b}(x)C_b(\sqrt{3}x))$
F_3	$2(S_{3a+2b}(y)C_c(z)-S_{3a+b}(y)C_c(z)-S_b(y)C_c(z))$	0
F_4	$(C_a(x)S_{3a+2b}(-\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{3})-C_{a+b}(x)S_{3a+b}(-\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{3})-C_{2a+b}(x)S_b(-\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{3}))C_c(z)$	0
F_5	$C_a(x)S_{3a+2b}(y)C_c(\frac{\sqrt{2}}{2})-C_{a+b}(x)S_{3a+b}(y)C_c(\frac{\sqrt{2}}{2})-C_{2a+b}(x)S_b(y)C_c(\frac{\sqrt{2}}{2})$	0

TABLE 5. The values of C-, S-, S^l- and S^s-functions on the boundaries of fundamental region F of $G_2 \times A_1$. The separation constants k_i, l_i, m_i, $i = 1, 2, 3$ are given by (9) in § 5.2.

$A_1 \times A_1 \times A_1$	CCC		SSS	
	D	**N**	**D**	**N**
F_1	$C_a(x)C_b(y)C_c(0)$	0	0	$-\sqrt{2}\pi i k_3 S_a(x)S_b(y)C_c(0)$
F_2	$C_a(x)C_b(0)C_c(z)$	0	0	$-\sqrt{2}\pi i k_2 S_a(x)C_b(0)S_c(z)$
F_3	$C_a(0)C_b(y)C_c(z)$	0	0	$-\sqrt{2}\pi i k_1 C_a(0)S_b(y)S_c(z)$
F_4	$C_a(x)C_b(\frac{1}{\sqrt{2}})C_c(z)$	0	0	$\sqrt{2}\pi i k_2 S_a(x)C_b(\frac{1}{\sqrt{2}})S_c(z)$
F_5	$C_a(\frac{1}{\sqrt{2}})C_b(y)C_c(z)$	0	0	$\sqrt{2}\pi i k_1 C_a(\frac{1}{\sqrt{2}})S_b(y)S_c(z)$
F_6	$C_a(x)C_b(y)C_c(\frac{1}{\sqrt{2}})$	0	0	$\sqrt{2}\pi i k_3 S_a(x)S_b(y)C_c(\frac{1}{\sqrt{2}})$

$A_1 \times A_1 \times A_1$	CCS		SSC	
	D	**N**	**D**	**N**
F_1	0	$-\sqrt{2}\pi i k_3 C_a(x)C_b(y)C_c(0)$	$S_a(x)S_b(y)C_c(0)$	0
F_2	$C_a(x)C_b(0)S_c(z)$	0	0	$-\sqrt{2}\pi i k_2 S_a(x)C_b(0)C_c(z)$
F_3	$C_a(0)C_b(y)S_c(z)$	0	0	$-\sqrt{2}\pi i k_1 C_a(0)S_b(y)C_c(z)$
F_4	$C_a(x)C_b(\frac{1}{\sqrt{2}})S_c(z)$	0	0	$\sqrt{2}\pi i k_2 S_a(x)C_b(\frac{1}{\sqrt{2}})C_c(z)$
F_5	$C_a(\frac{1}{\sqrt{2}})C_b(y)S_c(z)$	0	0	$\sqrt{2}\pi i k_1 C_a(\frac{1}{\sqrt{2}})S_b(y)C_c(z)$
F_6	0	$\sqrt{2}\pi i k_3 C_a(x)C_b(y)C_c(\frac{1}{\sqrt{2}})$	$S_a(x)S_b(y)C_c(\frac{1}{\sqrt{2}})$	0

$A_1 \times A_1 \times A_1$	CSC		SCS	
	D	**N**	**D**	**N**
F_1	$C_a(x)S_b(y)C_c(0)$	0	0	$-\sqrt{2}\pi i k_3 S_a(x)C_b(y)C_c(0)$
F_2	0	$-\sqrt{2}\pi i k_2 C_a(x)C_b(0)C_c(z)$	$S_a(x)C_b(0)S_c(z)$	0
F_3	$C_a(0)S_b(y)C_c(z)$	0	0	$-\sqrt{2}\pi i k_1 C_a(0)C_b(y)S_c(z)$
F_4	0	$\sqrt{2}\pi i k_2 C_a(x)C_b(\frac{1}{\sqrt{2}})C_c(z)$	$S_a(x)C_b(\frac{1}{\sqrt{2}})S_c(z)$	0
F_5	$C_a(\frac{1}{\sqrt{2}})S_b(y)C_c(z)$	0	0	$\sqrt{2}\pi i k_1 C_a(\frac{1}{\sqrt{2}})C_b(y)S_c(z)$
F_6	$C_a(x)S_b(y)C_c(\frac{1}{\sqrt{2}})$	0	0	$\sqrt{2}\pi i k_3 S_a(x)C_b(y)C_c(\frac{1}{\sqrt{2}})$

$A_1 \times A_1 \times A_1$	SCC		CSS	
	D	**N**	**D**	**N**
F_1	$S_a(x)C_b(y)C_c(0)$	0	0	$-\sqrt{2}\pi i k_3 C_a(x)S_b(y)C_c(0)$
F_2	$S_a(x)C_b(0)C_c(z)$	0	0	$-\sqrt{2}\pi i k_2 C_a(x)C_b(0)S_c(z)$
F_3	0	$-\sqrt{2}\pi i k_1 C_a(0)C_b(y)C_c(z)$	$C_a(0)S_b(y)S_c(z)$	0
F_4	$S_a(x)C_b(\frac{1}{\sqrt{2}})C_c(z)$	0	0	$\sqrt{2}\pi i k_2 C_a(x)C_b(\frac{1}{\sqrt{2}})S_c(z)$
F_5	0	$\sqrt{2}\pi i k_1 C_a(\frac{1}{\sqrt{2}})C_b(y)C_c(z)$	$C_a(\frac{1}{\sqrt{2}})S_b(y)S_c(z)$	0
F_6	$S_a(x)C_b(y)C_c(\frac{1}{\sqrt{2}})$	0	0	$\sqrt{2}\pi i k_3 C_a(x)S_b(y)C_c(\frac{1}{\sqrt{2}})$

Table 6. The values of six families of functions on the boundaries of fundamental region F of $A_1 \times A_1 \times A_1$. The separation constants k_i, $i = 1, 2, 3$ are given by (10) in § 5.3.