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BETA CANTOR SERIES EXPANSION AND ADMISSIBLE SEQUENCES

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Abstract. We introduce a numeration system, called the *beta Cantor series expansion*, that generalizes the classical positive and negative beta expansions by allowing non-integer bases in the Q-Cantor series expansion. In particular, we show that for a fix $\gamma \in \mathbb{R}$ and a sequence B of real number bases, every element of the interval $[\gamma, \gamma + 1)$ has a *beta Cantor series expansion* with respect to B where the digits are integers in some alphabet $\mathcal{A}(B)$. We give a criterion in determining whether an integer sequence is admissible when B satisfies some condition. We provide a description of the reference strings, namely the expansion of γ and $\gamma + 1$, used in the admissibility criterion.

Keywords: Beta expansion, Q-Cantor series expansion, numeration system, admissibility.

1. Introduction

The subject of representations of real numbers is an extensively studied research field. In the seminal work [1], Renyi introduced the now well-known concept of beta expansions. Beta expansions are representations of real numbers using an arbitrary positive real base $\beta > 1$ obtained via the beta transformation $T_\beta : [0, 1) \longrightarrow [0, 1)$ given by

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor.$$

The iterates of T induce a numeration system on $[0, 1)$ wherein the expansion of an element $x \in [0, 1)$ is given by the sequence $d(\beta; x) = (d_1, d_2, \dots)$ with $d_i = \lfloor \beta T^{i-1}(x) \rfloor$. Thus, the digits d_i belong to the alphabet $\mathcal{A} = \{0, 1, \dots, \lfloor \beta \rfloor\}$ if $\beta \notin \mathbb{N}$ or $\mathcal{A} = \{0, 1, \dots, \beta - 1\}$ if $\beta \in \mathbb{N}$. Parry, in [2], considered the *admissibility* problem of determining the integer sequences over the alphabet $\mathcal{A}$ that appear as the beta expansion of a real number in the domain $[0, 1)$. Parry provided a necessary and sufficient condition (formulated in terms of the beta expansion of 1) for a sequence of integers to be *beta admissible*. In the subsequent paper [3], Parry extended the definition of the beta transformation to $T : [0, 1) \longrightarrow [0, 1)$ where $T(x) = \beta x + \alpha + \lfloor \beta x + \alpha \rfloor$ with $\beta > 1$ and $0 \le \alpha < 1$ and he also tackled the admissibility problem in this setting.

An important generalization of beta expansion is a positional numeration system that uses negative bases. As remarked by Frougny and Lai in [4], it appears that Grünwald was the first to introduce this idea in [5]. Here, we present a general formulation considered by Ito and Sadahiro in [6]. Let $1 < \beta \in \mathbb{R}$ and define $l_\beta := -\beta/(\beta + 1)$ and $r_\beta := 1/(\beta + 1)$. The negative beta transformation is the map $T_{-\beta} : [l_\beta, r_\beta) \longrightarrow [l_\beta, r_\beta)$ given by

$$T_{-\beta}(x) = -\beta x - \lfloor -\beta x - l_\beta \rfloor.$$

The map $T_{-\beta}$ also induces an expansion on the domain $[l_\beta, r_\beta)$, where the digits are given by $\lfloor -\beta T^i(x) - l_\beta \rfloor$. An admissibility criterion was also given in [6, Theorem 10]. (In [7], Liao and Steiner introduced the self-map $\hat{T} : (0, 1] \longrightarrow (0, 1]$ given by $\hat{T}(x) = -\beta x + \lfloor \beta x \rfloor + 1$. This transformation is conjugate to the one defined by Ito and Sadahiro and, as such, the results for the negative beta expansion can be restated using the map $\hat{T}$.)

As with the positive beta transformations, Dombek, et.al, in [8] introduced a parameter α to generalize the negative beta transformation defined by Ito and Sadahiro. They considered the map $T : [\alpha, \alpha + 1) \longrightarrow [\alpha, \alpha + 1)$ given by $T(x) = -\beta x - \lfloor -\beta x - \alpha \rfloor$ where $\beta > 1$ and $\alpha \in (-1, 0]$. (See also [9, 10] for other transformations inducing an expansion in a negative base.)

The motivation of the current study originates from a certain class of rotational beta expansions in dimension two (see [11, 12]). Rotational beta expansions generalize the notion of beta expansions in higher dimensions. Let $\mathcal{Z} = [0, 1) \times [0, 1)$ and $1 < \beta \in \mathbb{R}$. Define the four-fold rotational beta transformation $T : \mathcal{Z} \longrightarrow \mathcal{Z}$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -\beta y - \lfloor -\beta y \rfloor \\ \beta x - \lfloor \beta x \rfloor \end{bmatrix}.$$

It is easy to see that if we wish to keep track of the itinerary of a point $z \in \mathcal{Z}$ under T, then we need to alternatingly apply the functions $f_1(x) = -\beta x - \lfloor -\beta x \rfloor$ and $f_2(x) = \beta x - \lfloor \beta x \rfloor$ to an element $x \in [0, 1)$. This series of applications of the maps f_1 and f_2 yields a numeration system in $[0, 1)$ in two bases $-\beta$ and β as discussed in Section 2 below.

This numeration system is akin to the Q-Cantor series expansion [13]. Given a sequence $Q = (q_n)_{n \ge 1}$ of integers $q_n \ge 2$, the *Q-Cantor expansion* of a real number x is the unique expansion of the form

$$x = E_0 + \sum_{n\geq 1} \frac{E_n}{\Pi_{j=1}^{n} q_j}$$

where $E_0 = \lfloor x \rfloor$ and $E_n \in \{0, 1, \ldots, q_n - 1\}$ for all $n \geq 1$ such that $E_n \neq q_n - 1$ infinitely many number of times.

We call the numeration system considered in this paper the *beta Cantor series expansion* as it marries the notions of beta expansion and Q-Cantor series expansion. As mentioned in Section 2, the beta expansion of Parry and the negative beta expansion of Ito and Sadahiro are examples of beta Cantor series expansion. It is the hope of the authors that the beta Cantor series expansion provides a unified formulation for the positive and negative beta numeration systems to further highlight their similarities. After all, the positive and negative beta expansions share many similar properties (see e.g. [9, 14, 15]). Our goal is to extend the work of Parry on admissibility to beta Cantor series expansions. In Section 2, we define the transformations that induce the beta Cantor series expansion. In Section 3, we provide a discussion on the relationship between two different definitions of the expansion of $\gamma + 1$ (similar to the expansion of 1 in [2] and the expansion of r_β in [6]). In Section 4, we tackle the problem of finding a necessary and sufficient condition for a sequence to be admissible with respect to the beta Cantor series expansion.

2. B-expansion maps

Fix $\gamma \in \mathbb{R}$ and let $B = (\beta_1, \beta_2, \ldots)$ where $\beta_i \in \mathbb{R}$ for all $i \in \mathbb{N}$. For $j \in \mathbb{N}$, we define $f_j : [\gamma, \gamma + 1) \longrightarrow [\gamma, \gamma + 1)$ by $f_j(x) = \beta_j x - \lfloor \beta_j x - \gamma \rfloor$. For $m \in \mathbb{N}$, consider the transformation $T^m = T^m_B = T^m_{B,\gamma}$ on $[\gamma, \gamma + 1)$ given by

$$T^m(x) = f_m(\ldots f_3(f_2(f_1(x))) \ldots).$$

Hence,

$$T^m(x) = \beta_m T^{m-1}(x) - a_m(x)$$

where

$$a_m(x) = \left\lfloor \beta_m T^{m-1}(x) - \gamma \right\rfloor.$$

For $\beta = \beta_m$, we also define

$$u_\beta := \min\{\lfloor \beta\gamma - \gamma \rfloor, \lfloor \beta(\gamma + 1) - \gamma \rfloor\},$$

$$v_\beta := \max\{\lfloor \beta\gamma - \gamma \rfloor, \lfloor \beta(\gamma + 1) - \gamma \rfloor\}$$

and

$$\mathcal{A}(\beta) := \begin{cases} [u_\beta, v_\beta) \cap \mathbb{Z} & \text{if } \beta > 0 \text{ and } \beta + \gamma(\beta - 1) \in \mathbb{Z} \\ [u_\beta, v_\beta] \cap \mathbb{Z} & \text{otherwise.} \end{cases}$$

Then $a_m(x) \in \mathcal{A}(\beta_m)$. Define $\mathcal{A}(B) := \Pi_{m=1}^{\infty} \mathcal{A}(\beta_m)$, which is the set of all sequences $(d_1, d_2, \ldots)$ where $d_m \in \mathcal{A}(\beta_m)$. For ease of notation, we define $B[i, j] := \Pi_{m=i}^{j} \beta_m$. When $i = 1$, we write $B[j]$ instead of $B[1, j]$ with the convention that $B[0] := 1$. Observe that $B[m + i] = B[m]B[m + 1, m + i]$.

The transformations T^m induce a numeration system on the interval $[\gamma, \gamma + 1)$ over the alphabet $\mathcal{A}(B)$ if $\lim_{m\to\infty} |B[m]| = \infty$.

Proposition 2.1. *Let $B = (\beta_1, \beta_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$ and $x \in [\gamma, \gamma + 1)$. If $\lim_{m\to\infty} |B[m]| = \infty$, then*

$$x = \sum_{i=1}^{\infty} \frac{a_i(x)}{B[i]}.$$

Proof. For simplicity, let $a_j = a_j(x)$. Note that

$$T^{j-1}(x) = \frac{T^j(x) + a_j}{\beta_j}.$$

Hence,

$$x = \frac{a_1}{\beta_1} + \frac{T(x)}{\beta_1} = \frac{a_1}{\beta_1} + \frac{a_2}{\beta_1\beta_2} + \frac{T^2(x)}{\beta_1\beta_2}.$$

In general,

$$x = \sum_{i=1}^{m} \frac{a_i}{B[i]} + \frac{T^m(x)}{B[m]}.$$

This implies that as $m \to \infty$,

$$\left| x - \sum_{i=1}^{m} \frac{a_i}{B[i]} \right| = \left| \frac{T^m(x)}{B[m]} \right| \leq \frac{\max\{|\gamma|, |\gamma + 1|\}}{|B[m]|} \to 0.$$

∎

We write $x = \sum_{i=1}^{\infty} \frac{a_i}{B[i]}$ as $(a_1, a_2, \ldots)_B$. We call the sequence $d(B; x) := (a_1, a_2, \ldots)$ the B-expansion of x. Let $1 < \beta \in \mathbb{R}$. Note that if $\gamma = 0$ and $B = (\overline{\beta})$, then the B-expansion of x coincides with the classical β-expansion. (Here, $\overline{v}$ stands for the periodic repetition of a word v.) When $\gamma = -\beta/(\beta + 1)$ and $B = (\overline{-\beta})$, then the B-expansion coincides with the $(-\beta)$-expansion. If B is *periodic*, say $B = (\overline{\beta_1, \beta_2, \ldots, \beta_N})$ for some $N \in \mathbb{N}$, we also call the B-expansion as the $\{\beta_1, \ldots, \beta_N\}$-expansion of x. In this case, we may write $d(B; x)$ as $d(\beta_1, \ldots, \beta_n; x)$.

We may extend the definition of T^j to $\gamma + 1$ as has been done in [2] and [6]. For all $j \in \mathbb{N}$, define

$$T^j(\gamma + 1) := \beta_j T^{j-1}(\gamma + 1) - \left\lfloor \beta_j T^{j-1}(\gamma + 1) - \gamma \right\rfloor.$$

As in Proposition 2.1, we have $\gamma + 1 = \sum_{i=1}^{\infty} \frac{c_i}{B[i]}$ where $c_i := \left\lfloor \beta_i T^{i-1}(\gamma + 1) - \gamma \right\rfloor$. We also write $d(B; \gamma+1) = (c_1, c_2, \ldots)$. Note that $c_1 \in [u_{\beta_1}, v_{\beta_1}] \cap \mathbb{Z}$ and for $j > 1$, $c_j \in \mathcal{A}(\beta_j)$ since $T^{j-1}(\gamma + 1) < \gamma + 1$.

Example. Let $\alpha = -\beta = (1 + \sqrt{5})/2$ be the golden mean. Table 1 gives some information on the $\{\alpha, \beta\}$-transformations for various values of γ.

γ	$\mathcal{A}(\alpha)$	$\mathcal{A}(\beta)$	$d(\alpha,\beta;\gamma)$	$d(\alpha,\beta;\gamma+1)$
0	$\{0,1\}$	$\{-2,-1,0\}$	$(\overline{0})$	$(1,-1,\overline{0})$
$1/\alpha$	$\{0,1\}$	$\{-4,-3,-2\}$	$(0,-3,1,-3,\overline{1,-2})$	$(2,\overline{-2,1})$
2	$\{1,2\}$	$\{-7,-6\}$	$(1,-6,\overline{1,-7})$	$(2,\overline{-7,1})$

TABLE 1. The expansion of γ and $\gamma+1$ under various values of γ when $\alpha=-\beta=(1+\sqrt{5})/2$

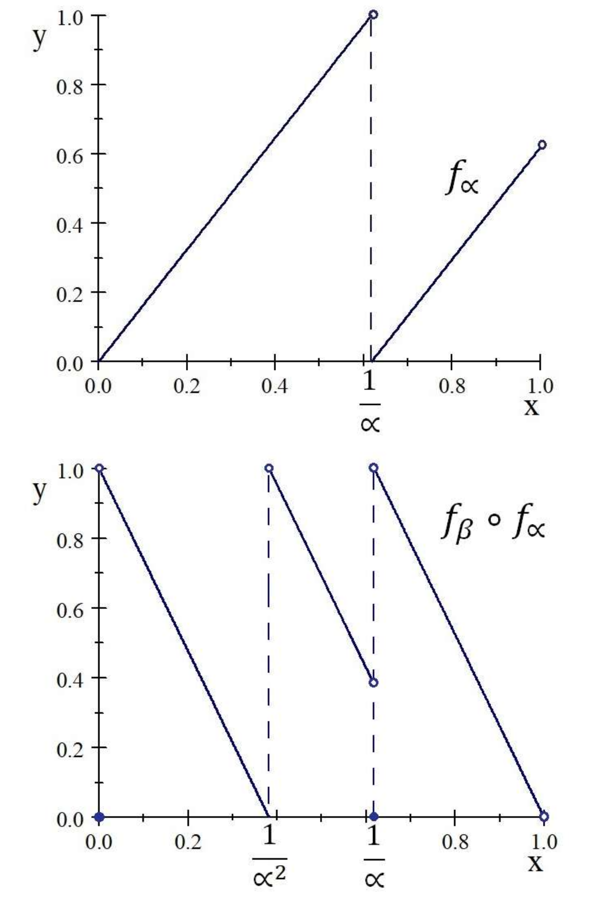

FIGURE 1. The maps T and T^2 when $\gamma=0$, $\alpha=-\beta=(1+\sqrt{5})/2$

Let us consider the particular case of $\gamma=0$. Then

$$f_\alpha(x)=\begin{cases}\alpha x & \text{if } x\in[0,1/\alpha)\\ \alpha x-1 & \text{if } x\in[1/\alpha,1)\end{cases}$$

and

$$f_\beta(x)=\begin{cases}-\alpha x & \text{if } x=0\\ -\alpha x+1 & \text{if } x\in(0,1/\alpha]\\ -\alpha x+2 & \text{if } x\in(1/\alpha,1)\ .\end{cases}$$

Figure 1 depicts the shape of the $\{\alpha,\beta\}$-transformations T and T^2. From these, we obtain a graph G (Figure 2) describing the dynamics of the map T^m. In this graph, the vertices are subintervals of $[\gamma,\gamma+1)$ that form its partition and there is a directed edge (dashed, if $\tau=\alpha$; and solid, otherwise) from vertex V_1 to vertex V_2 labelled d if and only if $V_2\subset f_\tau(V_1)$ and the corresponding digit is d.

Now, let $\gamma=1/\alpha$. We have

$$f_\alpha(x)=\begin{cases}\alpha x & \text{if } x\in[1/\alpha,1)\\ \alpha x-1 & \text{if } x\in[1,\alpha)\end{cases}$$

and

$$f_\beta(x)=\begin{cases}-\alpha x+2 & \text{if } x\in[1/\alpha,3/\alpha-1]\\ -\alpha x+3 & \text{if } x\in(3/\alpha-1,4/\alpha-1]\\ -\alpha x+4 & \text{if } x\in(4/\alpha-1,\alpha)\ .\end{cases}$$

Figure 3 gives a graph corresponding to $\gamma=1/\alpha$ where $J:=(1/\alpha,4-2\alpha)$, $K:=(4-2\alpha,3/\alpha-1)$, $L:=(3/\alpha-1,1)$, $M:=(1,2\alpha-2)$, $P:=(2\alpha-2,3-\alpha)$, $Q:=(3-\alpha,4/\alpha-1)$ and $R:=(4/\alpha-1,\alpha)$.

3. EXPANSION OF $\gamma+1$

The expansion of $\gamma+1$ defined in the previous section proves to be insufficient for our purposes and hence, the definition needs to be modified. In this section, we present another definition of the expansion of $\gamma+1$ analogous to those defined in [2] and [6, Lemma 6]. Hereafter, we assume $B=(\beta_1,\beta_2,\dots)\in\mathbb{R}^{\mathbb{N}}$ with $\lim_{m\to\infty}|B[m]|=\infty$.

Definition 1. *We define $d^*(B;\gamma+1)=(c_1^*,c_2^*,\dots)$ as the limit*

$$\lim_{x\to(\gamma+1)^-} d(B;x).$$

That is, for any $n\in\mathbb{N}$, there exists an $\epsilon_n>0$ such that for all $x\in(\gamma+1-\epsilon_n,\gamma+1)$ and for all $i<n$, the i-th digit of $d(B;x)$ is c_i^* where $\epsilon_{n+1}<\epsilon_n$ and $\epsilon_n\to0$ as $n\to\infty$.

Example. Let β be a quadratic Pisot number. Then β satisfies the minimal polynomial x^2-bx-c where $b\in\mathbb{N}$ and $1\le c\le b$; or $b\in\mathbb{N}-\{1,2\}$ and $2-b\le c\le -1$. Let $\gamma=0$. We compute for $d^*(\beta,-\beta;\gamma+1)=d^*(\beta,-\beta;1)$. Let $\epsilon>0$ be arbitrarily small.
Case 1. Let $1\le c\le b$. Then $b<\beta<b+1$. We have

$$\begin{aligned}T(1-\epsilon) &= \beta(1-\epsilon)-\lfloor\beta(1-\epsilon)\rfloor=\beta-\epsilon-b\\ T^2(1-\epsilon) &= -\beta(\beta-b-\epsilon)-\lfloor-\beta(\beta-b-\epsilon)\rfloor=\epsilon\\ T^3(1-\epsilon) &= \beta\epsilon-\lfloor\beta\epsilon\rfloor=\epsilon\\ T^4(1-\epsilon) &= -\beta\epsilon-\lfloor-\beta\epsilon\rfloor=-\epsilon+1.\end{aligned}$$

Hence, $d^*(\beta,-\beta;\gamma+1)=(\overline{b,-c,0,-1})$. It can be shown that $d(\beta,-\beta;\gamma+1)=(b,-c,\overline{0})$.

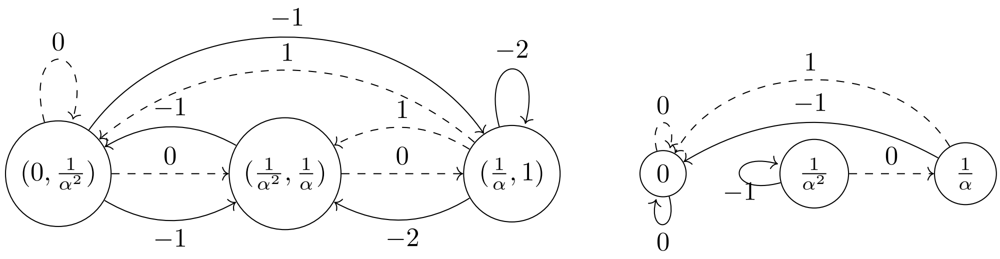

Figure 2. $T_{\alpha,\beta} : [0, 1) \longrightarrow [0, 1)$ with $\alpha = -\beta = (1 + \sqrt{5})/2$

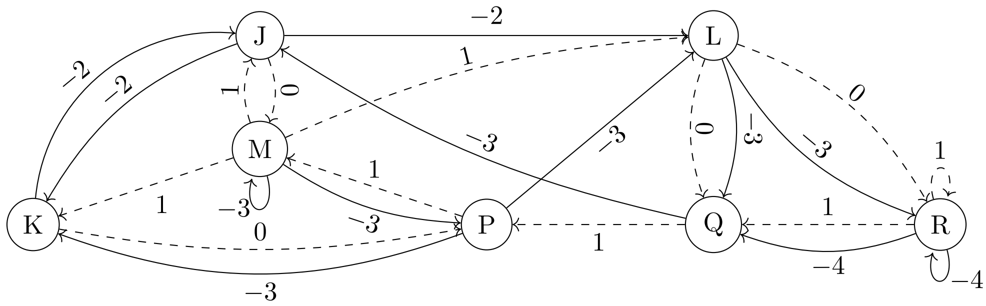

Figure 3. $T_{\alpha,\beta} : [1/\alpha, \alpha) \longrightarrow [1/\alpha, \alpha)$ with $\alpha = -\beta = (1 + \sqrt{5})/2$

Case 2. Let $2 - b \leq c \leq -1$. Then $b - 1 < \beta < b$. So

$$\begin{aligned}
T(1-\epsilon) &= \beta(1-\epsilon) - \lfloor \beta(1-\epsilon) \rfloor = \beta - \epsilon - b + 1 \\
T^2(1-\epsilon) &= -\beta(\beta - b + 1 - \epsilon) \\
&\quad - \lfloor -\beta(\beta - b + 1 - \epsilon) \rfloor = -\beta + b + \epsilon \\
T^3(1-\epsilon) &= \beta(-\beta + b + \epsilon) - \lfloor \beta(-\beta + b + \epsilon) \rfloor = \epsilon \\
T^4(1-\epsilon) &= -\beta\epsilon - \lfloor -\beta\epsilon \rfloor = -\epsilon + 1.
\end{aligned}$$

Hence, $d^*(\beta, -\beta; \gamma + 1) = (\overline{b-1, -b-c, -c, -1})$. Also, we have $d(\beta, -\beta; \gamma + 1) = (b - 1, -b - c, -c, \overline{0})$.

Example. Let $B = (\overline{\alpha, \beta})$ where $\alpha \in \mathbb{Z} < 0$ and $0 > \beta \in \mathbb{R}$. Suppose $\alpha(\gamma + 1) - \gamma \in \mathbb{Z}$ and $T^{2n}(\gamma + 1) = f_n(\beta) - \lfloor f_n(\beta) - \gamma \rfloor$ and $T^{2n+1}(\gamma + 1) = g_n(\beta) - \lfloor g_n(\beta) - \gamma \rfloor$ for some polynomials f_n and g_n of degree n in $\mathbb{Z}[x]$ where $f_n(\beta) - \gamma \notin \mathbb{Z}$ and $g_n(\beta) - \gamma \notin \mathbb{Z}$ for all $n \in \mathbb{N}$ (e.g. β may be taken to be transcendental over $\mathbb{Q}$ and $\gamma \in \mathbb{Q}$). Let $d(\alpha, \beta; \gamma + 1) = (c_1, c_2, \dots)$ and $d^*(\alpha, \beta; \gamma + 1) = (c_1^*, c_2^*, \dots)$. Then, for small $\epsilon > 0$, $c_1^* = \lfloor \alpha(\gamma + 1) - \gamma + \epsilon \rfloor = \alpha(\gamma + 1) - \gamma = c_1$. Since $f_n(\beta) - \gamma$ and $g_n(\beta) - \gamma$ are not integers, we can show that $T^n(\gamma + 1 - \epsilon) = T^n(\gamma + 1) + (-1)^{n+1}\epsilon$. Moreover, $c_{2n}^* = \lfloor f_n(\beta) - \gamma - \epsilon \rfloor = \lfloor f_n(\beta) - \gamma \rfloor = c_{2n}$ and $c_{2n+1}^* = \lfloor g_n(\beta) - \gamma + \epsilon \rfloor = \lfloor g_n(\beta) - \gamma \rfloor = c_{2n+1}$. Hence, $d(\alpha, \beta; \gamma + 1) = d^*(\alpha, \beta; \gamma + 1)$.

From the examples above, we see that $d(B; \gamma + 1)$ may or may not be equal to $d^*(B; \gamma + 1)$. In what follows, we characterize the B-expansions such that $d(B; \gamma + 1) = d^*(B; \gamma + 1)$.

Let sgn denote the signum function. Define $I_B := \{n \in \mathbb{N} \cup \{0\} \mid \operatorname{sgn}(B[n + 1]) > 0\}$ and

$$\mathcal{C}_B := \{\gamma \in \mathbb{R} \mid \beta_{n+1}T^n(\gamma + 1) - \gamma \notin \mathbb{Z} \text{ for } n \in I_B\}.$$

Proposition 3.1. *If $\gamma \in \mathcal{C}_B$, then $d(B; \gamma + 1) = d^*(B; \gamma + 1)$.*

Proof. We show by induction on $n \in \mathbb{N} \cup \{0\}$ that, for arbitrarily small constant, $\epsilon > 0$,

$$T^n(\gamma + 1 - \epsilon) = T^n(\gamma + 1) - B[n]\epsilon. \qquad (\star)$$

The case where $n = 0$ is clear. Suppose $(\star)$ holds for some $n \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
&T^{n+1}(\gamma + 1 - \epsilon) \\
&= \beta_{n+1}T^n(\gamma + 1 - \epsilon) - \lfloor \beta_{n+1}T^n(\gamma + 1 - \epsilon) - \gamma \rfloor \\
&= \beta_{n+1}T^n(\gamma + 1) - B[n + 1]\epsilon \\
&\quad - \lfloor \beta_{n+1}T^n(\gamma + 1) - B[n + 1]\epsilon - \gamma \rfloor .
\end{aligned}$$

Since $\gamma \in \mathcal{C}_B$ and ϵ is arbitrarily small, then $\lfloor \beta_{n+1}T^n(\gamma + 1) - B[n + 1]\epsilon - \gamma \rfloor = \lfloor \beta_{n+1}T^n(\gamma + 1) - \gamma \rfloor$. Therefore,

$$\begin{aligned}
T^{n+1}(\gamma + 1 - \epsilon) &= \beta_{n+1}T^n(\gamma + 1) - B[n + 1]\epsilon \\
&\quad - \lfloor \beta_{n+1}T^n(\gamma + 1) - \gamma \rfloor \\
&= T^{n+1}(\gamma + 1) - B[n + 1]\epsilon.
\end{aligned}$$

Thus, we get $d(B; \gamma + 1) = d^*(B; \gamma + 1)$. ∎

Proposition 3.2. *If $d(B; \gamma + 1) = d^*(B; \gamma + 1)$, then $\gamma \in \mathcal{C}_B$.*

Proof. Suppose $d(B; \gamma + 1) = (c_1, c_2, \dots)$ and $d^*(B; \gamma + 1) = (c_1^*, c_2^*, \dots)$ are equal. We show, by induction on $n \in \mathbb{N} \cup \{0\}$, that

$$T^n(\gamma + 1 - \epsilon) = T^n(\gamma + 1) - B[n]\epsilon. \qquad (\star)$$

The base case $n = 0$ is clear. Suppose ($\star$) holds for some $n \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned} & T^{n+1}(\gamma + 1 - \epsilon) \\ = \; & \beta_{n+1}(T^n(\gamma+1) - B[n]\epsilon) - c^*_{n+1} \\ = \; & \beta_{n+1}T^n(\gamma+1) - B[n+1]\epsilon - c_{n+1} \\ = \; & T^{n+1}(\gamma+1) - B[n+1]\epsilon. \end{aligned}$$

Thus, for all $n \in \mathbb{N} \cup \{0\}$ and $\epsilon > 0$ sufficiently small,

$$\begin{aligned} c^*_{n+1} &= \lfloor \beta_{n+1}T^n(\gamma+1) - B[n+1]\epsilon - \gamma \rfloor \\ &= \lfloor \beta_{n+1}T^n(\gamma+1) - \gamma \rfloor \\ &= c_{n+1}. \end{aligned}$$

If $n \in I_B$, then $\beta_{n+1}T^n(\gamma+1) - \gamma \notin \mathbb{Z}$. Thus, $\gamma \in \mathcal{C}_B$. ∎

Combining Prop. 3.1 and Prop. 3.2, we have the following theorem.

Theorem 3.3. *$\gamma \in \mathcal{C}_B$ if and only if $d(B;\gamma+1) = d^*(B;\gamma+1)$.*

Theorem 3.3 and [2, Theorem 3] imply Corollary 3.3.1 while Theorem 3.3 and [6, Lemma 6] imply Corollary 3.3.2.

Corollary 3.3.1. *Let $1 < \beta \in \mathbb{R}$. Let $T : [0,1) \longrightarrow [0,1)$ be the beta transformation given by $T(x) = \beta x - \lfloor \beta x \rfloor$. Then the following are equivalent:*

(1.) *$d(B;1) = d^*(B;1)$;*

(2.) *$\beta T^j(1) \notin \mathbb{Z}$ for all $j \in \mathbb{N} \cup \{0\}$;*

(3.) *$d(B;1)$ is infinite.*

Corollary 3.3.2. *Let $1 < \beta \in \mathbb{R}$. Let $T_{-\beta}$ be the negative beta transformation on $[l_\beta, r_\beta)$ given by $T_{-\beta}(x) = -\beta x - \lfloor -\beta x - l_\beta \rfloor$. Then the following are equivalent:*

(1.) *$d(B;r_\beta) = d^*(B;r_\beta)$;*

(2.) *$-\beta T^{2j+1}_{-\beta}(r_\beta) - l_\beta \notin \mathbb{Z}$ for all $j \in \mathbb{N} \cup \{0\}$;*

(3.) *$d(B;r_\beta)$ is not purely periodic of odd period.*

Next, we determine the relation between $d^*(B;\gamma+1)$ and $d(B;\gamma+1)$ when they are not equal (i.e., $\gamma \notin \mathcal{C}_B$). Define the propositional statement $E(B;k)$ to mean $\beta_{k+1}T^k(\gamma+1) - \gamma \in \mathbb{Z}$ and $\operatorname{sgn}(B[k+1]) > 0$.

Suppose $E(B;k)$ holds and k is minimal with such property. Then

$$T^{k+1}(\gamma+1) = \beta_{k+1}T^k(\gamma+1) - \lfloor \beta_{k+1}T^k(\gamma+1) - \gamma \rfloor = \gamma.$$

Thus, if $d(B;\gamma+1) = (c_1, c_2, \dots)$, then

$$d(B;\gamma+1) = (c_1, c_2, \dots, c_{k+1}) \circ d(\sigma^{k+1}(B);\gamma),$$

where $\circ$ denotes the usual word concatenation and σ^j ($j \in \mathbb{N}$) is the shift operator in $\mathbb{R}^{\mathbb{N}}$ given by

$$\sigma^j(r_1, r_2, \dots) = (r_{j+1}, r_{j+2}, \dots).$$

Moreover, from the proof of Proposition 3.1, we see that

$$\begin{aligned} & T^{k+1}(\gamma + 1 - \epsilon) \\ = \; & \beta_{k+1}T^k(\gamma+1) - B[k+1]\epsilon \\ & - \lfloor \beta_{k+1}T^k(\gamma+1) - B[k+1]\epsilon - \gamma \rfloor \\ = \; & \gamma + 1 - \epsilon. \end{aligned}$$

Therefore,

$$d^*(B;\gamma+1) = (c_1, c_2, \dots, c_{k+1} - 1) \circ d^*(\sigma^{k+1}(B);\gamma+1).$$

From the computation above, we see that the process of determining $d^*(B;\gamma+1)$ depends on the other sequences $d^*(\sigma^i(B);\gamma+1)$, $i \in \mathbb{N}$.

To illustrate this process, we present the two-base expansion case where we set $\alpha := \beta_1 > 0$ and $\beta := \beta_2 > 0$. We easily compute I_B to be $\mathbb{N} \cup \{0\}$. Suppose $E(B;k)$ is satisfied and k is minimal.

On the one hand, suppose k is odd. Then

$$d(\alpha,\beta;\gamma+1) = (c_1, c_2, \dots, c_{k+1}) \circ d(\alpha,\beta;\gamma)$$

and

$$d^*(\alpha,\beta;\gamma+1) = (c_1, c_2, \dots, c_{k+1} - 1) \circ d^*(\alpha,\beta;\gamma+1).$$

This implies that

$$d^*(\alpha,\beta;\gamma+1) = (\overline{c_1, c_2, \dots, c_{k+1} - 1}).$$

On the other hand, suppose k is even. Then

$$d(\alpha,\beta;\gamma+1) = (c_1, c_2, \dots, c_{k+1}) \circ d(\beta,\alpha;\gamma)$$

and

$$d^*(\alpha,\beta;\gamma+1) = (c_1, c_2, \dots, c_{k+1} - 1) \circ d^*(\beta,\alpha;\gamma+1).$$

Note that $I_{\sigma(B)} = \mathbb{N} \cup \{0\}$. Suppose that there is no $m \in \mathbb{N}$ such that $E(\sigma(B);m)$ holds. Then

$$d^*(\beta,\alpha;\gamma+1) = d(\beta,\alpha;\gamma+1)$$

and so,

$$d^*(\alpha,\beta;\gamma+1) = (c_1, c_2, \dots, c_{k+1} - 1) \circ d(\beta,\alpha;\gamma+1).$$

Let $d(\beta,\alpha;\gamma+1) = (q_1, q_2, \dots)$. If there exists $m \in \mathbb{N} \cup \{0\}$ such that $E(\sigma(B);m)$ holds and m is minimal and odd, then

$$d^*(\beta,\alpha;\gamma+1) = (\overline{q_1, q_2, \dots, q_{m+1} - 1}).$$

Therefore,

$$\begin{aligned} & d^*(\alpha,\beta;\gamma+1) \\ = \; & (c_1, c_2, \dots, c_{k+1} - 1) \circ (\overline{q_1, q_2, \dots, q_{m+1} - 1}). \end{aligned}$$

Finally, if m is even, we have

$$d^*(\beta,\alpha;\gamma+1) = (q_1, q_2, \dots, q_{m+1} - 1) \circ d^*(\alpha,\beta;\gamma+1).$$

Hence,

$$\begin{aligned} & d^*(\alpha, \beta; \gamma+1) \\ = \; & (c_1, c_2, \dots, c_{k+1}-1) \circ \\ & (q_1, q_2, \dots, q_{m+1}-1) \circ d^*(\alpha, \beta; \gamma+1) \\ = \; & (\overline{c_1, c_2, \dots, c_{k+1}-1, q_1, q_2, \dots, q_{m+1}-1}). \end{aligned}$$

To sum up, we have the following proposition.

Proposition 3.4. *Let $B = (\overline{\alpha, \beta})$ where $\alpha, \beta \in \mathbb{R} > 0$ and $\alpha\beta > 1$. Let $d(\alpha, \beta; \gamma+1) = (c_1, c_2, \dots)$ and $d(\beta, \alpha; \gamma+1) = (q_1, q_2, \dots)$. Then $d^*(\alpha, \beta; \gamma+1)$ can only assume one of the following forms:*

(1.) $(\overline{c_1, c_2, \dots, c_{2k}-1})$

(2.) $(c_1, c_2, \dots, c_{2k+1}-1, q_1, q_2, \dots)$

(3.) $(c_1, c_2, \dots, c_{2k+1}-1, \overline{q_1, q_2, \dots, q_{2m}-1})$

(4.) $(\overline{c_1, c_2, \dots, c_{2k+1}-1, q_1, q_2, \dots, q_{2m+1}-1})$

(5.) $(c_1, c_2, \dots)$

Examples. We now give examples to illustrate Prop. 3.4 (1–5) by providing values of $\alpha > 0$ and $\beta > 0$ with $\alpha\beta > 1$ and $\gamma = 0$ for each case. Let $r, s \in \mathbb{N}$.

(1.) Let $\alpha = r/s \notin \mathbb{Z}$ and $\beta = s$.

$$\begin{aligned} d(\alpha, \beta; \gamma+1) &= (\lfloor r/s \rfloor, r - s\lfloor r/s \rfloor, \overline{0}) \\ d^*(\alpha, \beta; \gamma+1) &= (\overline{\lfloor r/s \rfloor, r - s\lfloor r/s \rfloor - 1}) \end{aligned}$$

(2.) Let $\alpha = r$ and β be transcendental over $\mathbb{Q}$.

$$\begin{aligned} d(\alpha, \beta; \gamma+1) &= (r, \overline{0}) \\ d(\beta, \alpha; \gamma+1) &= d^*(\beta, \alpha; \gamma+1) = (q_1, q_2, \dots) \\ d^*(\alpha, \beta; \gamma+1) &= (r-1, q_1, q_2, \dots) \end{aligned}$$

(3.) Let $\alpha = (1+\sqrt{5})/2$ and $\beta = \alpha^2$.

$$\begin{aligned} d(\alpha, \beta; \gamma+1) &= (1, 1, 1, \overline{0}) \\ d(\beta, \alpha; \gamma+1) &= (2, 1, \overline{0}) \\ d^*(\alpha, \beta; \gamma+1) &= (1, 1, 0, \overline{2, 0}) \end{aligned}$$

(4.) Let $\beta = \alpha + 1$ where α is the (smallest) Pisot number which satisfies $\alpha^3 - \alpha - 1 = 0$.

$$\begin{aligned} d(\alpha, \beta; \gamma+1) &= (1, 0, 1, \overline{0}) \\ d(\beta, \alpha; \gamma+1) &= (2, 0, 1, \overline{0}) \\ d^*(\alpha, \beta; \gamma+1) &= (\overline{1, 0, 0, 2, 0, 0}) \end{aligned}$$

(5.) Let α be transcendental over $\mathbb{Q}$ and $\beta = r$. Then $d(\alpha, \beta; \gamma+1) = d^*(\alpha, \beta; \gamma+1)$.

To end this section, we recover the classical results for beta and negative beta expansions. Let $1 < \beta \in \mathbb{R}$. For the positive beta expansion, we see that $I_B = \mathbb{N} \cup \{0\}$ and $\mathcal{C}_B = \{\gamma \in \mathbb{R} \mid \beta T^n(\gamma+1) - \gamma \notin \mathbb{Z} \text{ for all } n \in I_B\}$. Suppose $0 \notin \mathcal{C}_B$. Then there exists a minimal $k \in \mathbb{N} \cup \{0\}$ such that $E(B; k)$ holds. We have

$$d(B; 1) = (c_1, \dots, c_{k+1}) \circ d(B; 0) = (c_1, \dots, c_{k+1}, \overline{0})$$

and

$$d^*(B; 1) = (\overline{c_1, \dots, c_{k+1}-1}).$$

In other words, $d^*(B; 1) =$

$$\begin{cases} d(B; 1) & \text{if } d(B; 1) \text{ is infinite} \\ (\overline{c_1, c_2, \dots, c_n - 1}) & \text{if } d(B; 1) = (c_1, \dots, c_n, \overline{0}). \end{cases}$$

For the negative beta expansion, we have $I_B = 2\mathbb{N} - 1$ and $\mathcal{C}_B = \{\gamma \in \mathbb{R} \mid -\beta T^{2n-1}(\gamma+1) - \gamma \notin \mathbb{Z} \text{ for all } n \in \mathbb{N}\}$. Suppose $l_\beta \notin \mathcal{C}_B$. Then there exists minimal $k \in \mathbb{N}$ such that $E(B; 2k-1)$ holds. Let $d(B; l_\beta) = (a_1, a_2, \dots)$. It is easy to see that $d(B; r_\beta) = (0, a_1, a_2, \dots)$. From $E(B; 2k-1)$, it follows that

$$d(B; r_\beta) = (c_1, \dots, c_{2k}, a_1, a_2, \dots).$$

This means that $c_1 = 0$; $c_i = a_{i-1}$ for all $i = 2, \dots, 2k$; and $a_i = a_{2k-1+i}$ for all $i \in \mathbb{N}$. Therefore,

$$d(B; l_\beta) = (\overline{a_1, \dots, a_{2k-1}})$$

and

$$\begin{aligned} d^*(B; r_\beta) &= (c_1, \dots, c_{2k} - 1) \circ d^*(B; r_\beta) \\ &= (\overline{c_1, \dots, c_{2k} - 1}) \\ &= (\overline{0, a_1, \dots, a_{2k-1} - 1}). \end{aligned}$$

This is equivalent to

$$d^*(B; r_\beta) = \begin{cases} (\overline{0, a_1, a_2, \dots, a_{2n-1} - 1}) \\ \quad \text{if } d(B; l_\beta) = (\overline{a_1, a_2, \dots, a_{2n-1}}) \\ d(B; r_\beta) \quad \text{otherwise.} \end{cases}$$

4. Admissible Sequences

Throughout this section, we let $B = (\beta_1, \beta_2, \dots) \in \mathbb{R}^{\mathbb{N}}$ with $\lim_{m \to \infty} |B[m]| = \infty$. A B-representation of a real number $x \in [\gamma, \gamma+1)$ is an expansion of the form

$$x = \sum_{i=1}^{\infty} \frac{d_i}{B[i]}$$

with $(d_1, d_2, \dots) \in \mathcal{A}(B)$. Note that the condition $\lim |B[m]| = \infty$ does not guarantee that any sequence $(d_1, d_2, \dots)$ in $\mathcal{A}(B)$ is a B-representation of a real number x since the series $\sum_{i=1}^{\infty} d_i/B[i]$ may not converge. If the sum converges, we adopt the notation $(d_1, d_2, \dots)_B = \sum_{i=1}^{\infty} d_i/B[i]$.

Now, the B-expansion of x is a particular B-representation of x. Deciding whether a sequence $(d_1, d_2, \dots)$ in $\mathcal{A}(B)$ is the B-expansion of an element of $[\gamma, \gamma+1)$, thus, entails showing that the series converges.

Definition 2. *An integer sequence $(d_1, d_2, \dots) \in \mathcal{A}(B)$ is B-admissible if there is an $x \in [\gamma, \gamma+1)$ such that $d(B; x) = (d_1, d_2, \dots)$.*

The admissibility of sequences with respect to the B-expansion map is related to the admissibility of sequences for a special class of rotational beta expansion map. Let $\mathcal{Z} = [0,1) \times [0,1)$ and $1 < \beta \in \mathbb{R}$. Define the map $\mathcal{T} : \mathcal{Z} \longrightarrow \mathcal{Z}$ by

$$\mathcal{T}((x,y)) = (-\beta y - \lfloor -\beta y \rfloor,\ \beta x - \lfloor \beta x \rfloor).$$

Let T be the B-expansion map on $[0,1)$ with $B = \overline{(-\beta,\beta)}$. It follows that for all $n \in \mathbb{N}$, we have

$$\mathcal{T}^{2n-1}(x,y) = \left(T_B^{2n-1}(y),\ T_{\sigma(B)}^{2n-1}(x)\right)$$

and

$$\mathcal{T}^{2n}(x,y) = \left(T_{\sigma(B)}^{2n}(x),\ T_B^{2n}(y)\right).$$

So, if $d(B;y) = (a_1,a_2,\dots)$ and $d(\sigma(B);x) = (b_1,b_2,\dots)$, then the expansion of (x,y) with respect to $\mathcal{T}$ is

$$((a_1,b_1),(b_2,a_2),(a_3,b_3),\dots).$$

Proposition 4.1. *Let $B = \overline{(-\beta,\beta)}$ with $\beta > 1$. Then $(a_1,a_2,\dots) \in \mathcal{A}(B)$ is B-admissible and $(b_1,b_2,\dots) \in \mathcal{A}(\sigma(B))$ is $\sigma(B)$-admissible if and only if $((a_1,b_1),(b_2,a_2),(a_3,b_3),\dots)$ is admissible with respect to $\mathcal{T}$.*

In this section, our goal is to provide an admissibility criterion for sequences in $\mathcal{A}(B)$. We first mention few results.

Lemma 4.2. *Let $x \in [\gamma,\gamma+1)$ such that $d(B;x) = (a_1,a_2,\dots)$. For $n \in \mathbb{N}$,*

$$T^n(x) = B[n]x - \sum_{i=1}^{n} \frac{a_i B[n]}{B[i]}.$$

Proof. We prove this lemma by induction. Let $x \in [\gamma,\gamma+1)$. Then $T(x) = B[1]x - a_1$. Suppose that for some $k \in \mathbb{N}$, $T^k(x) = B[k]x - \sum_{i=1}^{k} a_i B[k]/B[i]$. Thus,

$$\begin{aligned} T^{k+1}(x) &= \beta_{k+1}T^k(x) - a_{k+1} \\ &= B[k+1]x - \sum_{i=1}^{k} \frac{a_i B[k+1]}{B[i]} \\ &\quad - a_{k+1}\frac{B[k+1]}{B[k+1]} \\ &= B[k+1]x - \sum_{i=1}^{k+1} \frac{a_i B[k+1]}{B[i]}. \end{aligned}$$

∎

In the following lemma, we give certain conditions for a B-representation $(d_1,d_2,\dots)$ to be a B-expansion. Note that the convergence of the sum $(d_1,d_2,\dots)_B$ implies the convergence of $(d_{k+1},d_{k+2},\dots)_{\sigma^k(B)}$ for all $k \in \mathbb{N}\cup\{0\}$.

Lemma 4.3. *Let $(d_1,d_2,\dots)$ be a B-representation of $x \in [\gamma,\gamma+1)$. If $(d_{k+1},d_{k+2},\dots)_{\sigma^k(B)} \in [\gamma,\gamma+1)$ for all $k \in \mathbb{N}\cup\{0\}$, then $d(B;x) = (d_1,d_2,\dots)$.*

Proof. By induction on $n \in \mathbb{N}$, we prove that $d_n = \lfloor \beta_n T^{n-1}(x) - \gamma \rfloor$ and

$$T^n(x) = (d_{n+1},d_{n+2},\dots)_{\sigma^n(B)}.$$

Note that $\beta_1 x - d_1 = (d_2,d_3,\dots)_{\sigma(B)} \in [\gamma,\gamma+1)$. Hence, $d_1 = \lfloor \beta_1 T^0(x) - \gamma \rfloor$ and $T(x) = (d_2,d_3,\dots)_{\sigma(B)}$.

Suppose the claim holds for $n \le k$ where $k \in \mathbb{N}$. Then

$$\begin{aligned} &\beta_{k+1}T^k(x) - d_{k+1} \\ =\ & \beta_{k+1}\left(\frac{d_{k+1}}{\beta_{k+1}} + \frac{d_{k+2}}{\beta_{k+1}\beta_{k+2}} + \dots\right) - d_{k+1} \\ =\ & \frac{d_{k+2}}{\beta_{k+2}} + \frac{d_{k+3}}{\beta_{k+2}\beta_{k+3}} + \dots \\ =\ & (d_{k+2},d_{k+3},\dots)_{\sigma^{k+1}(B)} \in [\gamma,\gamma+1). \end{aligned}$$

Hence, $d_{k+1} = \lfloor \beta_{k+1}T^{k+1}(x) - \gamma \rfloor$ and $T^{k+1}(x) = (d_{k+2},d_{k+3},\dots)_{\sigma^{k+1}(B)}$. ∎

Corollary 4.3.1. *Let $x \in [\gamma,\gamma+1)$ such that $d(B;x) = (a_1,a_2,\dots)$. Then*

$$d(\sigma^n(B);T^n(x)) = (a_{n+1},a_{n+2},\dots).$$

Proof. By Lemma 4.2,

$$\begin{aligned} T^n(x) &= B[n]\left(x - \sum_{i=1}^{n}\frac{a_i}{B[i]}\right) \\ &= B[n]\sum_{i\ge n+1}\frac{a_i}{B[i]} \\ &= B[n]\sum_{i\ge 1}\frac{a_{n+i}}{B[n+i]} \\ &= \sum_{i\ge 1}\frac{a_{n+i}}{B[n+1,n+i]}. \end{aligned}$$

Thus, $(a_{n+1},a_{n+2},\dots)$ is a $\sigma^n(B)$-representation of $T^n(x)$. For all $k \in \mathbb{N}$, we have

$$\begin{aligned} &\sigma^k(a_{n+1},a_{n+2},\dots)_{\sigma^k(\sigma^n(B))} \\ =\ & \sigma^{n+k}(a_1,a_2,\dots)_{\sigma^{n+k}(B)} \\ =\ & T^{n+k}(x) \in [\gamma,\gamma+1). \end{aligned}$$

The conclusion then follows from Lemma 4.3. ∎

Remark. Proposition 2.1, Lemma 4.2, and Corollary 4.3.1 also hold when $x = \gamma + 1$.

From Lemma 4.3 and Corollary 4.3.1, we obtain the following proposition, which gives an admissibility criterion for a sequence $(d_1,d_2,\dots) \in \mathcal{A}(B)$ in terms of $\sigma^k(d_1,d_2,\dots)_{\sigma^k(B)}$.

Proposition 4.4. *A sequence $(d_1,d_2,\dots) \in \mathcal{A}(B)$ is B-admissible if and only if $\sigma^k(d_1,d_2,\dots)_{\sigma^k(B)} \in [\gamma,\gamma+1)$ for all $k \in \mathbb{N}\cup\{0\}$.*

Now, we provide another admissibility criterion – this time, in terms of the shifts of a sequence $(x_1,x_2,\dots) \in \mathcal{A}(B)$. To this end, we need to introduce an order $\prec_B$ on $\mathcal{A}(B)$.

Definition 3. *Let* $(a_1, a_2, \dots)$ *and* $(b_1, b_2, \dots)$ *be in* $\mathcal{A}(B)$. *We say*

$$(a_1, a_2, \dots) \prec_B (b_1, b_2, \dots)$$

if and only if there exists $k \in \mathbb{N}$ *such that* $b_i = a_i$ *for all* $i = 1, 2, \dots, k-1$ *and* $b_k \neq a_k$ *where*

$$(b_k - a_k)\operatorname{sgn}(B[k]) \geq 1.$$

If $(a_1, a_2, \dots) \prec_B (b_1, b_2, \dots)$ *or* $(a_1, a_2, \dots) = (b_1, b_2, \dots)$, *we write* $(a_1, a_2, \dots) \preceq_B (b_1, b_2, \dots)$.

Remark. For the classical positive and negative beta expansions, the order $\prec_B$ coincides with the orders defined in [2] and [6], respectively.

The following proposition tells us that the monotonicity of points in $[\gamma, \gamma+1)$ is carried over to the ordering of words with respect to $\prec_B$.

Proposition 4.5. *Let* $x, y \in [\gamma, \gamma+1)$. *Then* $d(B; x) \prec_B d(B; y)$ *if and only if* $x < y$.

Proof. Let $d(B; x) = (x_1, x_2, \dots)$ and $d(B; y) = (y_1, y_2, \dots)$. Let $k \in \mathbb{N}$ be the least integer such that $x_k \neq y_k$. Suppose $d(B; x) \prec_B d(B; y)$. Then

$$y - x = \frac{y_k - x_k}{B[k]} + \sum_{i \geq k+1} \frac{y_i - x_i}{B[i]}.$$

We have

$$\begin{aligned}
\sum_{i \geq k+1} \frac{y_i - x_i}{B[i]} &= \sum_{i \geq 1} \frac{y_{k+i} - x_{k+i}}{B[k+i]} \\
&= \frac{1}{B[k]} \sum_{i \geq 1} \frac{y_{k+i} - x_{k+i}}{B[k+1, k+i]} \\
&= \frac{T^k(y) - T^k(x)}{B[k]} \\
&= \frac{(T^k(y) - T^k(x))\operatorname{sgn}(B[k])}{|B[k]|} \\
&> \frac{-1}{|B[k]|}.
\end{aligned}$$

Thus,

$$y - x > \frac{(y_k - x_k)\operatorname{sgn}(B[k]) - 1}{|B[k]|} \geq 0.$$

For the reverse implication, suppose

$$0 < y - x = \frac{y_k - x_k + T^k(y) - T^k(x)}{B[k]}.$$

Note that $-1 < T^k(y) - T^k(x) < 1$. When $\operatorname{sgn}(B[k]) > 0$, then $y_k - x_k + 1 > 0$. This implies that $y_k - x_k \geq 0$ since both y_k and x_k are integers. But since $y_k \neq x_k$, then $y_k - x_k \geq 1$. However, when $\operatorname{sgn}(B[k]) < 0$, then $0 > y_k - x_k - 1$. Thus, $y_k - x_k \leq 0$. But since $y_k \neq x_k$, then $y_k - x_k \leq -1$. In both cases, $(y_k - x_k)\operatorname{sgn}(B[k]) \geq 1$. ∎

Proposition 4.5, together with Corollary 4.3.1, implies the following result.

Corollary 4.5.1. *If* $(d_1, d_2, \dots) \in \mathcal{A}(B)$ *is* B*-admissible, then, for all* $n \in \mathbb{N}$,

$$d(\sigma^n(B); \gamma) \preceq_{\sigma^n(B)} (d_{n+1}, d_{n+2}, \dots).$$

Analogous to Proposition 2.1 and Lemma 4.3, we provide a relation between $d^*(B; \gamma+1)$ and $\gamma+1$.

Proposition 4.6. *If* $d^*(B; \gamma+1) = (c_1^*, c_2^*, \dots)$, *then* $\gamma + 1 = (c_1^*, c_2^*, \dots)_B$ *and* $(c_{k+1}^*, c_{k+2}^*, \dots)_{\sigma^k(B)} \in [\gamma, \gamma+1]$ *for all* $k \in \mathbb{N}$.

Proof. Suppose $d^*(B; \gamma+1) = (c_1^*, c_2^*, \dots)$. Then there exist a sequence $\{\epsilon_n\}$ converging to 0 and a sequence $\{Y_n\}$ such that $Y_n \in (\gamma + 1 - \epsilon_n, \gamma+1)$ and $d(B; Y_n) = (c_1^*, \dots, c_n^*, y_{n,1}, y_{n,2}, \dots)$. Thus,

$$Y_n = \sum_{i=1}^{n} \frac{c_i^*}{B[i]} + \sum_{i \geq 1} \frac{y_{n,i}}{B[n+i]}.$$

Since

$$\sum_{i \geq 1} \frac{y_{n,i}}{B[n+i]} = \frac{1}{B[n]} \sum_{i \geq 1} \frac{y_{n,i}}{\sigma^n(B)[i]} \in \frac{1}{B[n]}[\gamma, \gamma+1),$$

then

$$\lim_{n \to \infty} \sum_{i \geq 1} \frac{y_{n,i}}{B[n+i]} = 0.$$

Hence,

$$\begin{aligned}
\gamma + 1 &= \lim_{n \to \infty} Y_n \\
&= \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{c_i^*}{B[i]} + \sum_{i \geq 1} \frac{y_{n,i}}{B[i]} \right) \\
&= \sum_{i \geq 1} \frac{c_i^*}{B[i]}.
\end{aligned}$$

Now, for $j, k \in \mathbb{N}$, let us consider

$$d(B; Y_{k+j}) = (c_1^*, c_2^*, \dots, c_{k+j}^*, y_{k+j,1}, y_{k+j,2}, \dots).$$

By Corollary 4.3.1,

$$\begin{aligned}
&d(\sigma^k(B); T^k(Y_{k+j})) \\
&= (c_{k+1}^*, \dots, c_{k+j}^*, y_{k+j,1}, y_{k+j,2}, \dots).
\end{aligned}$$

Hence, $(c_{k+1}^*, \dots, c_{k+j}^*, y_{k+j,1}, y_{k+j,2}, \dots)_{\sigma^k(B)} \in [\gamma, \gamma+1)$. That is,

$$\gamma \leq w_j := \sum_{i=1}^{j} \frac{c_{k+i}^*}{\sigma^k(B)[i]} + \sum_{i \geq 1} \frac{y_{k+j,i}}{\sigma^k(B)[j+i]} < \gamma + 1.$$

Since $\{w_j\}$ tends to $(c_{k+1}^*, c_{k+2}^*, \dots)_{\sigma^k(B)}$, then $\gamma \leq (c_{k+1}^*, c_{k+2}^*, \dots) \leq \gamma + 1$. ∎

Proposition 4.7. *If* $x \in [\gamma, \gamma+1)$, *then* $d(B; x) \prec_B d^*(B; \gamma+1)$.

Proof. Let $d^*(B;\gamma+1) = (c_1^*, c_2^*, \dots)$. Then there exist a sequence ϵ_n tending to zero and $Y_n \in (\gamma+1-\epsilon_n, \gamma+1)$ such that $d(B;Y_n) = (c_1^*, \dots, c_n^*, y_{n,1}, y_{n,2}, \dots)$ with $c_{n+1}^* \neq y_{n,1}$, so that $d(B;Y_n) \neq d^*(B;\gamma+1)$.

Suppose $d(B;Y_n) \succ_B d^*(B;\gamma+1)$. There exists $Y_{n+m} \in (Y_n, \gamma+1)$ where $m \geq 1$ such that

$$d(B;Y_{n+m}) = (c_1^*, \dots, c_{n+m}^*, y_{n+m,1}, y_{n+m,2}, \dots).$$

Since $d(B;Y_n) \succ_B d^*(B,\gamma+1)$, then $(y_{n,1} - c_{n+1}^*)\operatorname{sgn}(B[n+1]) \geq 1$, implying that $d(B;Y_n) \succ_B d(B;Y_{n+m})$. By Proposition 4.5, $Y_n > Y_{n+m}$ which is a contradiction since $Y_{n+m} \in (Y_n, \gamma+1)$. Hence, if $x < Y_n$, then $d(B;x) \prec_B d(B;Y_n) \prec_B d^*(B;\gamma+1)$. ∎

Definition 4. *A sequence $(d_1, d_2, \dots) \in \mathcal{A}(B)$ satisfies the* lexicographic restriction *if, for all $k \in \mathbb{N} \cup \{0\}$,*

$$\begin{aligned} d(\sigma^k(B);\gamma) &\preceq_{\sigma^k(B)} \sigma^k(d_1, d_2, \dots) \\ &\prec_{\sigma^k(B)} d^*(\sigma^k(B);\gamma+1). \end{aligned}$$

Combining Corollary 4.5.1 and Proposition 4.7 yields the following proposition.

Proposition 4.8. *Let $x \in [\gamma, \gamma+1)$. Then $d(B;x)$ satisfies the lexicographic restriction.*

We now show that the converse of Prop. 4.8 holds under some condition. To proceed, consider a sequence $z = (z_1, z_2, \dots) \in \mathcal{A}(B)$. For $i \in \mathbb{N}$, we define

$$z(i,j) = \begin{cases} (z_i, z_{i+1}, \dots, z_{i+j}) & \text{if } j \in \mathbb{N} \cup \{0\} \\ (z_i, z_{i+1}, \dots) & \text{if } j = \infty \end{cases}$$

and set

$$z(i,j)_{\sigma^{i-1}(B)} = \sum_{k=0}^{j} \frac{z_{i+k}}{B[i,i+k]},$$

provided that the sum converges if $j = \infty$. For $n \in \mathbb{N} \cup \{0\}$, let $u^{(n)} = d(\sigma^n(B);\gamma)$ and $v^{(n)} = d^*(\sigma^n(B);\gamma+1)$.

Lemma 4.9. *Let $w = (w_1, w_2, \dots) \in \mathcal{A}(B)$. If w satisfies the lexicographic restriction, then there are infinitely many n such that at least one of the two holds:*

(1.) *$B[n] > 0$ and $(w(1,n-1) \circ u^{(n)})_B \geq \gamma$;*

(2.) *$B[n] < 0$ and $(w(1,n-1) \circ v^{(n)})_B \geq \gamma$.*

Proof. Suppose $w \neq d(B;\gamma)$. For the base of the induction, we set $n = 0$ and define $w(1,-1)$ as the empty word. Then $(w(1,-1) \circ u^{(0)})_B = \gamma$. Likewise $(w(1,-1) \circ v^{(0)})_B = \gamma + 1 \geq \gamma$. Now, let $m \in \mathbb{N} \cup \{0\}$.

CASE 1 Suppose $B[m] > 0$ and $(w(1,m-1) \circ u^{(m)})_B \geq \gamma$ hold. By the lexicographic restriction,

$$u^{(m)} = d(\sigma^m(B);\gamma) \prec_{\sigma^m(B)} \sigma^m(w) = w(m+1,\infty).$$

Thus, there exists a least positive integer l such that $w_{m+i} = u^{(m)}(i,0)$ for all $i < l$ and

$$(w_{m+l} - u^{(m)}(l,0))\operatorname{sgn}(\sigma^m(B)[l]) \geq 1.$$

Since $B[m] > 0$, then $\operatorname{sgn}(\sigma^m(B)[l]) = \operatorname{sgn}(B[m+l])$.

CASE 1.1 Suppose $B[m+l] > 0$ so that $(w_{m+l} - u^{(m)}(l,0)) \geq 1$. We have

$$\begin{aligned} & (w(1,m+l-1) \circ u^{(m+l)})_B \\ - \; & (w(1,m-1) \circ u^{(m)})_B \\ = \; & \frac{w_{m+l} - u^{(m)}(l,0)}{B[m+l]} \\ + \; & \frac{(u^{(m+l)})_{\sigma^{m+l}(B)} - (u^{(m)}(l+1,\infty))_{\sigma^{m+l}(B)}}{B[m+l]}. \end{aligned}$$

Now,

$$\frac{w_{m+l} - u^{(m)}(l,0)}{B[m+l]} \geq \frac{1}{B[m+l]}.$$

Meanwhile,

$$\begin{aligned} & (u^{(m+l)})_{\sigma^{m+l}(B)} - (u^{(m)}(l+1,\infty))_{\sigma^{m+l}(B)} \\ = \; & \gamma - T^l_{\sigma^m(B)}(\gamma). \end{aligned}$$

Hence,

$$\frac{(u^{(m+l)})_{\sigma^{m+l}(B)} - (u^{(m)}(l+1,\infty))_{\sigma^{m+l}(B)}}{B[m+l]}$$

is greater than $-1/B[m+l]$ and less than or equal to 0. Therefore,

$$(w(1,m+l-1) \circ u^{(m+l)})_B - (w(1,m-1) \circ u^{(m)})_B \geq 0,$$

implying that

$$(w(1,m+l-1) \circ u^{(m+l)})_B > (w(1,m-1) \circ u^{(m)})_B \geq \gamma.$$

CASE 1.2 Suppose $B[m+l] < 0$. Then

$$\begin{aligned} & (w(1,m+l-1) \circ v^{(m+l)})_B \\ - \; & (w(1,m-1) \circ u^{(m)})_B \\ = \; & \frac{w_{m+l} - u^{(m)}(l,0)}{B[m+l]} \\ + \; & \frac{(v^{(m+l)})_{\sigma^{m+l}(B)} - (u^{(m)}(l+1,\infty))_{\sigma^{m+l}(B)}}{B[m+l]}. \end{aligned}$$

Since $B[m+l] < 0$, we have

$$\frac{w_{m+l} - u^{(m)}(l,0)}{B[m+l]} \geq \frac{-1}{B[m+l]}.$$

Moreover,

$$\begin{aligned} & (v^{(m+l)})_{\sigma^{m+l}(B)} - (u^{(m)}(l+1,\infty))_{\sigma^{m+l}(B)} \\ & = \gamma + 1 - T^l_{\sigma^m(B)}(\gamma). \end{aligned}$$

It follows that

$$\frac{(v^{(m+l)})_{\sigma^{m+l}(B)} - (u^{(m)}(l+1,\infty))_{\sigma^{m+l}(B)}}{B[m+l]}$$

is less than 0 but greater than or equal to $1/B[m+l]$. Therefore,

$$(w(1, m+l-1) \circ v^{(m+l)})_B - (w(1, m-1) \circ u^{(m)})_B \geq 0,$$

and consequently,

$$(w(1, m+l-1) \circ v^{(m+l)})_B > (w(1, m-1) \circ u^{(m)})_B \geq \gamma.$$

CASE 2 Suppose $B[m] < 0$ and $(w(1, m-1) \circ v^{(m)})_B \geq \gamma$ hold. By the lexicographic restriction,

$$\sigma^m(w) = w(m+1, \infty) \prec_{\sigma^m(B)} v^{(m)} = d^*(\sigma^m(B); \gamma+1).$$

Thus, there exists a least positive integer l such that $w_{m+i} = v^{(m)}(i, 0)$ for all $i < l$ and

$$(v^{(m)}(l, 0) - w_{m+l}) \operatorname{sgn}(\sigma^m(B)[l]) \geq 1.$$

Since $B[m] < 0$, we have $\operatorname{sgn}(\sigma^m(B)[l]) = -\operatorname{sgn}(B[m+l])$. As before, we have two subcases: $\operatorname{sgn}(B[m+l]) > 0$ and $\operatorname{sgn}(B[m+l]) < 0$. The proofs are similar. ∎

Analogous to Lemma 4.9, we have the following result.

Lemma 4.10. *Let $w \in \mathcal{A}(B)$. If w satisfies the lexicographic restriction, then there are infinitely many n such that at least one of the two holds:*

(1.) *$B[n] < 0$ and $(w(1, n-1) \circ u^{(n)})_B \leq \gamma + 1$;*

(2.) *$B[n] > 0$ and $(w(1, n-1) \circ v^{(n)})_B \leq \gamma + 1$.*

We now apply Lemmas 4.9 and 4.10 to prove the next proposition.

Proposition 4.11. *Let $w = (w_1, w_2, \dots) \in \mathcal{A}(B)$ such that the sum $\sigma^k(w)_{\sigma^k(B)}$ converges for all $k \in \mathbb{N} \cup \{0\}$. If w satisfies the lexicographic restriction, then $\sigma^k(w)_{\sigma^k(B)} \in [\gamma, \gamma+1)$.*

Proof. We show that $\gamma \leq w_B \leq \gamma + 1$. Let $E_n(w) := \sigma^n(w)_{\sigma^n(B)}$, which by assumption converges. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned} w_B &= \sum_{k=1}^{n} \frac{w_k}{B[k]} + \frac{E_n(w)}{B[n]} \\ &= w(1, n-1)_B + \frac{E_n(w)}{B[n]}. \end{aligned}$$

Thus, as n tends to ∞, the quotient $E_n(w)/B[n]$ tends to 0.

For sufficiently large n,

$$(w(1, n-1) \circ u^{(n)})_B \geq \gamma$$

or

$$(w(1, n-1) \circ v^{(n)})_B \geq \gamma$$

by Lemma 4.9. So,

$$\begin{aligned} w_B - (w(1, n-1) \circ t^{(n)})_B &= \frac{E_n(w)}{B[n]} - \frac{(t^{(n)})_{\sigma^n(B)}}{B[n]} \\ &= \frac{E_n(w)}{B[n]} - \frac{C}{B[n]} \longrightarrow 0 \end{aligned}$$

where $(t^{(n)}, C)$ is either $(u^{(n)}, \gamma)$ or $(v^{(n)}, \gamma+1)$. Therefore, $w_B \geq \gamma$. In general, observe that as w satisfies the lexicographic restriction with respect to B, then $\sigma^m(w)$ also satisfies the lexicographic restriction with respect to $\sigma^m(B)$. Consequently, Lemmas 4.9 and 4.10 apply. In other words, letting $\sigma^m(w)$, $\sigma^m(B)$, $(u^{(m)})_{\sigma^m(B)}$ and $(v^{(m)})_{\sigma^m(B)}$ take the role of w, B, γ and $\gamma+1$, respectively, in Lemma 4.9, we obtain the conclusion that $\sigma^m(w)_{\sigma^m(B)} \geq \gamma$.

Likewise, we have $\sigma^m(w)_{\sigma^m(B)} \leq \gamma + 1$ for all $m \in \mathbb{N} \cup \{0\}$ by Lemma 4.10. The only thing we are left to do is to show that $\sigma^m(w)_{\sigma^m(B)} \neq \gamma + 1$. Let $z = (z_1, z_2, \dots)$ denote the sequence $d^*(\sigma^{M-1}(B); \gamma+1)$. Let s be the least positive integer such that $w_{M+i-1} = z_i$ for $1 \leq i < s$ and

$$(z_s - w_{M+s-1}) \operatorname{sgn}(\sigma^{M-1}(B)[s]) \geq 1.$$

Note that there exists $Y \in [\gamma, \gamma+1)$ such that

$$d(\sigma^{M-1}(B); Y) = (z_1, \dots, z_s, y_{s+1}, y_{s+2} \dots).$$

Then,

$$\begin{aligned} &|Y(s+1, \infty)_{\sigma^{M+s-1}(B)} - w(M+s, \infty)_{\sigma^{M+s-1}(B)}| \\ &\qquad \leq (\gamma+1) - \gamma = 1. \end{aligned}$$

Therefore,

$$\begin{aligned} & Y - w(M, \infty)_{\sigma^{M-1}(B)} \\ = \; & \frac{(z_s - w_{M+s-1}) + Y(s+1, \infty)_{\sigma^{M+s-1}(B)}}{\sigma^{M-1}(B)[s]} \\ - \; & \frac{w(M+s, \infty)_{\sigma^{M+s-1}(B)}}{\sigma^{M-1}(B)[s]} \\ \geq \; & \frac{1 + \operatorname{sgn}(\sigma^{M-1}(B)[s])(Y(s+1, \infty)_{\sigma^{M+s-1}(B)}}{|\sigma^{M-1}(B)[s]|} \\ - \; & \frac{w(M+s, \infty)_{\sigma^{M+s-1}(B)})}{|\sigma^{M-1}(B)[s]|} \\ \geq \; & \frac{1-1}{|\sigma^{M-1}(B)[s]|} = 0. \end{aligned}$$

Since $\gamma + 1 > Y \geq w(M, \infty)_{\sigma^{M-1}(B)}$, then $w(M, \infty)_{\sigma^{M-1}(B)} \neq \gamma + 1$. ∎

In the previous proposition, an important part of the proof is the assumption that the sequence $w = (w_1, w_2, \dots) \in \mathcal{A}(B)$ has the property that the series $\sigma^k(w)_{\sigma^k(B)}$ converges for all $k \in \mathbb{N} \cup \{0\}$. It is clear that if the base $B = (\beta_1, \beta_2, \dots)$ is eventually periodic, then this property holds for w. We can say more. First, note that the digits are bounded by u_{β_i} and v_{β_i} (see Section 2), which, in turn, satisfy

$$\max(|u_{\beta_i}|, |v_{\beta_i}|) \leq (|\beta_i| + 1)(|\gamma| + 1).$$

Now, let us consider the following. For the base B, let $|B|$ be the sequence $(|\beta_1|, |\beta_2|, \dots)$. Suppose that

$$(|\beta_1| + 1, |\beta_2| + 1, \dots)_{|B|} = \sum_{n=1}^{\infty} \frac{|\beta_n| + 1}{|B[n]|} < \infty. \quad (\star\star)$$

Then for every sequence $w \in \mathcal{A}(B)$, the sum w_B is convergent. Indeed,

$$\begin{aligned}\left|\sum_{n=1}^{\infty} \frac{w_n}{B[n]}\right| &\leq \sum_{n=1}^{\infty} \left|\frac{w_n}{B[n]}\right| \\ &\leq \sum_{n=1}^{\infty} \frac{(|\beta_n|+1)(|\gamma|+1)}{|B[n]|} \\ &\leq (|\gamma|+1) \sum_{n=1}^{\infty} \frac{|\beta_n|+1}{|B[n]|} < \infty.\end{aligned}$$

Note that if B is eventually periodic, then $(\star\star)$ holds. However, if $B = (b_1, b_2, \dots)$ with $b_n = (n+1)/n$, then $(\star\star)$ does not hold.

We now state the main result of this article, which provides a sufficient and necessary condition for admissibility of integer sequence in $\mathcal{A}(B)$ with respect to the beta Cantor series expansion for a base sequence B satisfying $(\star\star)$. It would be interesting to know how the result can be extended beyond property $(\star\star)$.

Theorem 4.12. *Let $B \in \mathbb{R}^{\mathbb{N}}$ such that $\lim_{m\to\infty} |B[m]| = \infty$ and $(\star\star)$ holds. Let $(d_1, d_2, \dots) \in \mathcal{A}(B)$. Then $(d_1, d_2, \dots)$ is B-admissible if and only if $(d_1, d_2, \dots)$ satisfies the lexicographic restriction.*

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References

[1] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Math Acad Sci Hungar* **8**:477–493, 1957. DOI:10.1007/BF02020331.

[2] W. Parry. On the β-expansions of real numbers. *Acta Math Acad Sci Hungar* **11**:401–426, 1960. DOI:10.1007/BF02020954.

[3] W. Parry. Representations for real numbers. *Acta Math Acad Sci Hungar* **15**:95–105, 1964. DOI:10.1007/bf01897025.

[4] C. Frougny, A. Lai. On negative bases. *Development in Language Theory* pp. 252–263, 2009. DOI:10.1007/978-3-642-02737-6_20.

[5] V. Grünwald. Intorno all'aritmetica dei sistemi numerici a base negativa con particolare riguardo al sistema numerico a base negativo-decimale per lo studio delle sue analogie coll'aritmetica ordinaria (decimale). *Giornale di Matematiche di Battaglini* pp. 203–221, 1885.

[6] S. Ito, T. Sadahiro. Beta-expansions with negative bases. *Integers* **9**:239–259, 2009. DOI:10.1515/INTEG.2009.023.

[7] L. Liao, W. Steiner. Dynamical properties of the negative beta transformation. *Ergod Th & Dynam Sys* **32**(5):1673–1690, 2012. DOI:10.1017/S0143385711000514.

[8] D. Dombek, Z. Masáková, E. Pelantová. Number representation using the generalized $(-\beta)$-transformation. *Theoretical Computer Science* **412**(48):6653–6665, 2011. DOI:10.1016/j.tcs.2011.08.028.

[9] C. Kalle. Isomorphisms between positive and negative (β)-transformations. *Ergod Th & Dynam Sys* **34**(1):153–170, 2014. DOI:10.1017/etds.2012.127.

[10] K. Dajani, C. Kalle. Transformations generating negative $(-\beta)$-expansions. *Integers* **11**(B):1–18, 2011.

[11] S. Akiyama, J. Caalim. Rotational beta expansion: Ergodicity and soficness. *J Math Soc Japan* **69**(1):397–415, 2017. DOI:10.2969/jmsj/06910397.

[12] S. Akiyama, J. Caalim. Invariant measure of rotational beta expansion and Tarski's plank problem. *Discrete Comput Geom* **57**(2):357–370, 2017. DOI:10.1007/s00454-016-9849-4.

[13] G. Cantor. Üeber die einfachen zahlensysteme. *Zeitschrift für Math und Physik* **14**:121–128, 1869.

[14] Z. Masáková, E. Pelantová. Ito-Sadahiro numbers vs. Parry numbers. *Acta Polytechnica* **51**(4):59–64, 2011.

[15] K. Dajani, S. Ramawadh. Symbolic dynamics of $(-\beta)$-expansions. *Journal of Integer Sequences* **15**:12.2.6/1 – 12.2.6/21, 2012.