Fuzzy Dynamic Analysis of a 2D Frame

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This paper deals with the dynamic analysis of a 2D concrete frame with uncertainties which are an integral part of any real structure. The uncertainties can be modeled by a stochastic or a fuzzy approach. The fuzzy approach is used and the influence of uncertain input data (modulus of elasticity and density) on output data is studied. Fuzzy numbers are represented by α-cuts. In order to reduce the volume of computation in the fuzzy approach, the response surface function concept is applied. In this way the natural frequencies and mode shapes described by fuzzy numbers are obtained. The results of fuzzy dynamic analysis can be used, e.g., in seismic design of structures based on the response spectrum.

Keywords: fuzzy numbers, natural frequency, mode shape, response surface function.

1 Introduction

Concrete, as a convenient building material, inherently involves uncertainty about its composition, which is difficult to eliminate completely. However, this uncertainty can be assessed by statistical, fuzzy, or other suitable tools. In the case of concrete structures, such as frames made of reinforced concrete, it is costly to obtain a sufficient experimental data set which would yield the desired statistical characteristics of the material parameters. Instead, the knowledge gained by practicing engineers can be included as fuzzy numbers in the material modeling.

For design purposes, traditionally, we may wish to conduct a statistical analysis, using the statistical characteristics of several measured events. In the case of an earthquake, however, the measured data for each site of interest is not particularly dense, leaving the statistical characteristics with little relevance. On the other hand, the expected seismic load at a site can alternatively be expressed by fuzzy sets [1], which take into account the scarcity of seismic stations and information about local sub-soil composition.

In this paper, an approach to dynamic analysis based on fuzzy set theory is presented as a germane alternative to classical stochastic dynamic analysis. The material parameters of reinforced concrete are considered to be fuzzy quantities with a given distribution, i.e., fuzzy numbers with a desired shape of the membership function [2]. The dynamic analysis is, then, performed with the help of fuzzy arithmetic on either α-cuts or computation-efficient (L, R) numbers [3]. The result of such an analysis is in the form of fuzzy numbers, which is less expensive than the stochastic approach in terms of computation time, but still provides an idea of the distribution of the sought quantity. In order to further improve the computational efficiency, inspired by [4], the concept of a surface response function is utilized, [5, 6]. This approach is demonstrated in an illustrative example of a 2D frame, where the effect of uncertain material parameters transpires in the corresponding distributions of the natural modal shapes and natural frequencies of the analyzed two-dimensional frame. A methodology for a possible application to seismic design is also explained.

It is believed that this approach enables practicing engineers and other people with knowledge to contribute actively to analyses of seismic-sensitive structures.

2 Dynamic finite element analysis

The finite element method applied to dynamical problems of structures results in the form

$$\mathbf{M}\frac{\mathrm{d}^2\boldsymbol{r}(t)}{\mathrm{d}t^2}+\mathbf{C}\frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t}+\mathbf{K}\boldsymbol{r}(t)=\boldsymbol{f}(t), \tag{1}$$

where $\mathbf{M}$ denotes the mass matrix, $\mathbf{C}$ stands for the damping matrix, $\mathbf{K}$ denotes the stiffness matrix, $\boldsymbol{f}(t)$ expresses the load vector and $\boldsymbol{r}(t)$ is the vector of the nodal displacements which are computed, t stands for time. Eq. (1) represents a semidiscrete problem where the spatial coordinates are discretized while the time is still assumed to be continuous [7].

The analysis of the natural frequencies (eigenvalues) and natural mode shapes (eigenvectors) of an undamped structure is based on the simplified relation Eq. (1), which has the form

$$(\mathbf{K}-\omega_0^2\mathbf{M})\,\boldsymbol{v}=\mathbf{0}. \tag{2}$$

The nonzero vector $\boldsymbol{v}$ is the eigenvector containing the natural mode shapes and ω_0 stands for the natural frequency. Eq. (2) represents a generalized problem of eigenvalues. The most common method for solving of such problems is subspace iteration [8].

3 Fuzzification of dynamic finite element analysis

The uncertainty, that is present in the input parameters can be tackled with the help of fuzzy set theory [1]. In this theory, uncertain quantities are defined in terms of fuzzy sets. Unlike in the classical set theory, here membership of an element in a fuzzy set also assumes values between 0 and 1, where 0 means "does not belong" and 1 means "definitely belongs" to a fuzzy set. Usually, fuzzy sets represent a vague verbal evaluation. In cases when a fuzzy set represents a numeral, it is called a fuzzy number.

3.1 Fuzzy numbers

The notion of a fuzzy number arises from experience of everyday life where many phenomena which can be quantified are not characterized in terms of absolutely precise numbers.

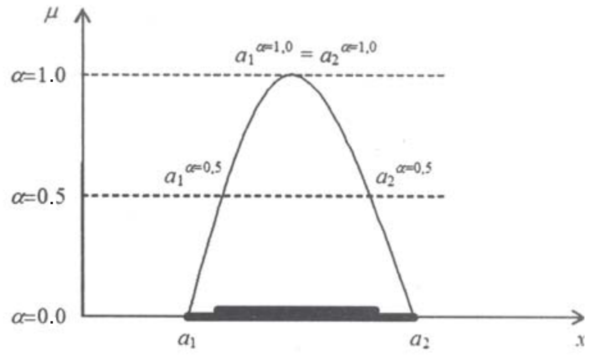

Fig. 1: A normal fuzzy number and its α-cuts

Fuzzy numbers are fuzzy sets which are defined on the set of real numbers. Their membership function assigns the degree of 1 to the central, also called nominal, modal or mean, value and lower degrees to other numbers which reflect their proximity to the central value according to the used membership function. The membership function should thus decrease from 1 to 0 on both sides of the central value. Such fuzzy sets are called fuzzy numbers. An example of a fuzzy number is shown in Fig. 1, where μ represents the membership function and a_1 and a_2 stand for two real numbers on the real axis. The intervals defined for a specific value of the membership function, e.g. $\alpha = 0.7$, represent the so-called α-cuts. A fuzzy number can be equally expressed by either a nominal value and a membership function on each side of the nominal value or by a set of α-cuts.

3.2 Fuzzy arithmetic

A fuzzy arithmetic operation depends on the definition of a fuzzy number. In the cases when fuzzy numbers are defined by a set of α-cuts, the problem of fuzzy arithmetic is reduced to the well-known arithmetic operations on intervals, which are applied to each α-cut. Implicitly, this means a sequence of binary combinations on each α-cut in order to obtain the minimum and the maximum value for each α-cut. The finite element method converts a problem into a system of linear equations, in this case a system of fuzzy linear equations, which comprises an extensive number of arithmetic operations. This fact makes the formulation in the above terms merely unsolvable due to the number of all necessary binary operations.

To eliminate the drawback of the α-cut formulation, new techniques for solving fuzzy linear equation systems have been developed, e.g. [9]. However, these techniques are not easily applicable to robust problems, such as fuzzy dynamic finite element analysis. Therefore, another technique for reducing the large number of binary combinations has been exploited.This technique was originally developed for other problems, such as statistical analysis.

3.3 Surface response function

Fuzzy analyses, as well as stochastic analyses, suffer from non-occurrence of analytical solutions in the case of non-deterministic input data. This situation can be remedied by the following. Let $\widetilde{\boldsymbol{x}} \in \widetilde{\boldsymbol{X}}$ denote the vector of input data from the space of input data $\widetilde{\boldsymbol{X}}$ and $\widetilde{\boldsymbol{y}} \in \widetilde{\boldsymbol{Y}}$ denote the vector of output data from the space of output data $\widetilde{\boldsymbol{Y}}$. Both, stochastic and fuzzy analyses require knowledge of the response, which can be written in the form

$$\widetilde{\boldsymbol{y}} = \mathsf{F}(\widetilde{\boldsymbol{x}}), \tag{3}$$

where F denotes the response of a system (structure) to the input data collected in vector $\widetilde{\boldsymbol{x}}$. This represents a mapping from the space $\widetilde{\boldsymbol{X}}$ to the space $\widetilde{\boldsymbol{Y}}$. The non-occurrence of an analytical solution requires the application of a suitable numerical method which discretizes the problem and solves it numerically. The space $\widetilde{\boldsymbol{X}}$ is discretized by an n-dimensional space, $\boldsymbol{X}$, and similarly the space $\widetilde{\boldsymbol{Y}}$ by an m-dimensional space, $\boldsymbol{Y}$. A stochastic analysis based on simulation methods generates thousands or millions of samples of input data (the vectors $\boldsymbol{x}$) and then deterministic computation follows. Fuzzy analysis based on α-cuts requires computation of all combinations of input data, which also leads to thousands or millions of samples. Both approaches yield the response of a system based on a huge amount of output data (thousands or millions of vectors $\boldsymbol{y}$) obtained from many executions of standard (deterministic or crisp) solutions.

In order to reduce the necessary number of computation runs, the concept of a response surface function has been used many times. The basic idea of the response function is to approximate operator F by a suitable function which should be as simple as possible. The function for the k-th output parameter can be written in the form

$$f^{(k)}(x) = a^{(k)} + \sum_{i=1}^{n} b_i^{(k)} x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^{(k)} x_i\, x_j, \tag{4}$$

where the superscript identifies an output parameter and n denotes the dimension of the space of the input data, $\boldsymbol{X}$. The unknown coefficients are obtained from the least square method in the following way. Let the set of input parameters contain s samples. Each sample is located in the vector $\boldsymbol{x}^{[i]}$, where the superscript identifies a sample. The standard computation gives the output data, which are collected in the vectors $\boldsymbol{y}^{[i]}$. The coefficients of the response function minimize the following expression

$$F^{(k)}(a^{(k)}, b_i^{(k)}, c_{ij}^{(k)}) = \sum_{i=1}^{s} \left(f^{(k)}(\boldsymbol{x}^{[i]}) - y_k^{[i]} \right)^2. \tag{5}$$

In many cases, it is not necessary to use quadratic terms. Considering only the linear terms simplifies further computations.

4 Numerical example

As an example, the natural frequency analysis of a two-dimensional frame with four floors made of reinforced concrete is considered. The overall height of the frame is 16 meters and the width is 5 + 5 meters. The dimensions of the beams and columns are identical (0.5×0.5 m). It is assumed that the building was erected in four consecutive lifts. Each lift consists of placing concrete in three columns and in the beam which connects the upper ends of the columns.

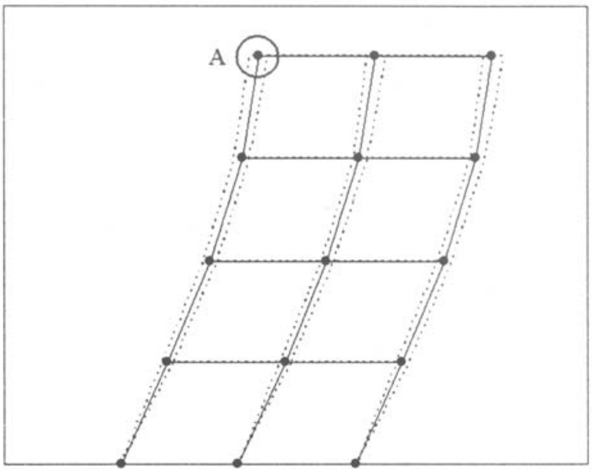

Fig. 2: Mode shape

Therefore, it is further assumed that there are only four types of concrete whose composition can possibly differ. The influencing material parameters are the modulus of elasticity,

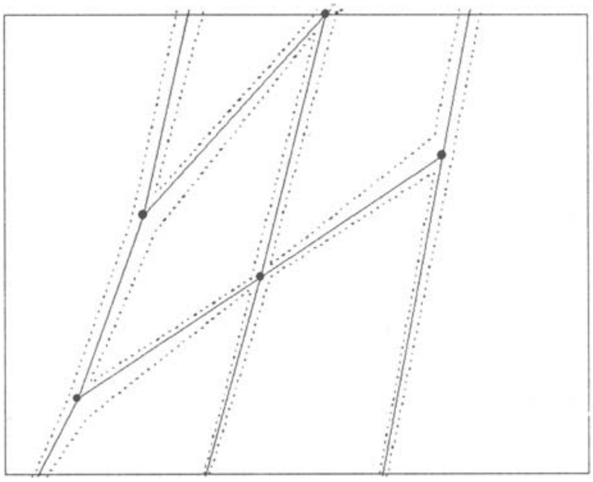

Fig. 3: Enlarged section of mode shape 1

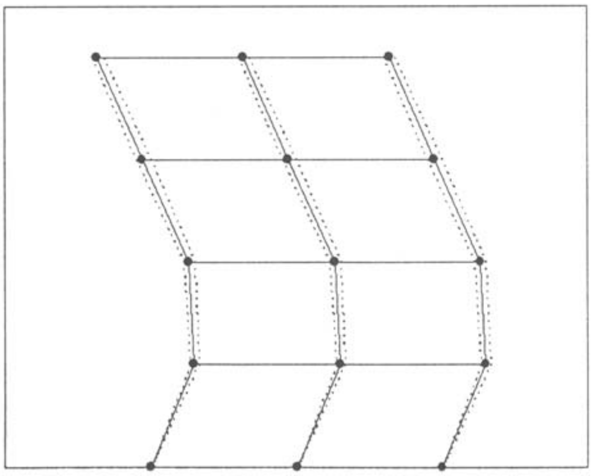

Fig. 4: Mode shape 2

E and the density, ρ. E and ρ are fuzzy input parameters with nominal values of 30 GPa and 2500 kg/m^3, respectively, which can change by ±10 % and with a linear membership function (triangular fuzzy numbers).

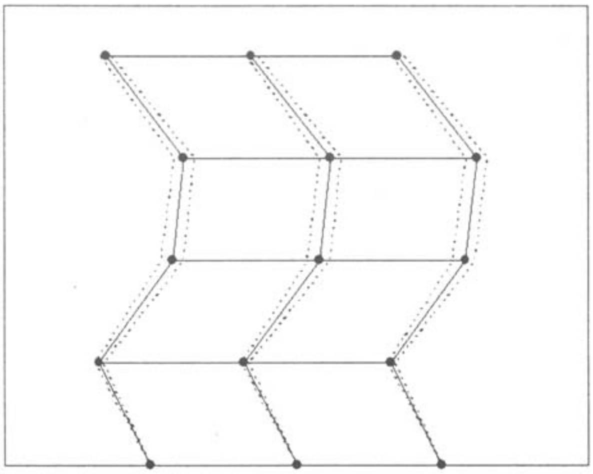

Fig. 5: Mode shape 3

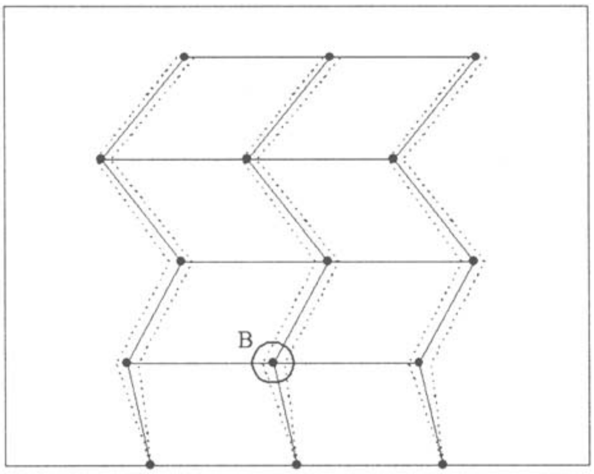

Fig. 6: Mode shape 4

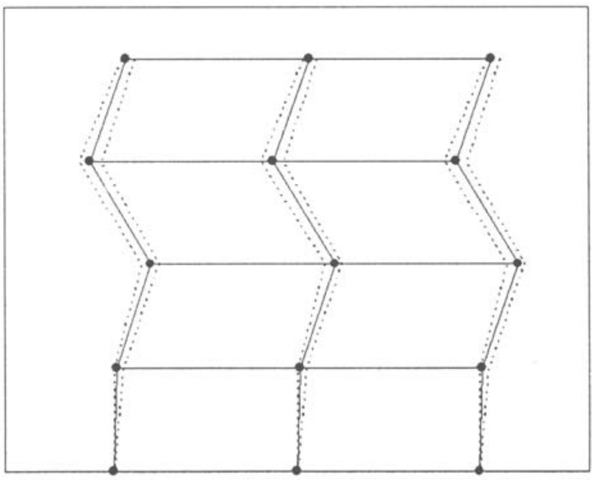

Fig. 7: Mode shape 5

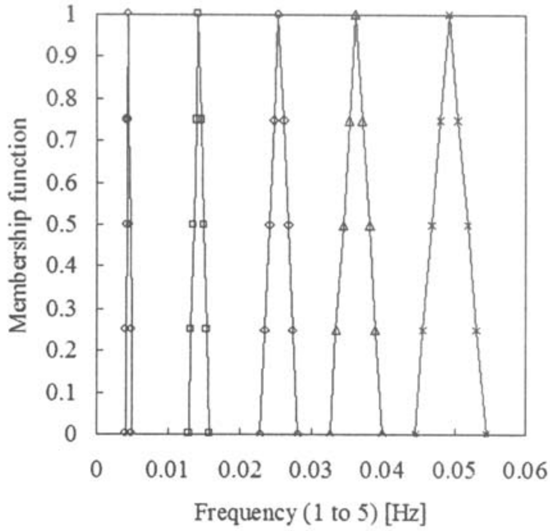

Fig. 8: Distribution of natural frequencies

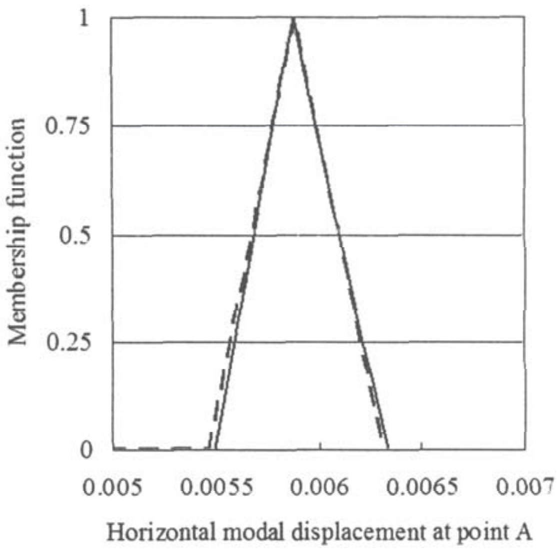

Fig. 9: Distribution of horizontal displacement

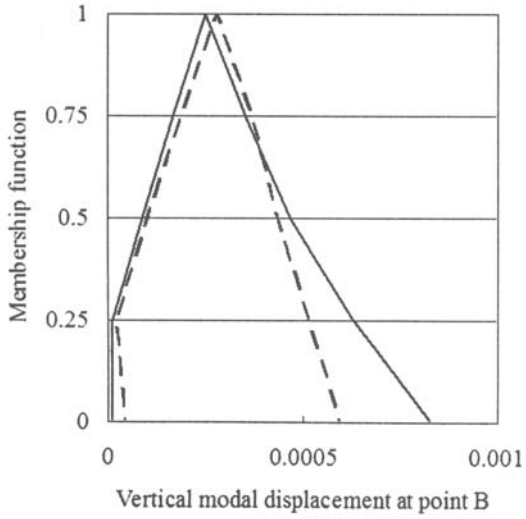

Fig. 10: Distribution of vertical displacement

For illustrative purposes, we need 125 response surface functions to describe the first five natural vibration modes, i.e. a response surface function to express each natural frequency and the horizontal and vertical displacements in each joint (three joints on each of the four floors) for each natural mode shape. In order to obtain sufficient input and output data for calculation of the coefficients of the response surface functions it was decided to take three values (minimum, modal value, maximum) for each material parameter, E and ρ, i.e., $(=6561)$ independent runs of the dynamic finite element analysis. The specific form of Eq. (4) in this example was

$$f^{(k)}(x) = b_1^{(k)}E_1 + b_2^{(k)}E_2 + b_3^{(k)}E_3 + b_4^{(k)}E_4 + \\ + b_5^{(k)}\rho_1 + b_6^{(k)}\rho_2 + b_7^{(k)}\rho_3 + b_8^{(k)}\rho_4 + b_9^{(k)}. \tag{6}$$

The first five mode shapes are shown in Figs. 2 to 7, where the dotted lines represent all possible envelopes of response, in other words, the minimum and maximum values, which correspond to the values obtained for α-cuts ($\alpha = 0$). The finite element model of this frame discretized each frame section (beam and column) by five beam elements, however, only the joint displacements are shown. This is why no significant difference is distinguishable in Figs. 6 and 7. In Fig. 3, a section of the frame is enlarged and the vertical displacements are 1000 times increased compared to the horizontal displacements so that we can see the distribution of possible displacements of the frame. The distribution of the first five natural frequencies is shown in Fig. 8. It was observed that the response function gives very good results for the dominant displacements (at point A, which is the top left joint) in the lower natural mode shapes, which are important for seismic design. For vertical displacements, which do not play an important role in seismic design (at point B, which is the intermediate joint of the first floor), the response function could not fit the proper shape of the membership function, which is evidenced in Figs. 9 and 10.

5 Possible applications

In the design of earthquake resistant structures, it is essential not to neglect any uncertainty as it may lead to an erroneous conclusion due to the dynamic simulation which may amplify such uncertainty beyond all limits. For these reasons, it seems reasonable to express uncertain numerical data in terms of fuzzy numbers and to use them as such in analyses to cover all possible solutions.

In the previous section, an approach to natural vibration analysis was shown which provides input data for further analyses considering, e.g. earthquake induced vibrations. Spectral-analysis based methods require only the maximum values obtained for each natural mode in order to evaluate the excited vibration. Therefore, it is desirable to verify whether these values can be satisfactorily expressed by surface response functions which were obtained only by binary combinations of material parameters with three values (minimum, modal value, maximum). The resulting surface response function was also obtained for five values, corresponding to the α-cut values with a equal to 0, 0.5 and 1. However, this meant $5^{2\times4}$ $(=390625)$ independent runs of the dynamic finite element analysis. The improvement was negligible, and compared with the computational effort it proved truly unnecessary.

6 Conclusions

The fuzzy approach has been applied to dynamic analysis of a 2D concrete frame with uncertainties. The natural frequencies and the natural mode shapes have been computed and described by fuzzy numbers. The response surface function has been applied in order to reduce the number of required computations. The results have been compared with a full analysis based on an evaluation of all combinations (in the order of thousands) and very good accordance has been obtained. The first lower natural mode shapes are naturally described more precisely than the higher ones. The errors of the obtained results for lower natural mode shapes are less than 5%. There are some quantities belonging to higher modes where the response surface function gives results unacceptable from the point of the fuzzy set theory. These difficulties should be studied in future.

7 Acknowledgment

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