

MODIFIED KORTEWEG-DE VRIES EQUATION AS A SYSTEM WITH BENIGN GHOSTS

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ABSTRACT. We consider the modified Korteweg-de Vries equation, $u_{xxx} + 6u^2u_x + u_t = 0$, and explore its dynamics in *spatial* direction. Higher x derivatives bring about the *ghosts*. We argue that these ghosts are benign, i.e., the classical dynamics of this system does not involve a blow-up. This probably means that the associated quantum problem is also well defined.

KEYWORDS: Benign ghosts, KdV equation, integrability.

1. INTRODUCTION

A system with ghosts is, by definition, a system where the quantum Hamiltonian has no ground state so its spectrum involves the states with arbitrarily low and arbitrarily high energies. In particular, all non-degenerate theories with higher derivatives in the Lagrangian (but not only them!) involve ghosts. The ghosts show up there already at the classical level: the Ostrogradsky Hamiltonians of higher derivative systems [1] include the linear in momenta terms and are thus not positive definite [2]. This brings about the ghosts in the quantum problem [3, 4].

In many cases, ghost-ridden systems are sick – the Schrödinger problem is not well posed and unitarity is violated. Probably, the simplest example of such a system is a system with the Hamiltonian describing the 3-dimensional motion of a particle in an attractive $\frac{1}{r^2}$ potential:

$$H = \frac{\vec{p}^2}{2m} - \frac{\kappa}{r^2}. \quad (1)$$

For certain initial conditions, the particle falls to the center in a finite time, as is shown in Figure 1.

The quantum dynamics of this system depends on the value of κ . If $m\kappa < 1/8$, the ground state exists and unitarity is preserved. If $m\kappa > 1/8$, the spectrum is not bounded from below and, what is worse, the quantum problem cannot be well posed until the singularity at the origin is smoothed out [5–7]. One can say that for $m\kappa < 1/8$, the quantum fluctuations cope successfully with the attractive force of the potential and prevent the system from collapsing.

The latter example suggests that quantum fluctuations can only make a ghost-ridden system better, not worse. We, therefore, *conjecture* that, if the classical dynamics of the system is benign, i.e., the system does not run into singularity in finite time,¹ its quantum

¹We still call a system benign if it runs into a singularity at $t = \infty$. Such systems have well-defined quantum dynamics. This refers, for example, to the problem of motion in a uniform electric field (see e.g. [8], §24) and also to the inversed oscillator

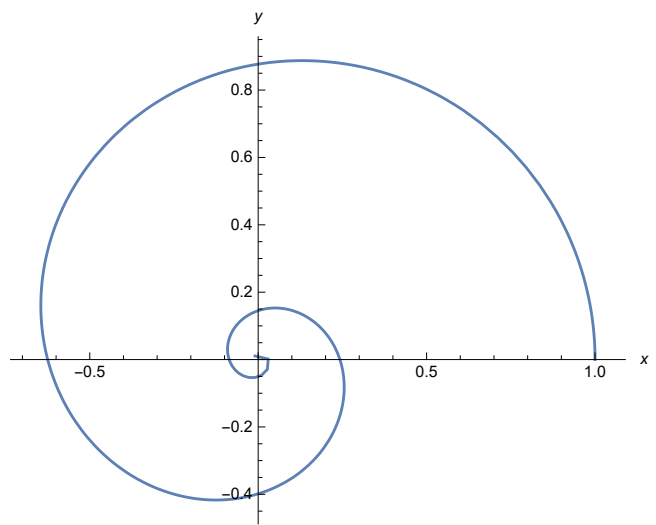


FIGURE 1. Falling on the center for the Hamiltonian (1) with $m = 1$ and $\kappa = .05$. The energy is slightly negative. The particles with positive energies escape to infinity.

dynamics will also be benign, irrespectively of whether the spectrum has, or does not have, a bottom.

This all refers to ordinary mechanical or field theory systems, where energy is conserved and the notion of Hamiltonian exists. The ghosts in gravity (especially, in higher-derivative gravity) are special issue that we are not discussing here.

Besides malignant ghost-ridden systems, of which the system (1) with $m\kappa > 1/8$ represents an example, there are also many systems with ghosts, which are *benign* – unitarity is preserved and the quantum Hamiltonian is self-adjoint with a well-defined real spectrum. To begin with, such is the famous *Pais-Uhlenbeck oscillator* [10] – a higher derivative system

with the Hamiltonian $H = (p^2 - x^2)/2$. In the latter problem, the classical trajectories $x(t)$ grow exponentially with time, but the quantum problem is still benign (see e.g. [9], Ch. 3, corollary 13). The spectrum in this case is continuous, as it is for the uniform field problem.

with the Lagrangian

$$L = \frac{1}{2} [\dot{x}^2 - (\omega_1^2 + \omega_2^2)x^2 + \omega_1^2\omega_2^2x^2]. \quad (2)$$

This system is free, its canonical Hamiltonian can be reduced to the difference of the two oscillator Hamiltonians by a canonical transformation [11]. The first example of a nontrivial benign ghost system involving nonlinear interactions was built up in [12]. For other such examples, see Refs. [13–17].

In recent [18], we outlined two wide classes of benign ghost systems: (i) the systems obtained by a variation of ordinary systems and involving, compared to them, a double set of dynamic variables and (ii) the systems describing geodesic motion over Lorentzian manifolds. In addition, we noticed that the evolution of the modified Korteweg-de Vries (MKdV) system (9) in the *spatial* direction also exhibits a benign ghost dynamics. This report is mostly based on section 4 of [18] that deals with MKdV dynamics.

2. SPATIAL DYNAMICS OF KdV AND MKdV EQUATIONS

First, consider the ordinary KdV equation,

$$u_{xxx} + 6uu_x + u_t = 0, \quad (3)$$

where $u_x = \partial u / \partial x$, $u_t = \partial u / \partial t$ etc. It has an infinite number of integrals of motion and is exactly soluble.² The KdV equation is derived from the field Lagrangian

$$L[\psi(t, x)] = \frac{1}{2}\psi_{xx}^2 - \psi_x^3 - \frac{1}{2}\psi_t\psi_x \quad (4)$$

if one denotes $u(t, x) \equiv \psi_x$ after having varied over $\psi(t, x)$. This Lagrangian involves higher spatial derivatives, but not higher time derivatives and does not involve ghosts in the ordinary sense. We can, however, simply *rename*

$$t \rightarrow X, \quad x \rightarrow T, \quad (5)$$

in which case the equation acquires the form

$$u_{TTT} + 6uu_T + u_X = 0 \quad (6)$$

and higher time derivatives appear. According to our conjecture, to study the question of whether the quantum Hamiltonian corresponding to the thus rotated Lagrangian (4) is Hermitian and unitarity is preserved, it is sufficient to study its classical dynamics: if it does not involve a blow-up and all classical trajectories exist at all times T , one can be sure that the quantum system is also benign.

Note that the question whether or not blowing up trajectories are present is far from being trivial. The ordinary Cauchy problem for the equation (4) consists

²Exact solvability always makes the behaviour of a system more handy. In particular, many mechanical models including benign ghosts, which were mentioned above, are exactly solvable.

in setting the initial value of $u(t_0, x)$ at a given time moment, say, $t_0 = 0$. And we are now interested [staying with Eq. (4) and not changing the name of the variables according to (5)] in the Cauchy problem in x direction. The presence of third spatial derivatives in (4) makes it necessary to define, at the line $x = x_0$, *three* different functions: $u(t, x_0)$, $u_x(t, x_0)$ and $u_{xx}(t, x_0)$. The presence of three arbitrary functions makes the space of solutions to the spatial Cauchy problem much larger than for the ordinary Cauchy problem. The solutions to the latter represent a subset of measure zero in the set of the solutions in the former, and the fact that the solutions to the ordinary Cauchy problem are all benign does not mean that it is also the case for the rotated x -directed problem.

And, indeed, for the ordinary KdV equation (4), the problem is *not* benign. It is best seen if we choose a t -independent Ansatz $u(t, x) \rightarrow u(x)$ and plug it into (3). The equation is reduced to

$$\partial_x(u_{xx} + 3u^2) = 0 \implies u_{xx} + 3u^2 = C. \quad (7)$$

This equation describes the motion in the cubic potential $V(u) = u^3 - Cu$. It has blow-up solutions. If $C = 0$, they read

$$u(x) = -\frac{2}{(x - x_0)^2}. \quad (8)$$

However, the situation is completely different for the *modified* KdV equation,³

$$u_{xxx} + 6u^2u_x + u_t = 0. \quad (9)$$

This equation admits an infinite number of integrals of motion, as the ordinary KdV equation does. The first three local conservation laws are

$$\partial_t u = -\partial_x(u_{xx} + 2u^3), \quad (10)$$

$$\partial_t u^2 = -2\partial_x \left[\frac{3}{2}u^4 + uu_{xx} - \frac{1}{2}u_x^2 \right], \quad (11)$$

$$\partial_t \left(\frac{1}{2}u^4 - \frac{1}{2}u_x^2 \right) =$$

$$\partial_x \left[u_x(2u^2u_x + \frac{1}{2}u_{xxx}) - \frac{1}{2}u_{xx}^2 - 2u^3u_{xx} - 2u^6 \right] \quad (12)$$

For the time-independent Ansatz, we obtain, instead of (7),

$$\partial_x(u_{xx} + 2u^3) = 0 \implies u_{xx} + 2u^3 = C. \quad (13)$$

This describes the motion in a *quartic* potential $V(u) = u^4/2 - Cu$. This motion is bounded, the solutions being elliptic functions.

³In Ref. [18], we wrote this equation as

$$u_{xxx} + 12\kappa u^2u_x + u_t = 0$$

and kept κ in all subsequent formulas. But here, we have chosen, for simplicity, to fix $\kappa = 1/2$.

This observation presents an argument that the rotated Cauchy problem for the equation (9) with arbitrary initial conditions on the line $x = const$ might be benign.

Note that this behaviour is specific for the equation (9) with the positive sign of the middle term (the so-called *focusing* case). Plugging the time-independent Ansatz in the *defocusing* MKdV equation,⁴

$$u_{xxx} - 6u^2u_x + u_t = 0, \tag{14}$$

the problem would be reduced to the motion in the potential $V(u) = -u^4/2 - Cu$ characterized by a blow-up. This conforms to the well-known fact that any solution $u(t, x)$ of the ordinary KdV equation is related to a solution $v(t, x)$ of the defocusing MKdV equation by the Miura transformation,

$$u = -(v^2 + v_x). \tag{15}$$

A different (though related) analytic argument indicating the absence of real blow-up solutions for the focusing MKdV equation comes from the analysis of its scaling properties. It is easily seen that Eq. (9) is invariant under the rescalings $u = \lambda_u \bar{u}$, $x = \lambda_x \bar{x}$, $t = \lambda_t \bar{t}$ if

$$\lambda_t = \lambda_x^3, \quad \lambda_u = \lambda_x^{-1}. \tag{16}$$

The quantities xu and $x/t^{1/3}$ are invariant under these rescalings. Using also the space and time translational invariance of the MKdV equation, we can look for scaling solutions of the type

$$u(t, x) = \frac{1}{[3(t - t_0)]^{1/3}} w(z), \tag{17}$$

where

$$z = \frac{x - x_0}{[3(t - t_0)]^{1/3}}. \tag{18}$$

Inserting the ansatz (17) in Eq. (9), one easily verifies that the function $w(z)$ satisfies the equation

$$0 = w''' + (6w^2 - z)w' - w = \frac{d}{dz} [w'' + 2w^3 - zw] \tag{19}$$

Denoting the constant value of the bracket in the last right-hand side as C , we conclude that $w(z)$ satisfies a second-order equation,

$$w'' = -2w^3 + zw + C. \tag{20}$$

For the equation (14), the same analysis would give the equation

$$w'' = 2w^3 + zw + C. \tag{21}$$

These are Painlevé II equations [19]. In general, Painlevé equations have pole singularities. And indeed, a local analysis of Eq. (21) (keeping the leading-order terms $w'' \approx 2w^3$) shows that (21) admits simple poles, $w(z) \approx \pm 1/(z - z_0)$. The existence of

⁴The coefficient 6 is a convention. It can be changed by rescaling t and x . But the sign stays invariant under rescaling.

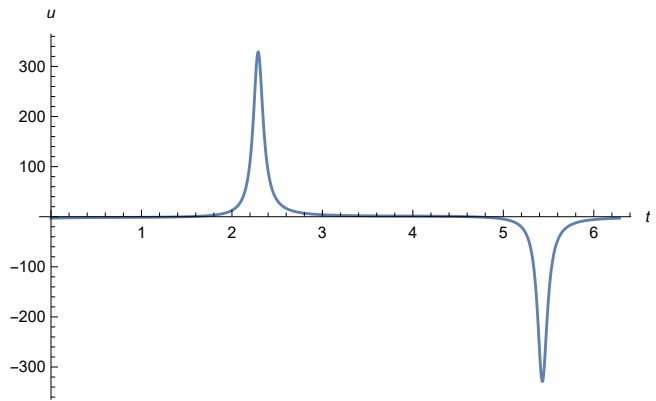


FIGURE 2. $u(t, x = 2.26)$ for the defocusing MKdV.

a real simple pole at $z = z_0$ would then correspond to a singular (blow-up) behavior of $u(t, x)$ of the form $u(t, x) \propto [x - x_0 - z_0[3(t - t_0)]^{1/3}]^{-1}$. But for the equation (20) [and hence for (9)], the singularities are absent.

The third argument in favor of the conjecture that the x evolution of sufficiently smooth Cauchy data on the line $x = const$ for the MKdV equation (9) does not bring about singularities in $u(t, x)$ comes from numerical simulations. To simplify the numerical analysis, we considered the problem on the band $0 \leq t \leq 2\pi$, where we imposed [as is allowed by Eq. (9)] periodic boundary conditions:

$$u(t + 2\pi, x) = u(t, x). \tag{22}$$

We have chosen the Cauchy data

$$u(t, 0) = \sin t, \quad u_x(t, 0) = u_{xx}(t, 0) = 0. \tag{23}$$

We first checked that the use of such Cauchy data for the defocusing MKdV equation (14) was leading to a blow-up rather fast (at $x = 2.2630\dots$). This is illustrated in Figure 2, where the function $u(t, x)$ is plotted just before the blow-up, at $x = 2.26$.

By contrast, our numerical simulations of the x evolution of the focusing MKdV equation showed that $u(t, x)$ stayed bounded for all the values of x that we explored. We met, however, another problem associated with the *instability* of Eq. (9) under high-frequency (HF) perturbations.

Suppressing the nonlinear term in the KdV or MKdV equations, we obtain

$$u_{xxx} + u_t = 0. \tag{24}$$

This equation describes the fluctuations around the solution $u(t, x) = 0$. Its analysis gives us an idea about the behaviour of fluctuations around other solutions. Decomposing $u(t, x)$ as a Fourier integral, in plane waves $e^{i(\omega t + kx)}$, we obtain the dispersion law

$$\omega = k^3. \tag{25}$$

If one poses the conventional Cauchy problem with some Fourier-transformable initial data

$$u(0, x) = v(x) \equiv \int \frac{dk}{2\pi} v(k)e^{ikx}, \quad (26)$$

the time evolution of the initial data $v(x)$ yields the solution

$$u(t, x) = \int \frac{dk}{2\pi} v(k)e^{i(k^3t+kx)}. \quad (27)$$

The important point here is that $u(t, x)$ is obtained from $v(k)$ by a purely oscillatory complex kernel $e^{i(k^3t+kx)}$ of unit modulus. It has been shown that this oscillatory kernel has *smoothing* properties (see, e.g., [20]). This allows one to take the initial data in low- s Sobolev spaces H^s (describing pretty rough initial data).

However, if one considers the x -evolution Cauchy problem, one starts from three independent functions of t along the $x = 0$ axis: $u(t, 0) = u_0(t)$, $u_x(t, 0) = u_1(t)$ and $u_{xx}(t, 0) = u_2(t)$, as in (23). Assuming that the three Cauchy data $u_a(t)$, $a = 0, 1, 2$, are Fourier-transformable, we can represent them as

$$u_a(t) = \int \frac{d\omega}{2\pi} u_a(\omega)e^{i\omega t}. \quad (28)$$

The three Cauchy data determine a unique solution which, when decomposed in plane waves, satisfies the same dispersion law (25) as before. However, the dispersion law (25) must now be solved for k in terms of ω . As it is a cubic equation in k , it has *three different roots*:

$$k_a(\omega) = \omega^{\frac{1}{3}}e^{2\pi ia/3} \quad a = 0, 1, 2. \quad (29)$$

This yields a solution for $u(t, x)$ of the form

$$u(t, x) = \sum_{a=0,1,2} \int \frac{d\omega}{2\pi} v_a(\omega)e^{i(\omega t+k_a x)}, \quad (30)$$

where the three coefficients $v_a(\omega)$ are uniquely determined by the three initial conditions at $x = 0$. The point of this exercise was to exhibit the fact that, when considering the x evolution with arbitrary Cauchy data $u_0(t)$, $u_1(t)$, $u_2(t)$, the solution involves *exponentially growing* modes in the x direction, linked to the imaginary parts of $k_1(\omega)$ and $k_2(\omega)$.

This can be avoided if the initial data are sufficiently smooth, not involving HF modes. As a minimum condition for a local existence theorem, one should require the Fourier transforms $v_a(\omega)$ to decrease like $e^{-\alpha|\omega|^{\frac{1}{3}}}$ for some positive constant α .⁵

However, it is difficult to respect these essential smoothness constraints on the behaviour of $u(t, x)$ in the numerical calculations. The standard Mathematica algorithms do not do so, and that is why we, starting from some values of x , observe the HF noise in our results.

⁵See Ref. [18] for more detailed discussion.

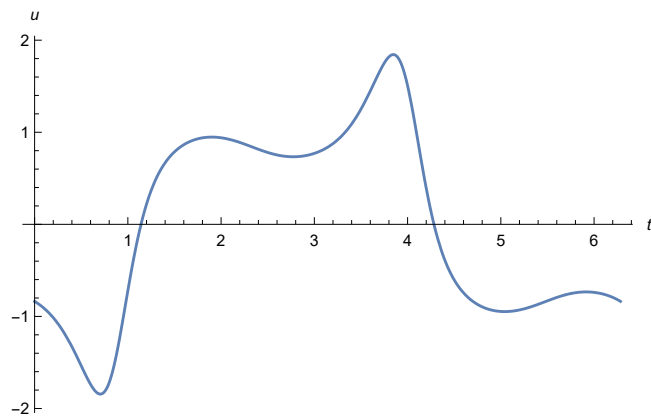


FIGURE 3. $u(t, x = 3)$ for the focusing MKdV.

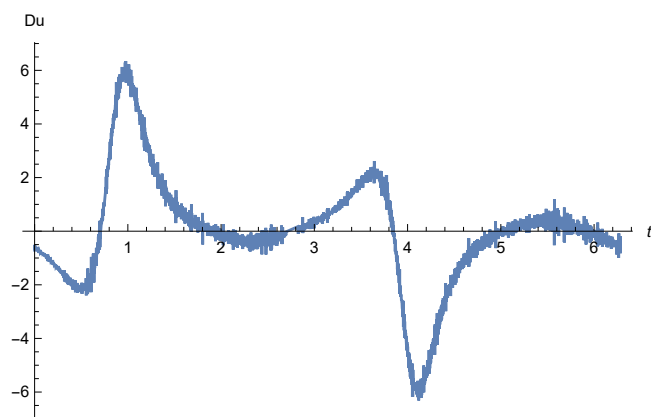


FIGURE 4. $u_x(t, x = 3)$ for the focusing MKdV.

In Figures 3, 4, we present the results of numerical calculations of $u(t, x)$ and $u_x(t, x)$ for $x = 3$. There is no trace of a blow-up. For the plot of $u(t, x)$, one also does not see a HF noise, but it is seen in the plot for $u_x(t, x)$. For larger values of x , the noise also shows up in the plot of $u(t, x)$. At $x \gtrsim 3.8$, the noise overwhelms the signal.

The observed noise is a numerical effect associated with a finite computer accuracy. To confirm this, we performed a different calculation choosing the initial conditions which correspond to the exact solitonic solution to Eq. (9).

The soliton is a travelling wave, $u(t, x) = u(x-ct) \equiv u(\bar{x})$. Plugging this Ansatz into (9), we obtain an ordinary differential equation

$$\frac{\partial}{\partial \bar{x}} [u_{\bar{x}\bar{x}} + 2u^3 - cu] = 0. \quad (31)$$

Denoting the constant quantity within the bracket as C' , we then get the following second-order equation for the function $u(\bar{x})$:

$$u_{\bar{x}\bar{x}} = -\frac{d}{du}\mathcal{V}(u), \quad (32)$$

with a potential function $\mathcal{V}(u)$ now given by

$$\mathcal{V}(u) = \frac{u^4 - cu^2}{2} - C'u. \quad (33)$$

As was also the case for the time-independent Ansatz, the problem is reduced to the dynamics of a particle moving in the confining quartic potential $\mathcal{V}(u)$. The trajectory of the particle depends on three parameters: the celerity c , the constant C' , and the particle energy,

$$E = \frac{1}{2}u_{\bar{x}}^2 + \mathcal{V}(u). \quad (34)$$

The usually considered solitonic solutions (such that $u(\bar{x})$ tends to zero when $\bar{x} \rightarrow \pm\infty$) are obtained by taking $c > 0$, $C' = 0$ (so that the potential represents a symmetric double-well potential) and $E = 0$. The zero-energy trajectory describes a particle starting at “time” $\bar{x} = -\infty$, at $u = 0$ with zero “velocity” $u_{\bar{x}}$, gliding down, say, to the right, reflecting on the right wall of the double well and then turning back to end up, again, at $u = 0$ when $\bar{x} = +\infty$. The explicit form of the corresponding solution defined on the infinite (t, x) plane is

$$u(t, x) = \frac{\sqrt{c}}{\cosh[\sqrt{c}(x - ct)]}. \quad (35)$$

However, to make contact with our numerical calculations, we need a *periodic* soliton solution. Such solutions can be easily constructed by considering bounded mechanical motions in the potential $\mathcal{V}(u)$ having a *non-zero energy*. Periodic solutions exist both for positive and negative c . The trajectories are the elliptic functions. It was more convenient for us to assume $c = -|c|$, in which case we could make contact with Ref. [12], where the expressions for the trajectories of motion in the same quartic potential were explicitly written, one only had to rename the parameters. Choosing $E = 1$ and $c = -1$, we obtain the following solution:

$$u(t, x) = \text{cn} \left[\sqrt{3}(x + t), m \right], \quad (36)$$

where $\text{cn}(z)$ is the Jacobi elliptic cosine function with the elliptic modulus $m = 1/3$. The function (36) is periodic both in t and x with the period

$$T = L = \frac{4}{\sqrt{3}}K \left(\frac{1}{3} \right) \approx 4. \quad (37)$$

We fixed the initial conditions for $x = 0$ and periodic conditions in time as is dictated by (36), and then numerically solved (9). The numerical solution should reproduce the exact one, and it does for $x \lesssim 4$. However, at larger values of x , the HF noise appears. The result of the calculation for $x = 4.5$ is given in Figure 5.

One can suppress the HF noise by increasing the step size, but then the form of the soliton is distorted. To find a numerical procedure that suppresses the noise and gives correct results for large values of x remains a challenge for future studies.

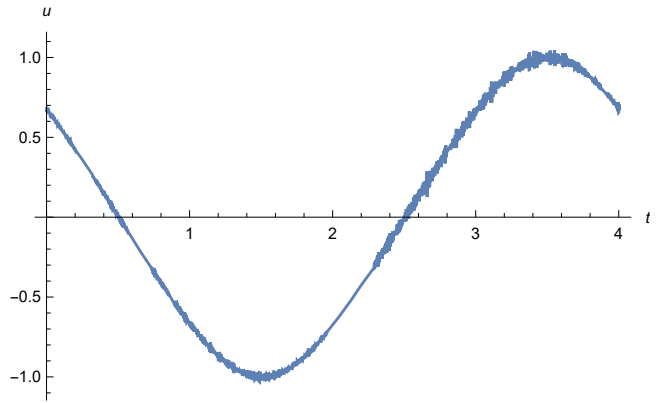


FIGURE 5. HF noise for the periodic soliton evolution. $x = 4.5$.

3. DISCRETE MODELS WITH BENIGN GHOSTS

One of the possible solutions to this numerical problem could consist in discretizing the model in time direction and assuming that the variable t takes only the discrete values $t = h, 2h, \dots, Nh$, for some integer $N \geq 3$ and by replacing the continuous time derivative ψ_t by a discrete (symmetric) time derivative $[\psi(t + h, x) - \psi(t - h, x)]/(2h)$. Then the Lagrangian⁶

$$L[\psi(t, x)] = \frac{\psi_{xx}^2 - \psi_x^4 - \psi_x \psi_t}{2} \quad (38)$$

acquires the form

$$L_N = \sum_{k=1}^N \left\{ \frac{[\psi_{xx}(kh, x)]^2 - [\psi_x(kh, x)]^4}{2} - \frac{1}{2} \psi_x(kh, x) \frac{\psi[(k+1)h, x] - \psi[(k-1)h, x]}{2h} \right\}, \quad (39)$$

where we impose the periodicity: $\psi(0, x) \equiv \psi(Nh, x)$ and $\psi[(N+1)h, x] \equiv \psi(h, x)$.⁷

The Lagrangian (39) includes a finite number of degrees of freedom and represents a mechanical system. This system involves higher derivatives in x (playing the role of time) and hence involves ghosts. Defining the new dynamical variables $a^k(x) = \psi_x(kh, x)$, the equations of motion derived from the Lagrangian (39) read

$$a_{xxx}^k + 6(a^k)^2 a_x^k + \frac{a^{k+1} - a^{k-1}}{2h} = 0. \quad (40)$$

There are two integrals of motion: the energy

$$E = \sum_{k=1}^N \left[\frac{(a_x^k)^2 - 3(a^k)^4}{2} - a^k a_{xx}^k \right] \quad (41)$$

⁶It is quite analogous to (4). After variation with respect to $\psi(t, x)$, one gets Eq. (9) after posing $u(t, x) = \psi_x(t, x)$.

⁷It is also possible to impose the Dirichlet-type boundary conditions, $\psi(0h, x) = \psi[(N+1)h, x] = 0$. For $N = 2$, periodicity cannot be imposed and Dirichlet conditions are the only option.

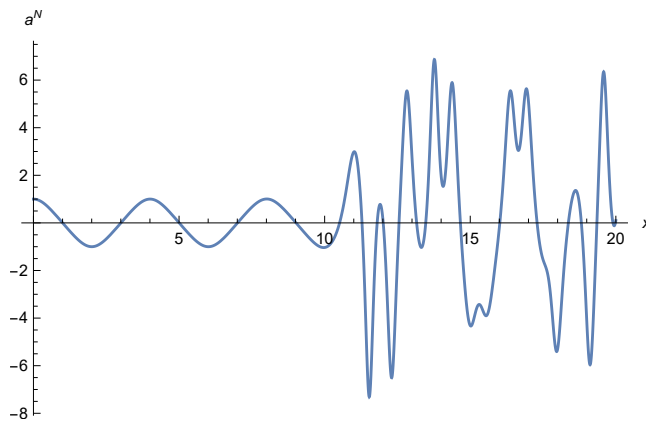


FIGURE 6. The solution of the system (40) for $a^N(x)$ ($N = 350$, $h = T/N$).

and

$$Q = \sum_{k=1}^N [a_{xx}^k + 2(a^k)^3]. \quad (42)$$

The expression (41) is the discretized version of the integral

$$\int dt \left[\frac{(u_x)^2 - 3u^4}{2} - uu_{xx} \right] \quad (43)$$

in the continuous MKdV system, which is conserved during the x evolution, as follows from the local conservation law (11). The second integral of motion is related to the conservation law (10). By contrast, the currents in the higher conservation laws of the MKdV equation, starting with Eq.(12), do not translate into integrals of motion of the discrete systems. We have only two integrals of motion and many variables, which means that the equation system (40) is not integrable and exhibits a chaotic behaviour.

We fed these equations to Mathematica and found out that their solution stays bounded up to $x = 10000$ and more – the ghosts are benign! This represents a further argument in favour of the conjecture that in the continuous theory the evolution in spatial direction is also benign. Indeed, one may expect that taking larger and larger values of N would allow one to simulate better and better the continuous theory (though the presence of chaos might make such a convergence non uniform in x).

Anyway, we tried the solitonic initial conditions and found out that the discrete system for large $N = 350$ (the limit of Mathematica skills) behaves better than the PDE. As is seen from Figure 6, the discrete solution stays close to the exact soliton solution up to $x \approx 10$, to be compared to $x \approx 4.5$, which was the horizon of the numerical procedure of the previous section. Hopefully, a clever mathematician, an expert in numerical calculations, would be able to increase the horizon even more...

Lastly, we note that, irrespectively to the relationship of the systems (39) to the MKdV equation, these systems represent an interest by their own because they provide a set of nontrivial interacting higher derivative systems with benign ghosts. Such systems were not known before.

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