

EXACT SOLUTIONS FOR TIME-DEPENDENT COMPLEX SYMMETRIC POTENTIAL WELL

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ABSTRACT. Using the pseudo-invariant operator method, we investigate the model of a particle with a time-dependent mass in a complex time-dependent symmetric potential well $V(x, t) = if(t)|x|$. The problem is exactly solvable and the analytic expressions of the Schrödinger wavefunctions are given in terms of the Airy function. Indeed, with an appropriate choice of the time-dependent metric operators and the unitary transformations, for each region, the two corresponding pseudo-Hermitian invariants transform into a well-known time-independent Hermitian invariant which is the Hamiltonian of a particle confined in a symmetric linear potential well. The eigenfunctions of the last invariant are the Airy functions. Then, the phases obtained are real for both regions and the general solution to the problem is deduced.

KEYWORDS: Non-Hermitian Hamiltonian, time-dependent Hamiltonian, pseudo-invariant method, PT-symmetry, pseudo-Hermiticity.

1. INTRODUCTION

The discovery of a class of non-Hermitian Hamiltonian that may have a real spectrum has prompted a revival of theoretical and applied research in quantum physics. In fact, in 1998, C.M. Bender and S. Boettcher showed that any non-Hermitian Hamiltonian invariant under the unbroken space-time reflection, or \mathcal{PT} -symmetry, has real eigenvalues and satisfies all the physical axioms of quantum mechanics [1–3]. In 2002, A. Mostafazadeh presented a more extended version of non-Hermitian Hamiltonians having a real spectrum, proving that the hermiticity of the Hamiltonian with respect to a positive definite inner product, $\langle \cdot, \cdot \rangle_\eta = \langle \cdot | \eta | \cdot \rangle$, is a necessary and sufficient condition for the reality of the spectrum, where η is the metric operator which is linear, Hermitian, invertible and positive. This condition requires that the Hamiltonian H satisfies the pseudo-Hermitian relation [4–6]:

$$H^\dagger = \eta H \eta^\dagger. \quad (1)$$

Moreover in recent years, a significant progress has been achieved in the study of time-dependent (TD) non-Hermitian quantum systems in several branches of physics. Finding exact solutions to the TD Schrödinger equation, which cannot be reduced to eigenvalues equation in general, is a problem of intriguing difficulty. Different methods are used to obtain solutions of Schrödinger’s equation for explicitly TD systems, such as unitary and non-unitary transformations, the pseudo-invariant method, Dyson’s maps, point transformations, Darboux transformations, perturbation theory and adiabatic approximation [7–31].

However, the emergence of a non-linear Ermakov-type auxiliary equation for several TD systems, which is difficult to solve, constitutes an additional constraint to obtain exact analytical solutions [32, 33]. This greatly reduces the number of exactly solvable time-dependent non-Hermitian systems [34–38]. In particular, other works have been concerned with studying exact solutions of TD Hamiltonians with a specific TD mass in the non-Hermitian case [39, 40] and also in the Hermitian case [41–45].

In the present work, we used the pseudo-invariant method [17] to obtain the exact solutions of the Schrödinger equation for a particle with TD mass moving in a TD complex symmetric potential well:

$$V(x, t) = if(t)|x|, \quad (2)$$

where $f(t)$ is an arbitrary real TD function.

The manuscript is organised as follows: In Section 2, we introduce some of the basic equations of the TD non-Hermitian Hamiltonians and their time-dependent Schrödinger equation (TDSE) with a TD metric. In Section 3, we discuss the use of the Lewis-Riesenfeld invariant method to address the Schrödinger equation for an explicitly TD non-Hermitian Hamiltonian. In Section 4, we use the Lewis-Riesenfeld method to solve the TD Schrödinger equation for a particle with TD mass in a TD complex symmetric potential well. Finally, in Section 5, we conclude with a brief review of the obtained results.

2. TD NON-HERMITIAN HAMILTONIAN WITH TD METRIC

Let $H(t)$ be a non-Hermitian TD Hamiltonian and $h(t)$ its associated TD Hermitian Hamiltonian. The two corresponding TD Schrödinger equations describing the quantum evolution are:

$$H(t) |\Phi^H(t)\rangle = i\hbar \partial_t |\Phi^H(t)\rangle, \quad (3)$$

$$h(t) |\Psi^h(t)\rangle = i\hbar \partial_t |\Psi^h(t)\rangle, \quad (4)$$

where the two Hamiltonians are related by the Dyson maps $\rho(t)$ as

$$H(t) = \rho^{-1}(t) h(t) \rho(t) - i\hbar \rho^{-1}(t) \dot{\rho}(t), \quad (5)$$

and their wavefunctions $|\Phi^H(t)\rangle$ and $|\Psi^h(t)\rangle$ as

$$|\Psi^h(t)\rangle = \rho(t) |\Phi^H(t)\rangle. \quad (6)$$

The hermiticity of $h(t)$ allowed us to establish the connection between the Hamiltonian $H(t)$ and its Hermitian conjugate $H^\dagger(t)$ as

$$H^\dagger(t) = \eta(t) H(t) \eta^{-1}(t) + i\hbar \dot{\eta}(t) \eta^{-1}(t), \quad (7)$$

which is a generalisation of the well-known conventional quasi-Hermiticity Equation (1), and the TD metric operator is Hermitian and defined as $\eta(t) = \rho^\dagger(t) \rho(t)$.

3. PSEUDO-INVARIANT OPERATOR METHOD

Let us start with the description of the Lewis-Riesenfeld theory [46] for a TD Hermitian Hamiltonian $h(t)$ with a Hermitian TD invariant $I^h(t)$. The dynamic invariant $I^h(t)$ satisfies:

$$\frac{dI^h(t)}{dt} = \frac{\partial I^h(t)}{\partial t} - \frac{i}{\hbar} [I^h(t), h(t)] = 0. \quad (8)$$

The eigenvalue equation for $I^h(t)$ is:

$$I^h(t) |\psi_n^h(t)\rangle = \lambda_n |\psi_n^h(t)\rangle, \quad (9)$$

where the eigenvalues λ_n of $I^h(t)$ are reals and time-independent, and the Lewis-Riesenfeld phase is defined as:

$$\hbar \frac{d}{dt} \varepsilon_n(t) = \langle \psi_n^h(t) | i\hbar \frac{\partial}{\partial t} - h(t) | \psi_n^h(t) \rangle, \quad (10)$$

and the solution of the TDSE of $h(t)$ is given as

$$|\Psi^h(t)\rangle = \exp[i\varepsilon_n(t)] |\psi_n^h(t)\rangle. \quad (11)$$

In the paper [17], we showed that any TD Hamiltonian $H(t)$ satisfying the TD quasi-hermiticity Equation (7) admits a pseudo-hermitian invariant $I^{ph}(t)$ such that:

$$\begin{aligned} I^{ph\dagger}(t) &= \eta(t) I^{ph}(t) \eta^{-1}(t) \Leftrightarrow \\ I^h(t) &= \rho(t) I^{ph}(t) \rho^{-1}(t) = I^{h\dagger}(t). \end{aligned} \quad (12)$$

Since the Hermitian invariant $I^h(t)$ satisfies the eigenvalues Equation (9), Equation (12) ensures that the pseudo-Hermitian invariant's spectrum is real with the same eigenvalues λ_n of $I^h(t)$:

$$I^h(t) |\psi_n^h(t)\rangle = \lambda_n |\psi_n^h(t)\rangle, \quad (13)$$

$$I^{ph}(t) |\phi_n^{ph}(t)\rangle = \lambda_n |\phi_n^{ph}(t)\rangle, \quad (14)$$

where the eigenfunctions $|\psi_n^h(t)\rangle$ and $|\phi_n^{ph}(t)\rangle$, of $I^h(t)$ and $I^{ph}(t)$, respectively, are related as

$$|\psi_n^h(t)\rangle = \rho(t) |\phi_n^{ph}(t)\rangle. \quad (15)$$

The inner products of the eigenfunctions associated with the non-Hermitian invariant $I^{ph}(t)$ can now be written as

$$\langle \phi_m^{ph}(t) | \phi_n^{ph}(t) \rangle_\eta = \langle \phi_m^{ph}(t) | \eta | \phi_n^{ph}(t) \rangle = \delta_{mn}, \quad (16)$$

and it corresponds to the conventional inner product associated to the Hermitian invariant $I^h(t)$.

It is easy to verify, by a direct substitution of the Hermitian Hamiltonian $h(t)$ and the Hermitian invariant $I^h(t)$ by their equivalents in the Expressions (5) and (12), respectively, that the pseudo Hermitian invariant $I^{ph}(t)$ satisfies:

$$\frac{\partial I^{ph}(t)}{\partial t} = \frac{i}{\hbar} [I^{ph}(t), H(t)]. \quad (17)$$

We should remark that the invariant operator's eigenstates and eigenvalues can be computed using the same procedure as the Hermitian case.

The solution $|\Phi^H(t)\rangle$ of the Schrödinger Equation (3) is different from $|\phi_n^{ph}(t)\rangle$ in Equation (14) only by the factor $e^{i\varepsilon_n^{ph}(t)}$ where $\varepsilon_n^{ph}(t)$ is a real phase given by:

$$\hbar \frac{d}{dt} \varepsilon_n^{ph}(t) = \langle \phi_n^{ph}(t) | \eta(t) \left[i\hbar \frac{\partial}{\partial t} - H(t) \right] | \phi_n^{ph}(t) \rangle. \quad (18)$$

4. PARTICLE IN TD COMPLEX SYMMETRIC POTENTIAL WELL

Let us consider a particle with a TD mass $m(t)$ in the presence of a pure imaginary TD symmetric potential well Equation (2), where its Hamiltonian can be written as:

$$H(t) = \begin{cases} \frac{p^2}{2m(t)} + if(t)x & \text{if } x \geq 0 \\ \frac{p^2}{2m(t)} - if(t)x & \text{if } x < 0 \end{cases}, \quad (19)$$

the associated TDSE of the system is:

$$\left[\frac{p^2}{2m(t)} + if(t)|x \right] \Psi(x, t) = i \frac{\partial}{\partial t} \Psi(x, t), \quad (20)$$

where $m(t)$ is the particle TD mass and $f(t)$ an arbitrary real TD function, and the unit of $\hbar = 1$. This model can be considered as the complex version of the Hermitian case of a particle, with TD mass and charge q , moving under the action of TD electric field $E(t)$ and confined in a pure imaginary symmetric linear potential well: $if(t)x$ for $x \geq 0$ and $-if(t)x$ for $x \leq 0$, where $f(t) = -qE(t)$.

According to the results in [17], the solution to the TD Schrödinger equation with a TD non-Hermitian Hamiltonian is easily found if a nontrivial TD pseudo-Hermitian invariant $I^{ph}(t)$ exists and satisfies the von-Neumann Equation (17).

In the current problem, in order to solve the TD Shrödinger Equation (20) we assume that the Hamiltonian $H(t)$ admits an invariant in each region: let $I_1^{ph}(t)$ for $x \geq 0$ and $I_2^{ph}(t)$ for $x \leq 0$.

For the region $x \geq 0$, let us look for a non-Hermitian TD invariant in the following quadratic form:

$$I_1^{ph}(t) = \beta_1(t)p^2 + \beta_2(t)x + \beta_3(t)p + \beta_4(t), \quad (21)$$

where $\beta_i(t)$ are arbitrary complex functions to be determined. By inserting the Expressions (19) and (21) in Equation 17, the following system of equations can be found:

$$\begin{cases} \dot{\beta}_1(t) = 0, \\ \dot{\beta}_2(t) = 0, \\ \dot{\beta}_3(t) = -\frac{\beta_2(t)}{m(t)} + 2if(t)\beta_1(t), \\ \dot{\beta}_4(t) = if(t)\beta_3(t), \end{cases} \quad (22)$$

to simplify the calculations, we take $\beta_1(t) = 1$ and $\beta_2(t) = 1$, so $\beta_3(t)$ and $\beta_4(t)$ are given by:

$$\beta_3(t) = g(t) + ik(t), \quad (23)$$

$$\beta_4(t) = s(t) + iw(t), \quad (24)$$

where

$$g(t) = -\int \frac{dt}{m(t)}, \quad k(t) = 2 \int f(t) dt,$$

$$s(t) = -\int f(t)k(t) dt \text{ and}$$

$$w(t) = \int f(t)g(t) dt.$$

Substituting Expressions (23) and (24) in Equation (21) we found:

$$I_1^{ph}(t) = p^2 + x + [g(t) + ik(t)]p + s(t) + iw(t). \quad (25)$$

Its eigenvalue equation is as follows:

$$I_1^{ph}(t) |\psi(t)\rangle = \lambda_1 |\psi(t)\rangle, \quad (26)$$

in order to show that the spectrum of $I_1^{ph}(t)$ is real, we search for a metric operator that fulfills the pseudo hermiticity relation:

$$I_1^{ph\dagger}(t) = \eta_1(t) I_1^{ph}(t) \eta_1^{-1}(t), \quad (27)$$

and we make the following choice for metric:

$$\eta_1(t) = \exp[-\alpha(t)x - \beta(t)p], \quad (28)$$

where $\alpha(t)$ and $\beta(t)$ are chosen as real functions in order that the metric operator $\eta_1(t)$ is Hermitian.

The position and momentum operators transform according to the transformation $\eta_1(t)$ as:

$$\eta_1(t)x\eta_1^{-1}(t) = x + i\beta(t), \quad (29)$$

$$\eta_1(t)p\eta_1^{-1}(t) = p - i\alpha(t), \quad (30)$$

incorporating these relationships into Equation (27), we found:

$$\alpha(t) = k(t), \quad (31)$$

$$\beta(t) = g(t)k(t) - 2w(t), \quad (32)$$

then the TD metric operator $\eta_1(t)$ is given by:

$$\eta_1(t) = \exp[-k(t)x - (g(t)k(t) - 2w(t))p], \quad (33)$$

according to the relation $\eta_1(t) = \rho_1^\dagger(t)\rho_1(t)$, and since $\rho_1(t)$ is not unique, we can take it as a Hermitian operator in order to simplify the calculations:

$$\rho_1(t) = \exp \left[-\frac{k(t)}{2}x - \left[\frac{g(t)k(t)}{2} - w(t) \right] p \right], \quad (34)$$

the Hermitian invariant $I_1^h(t)$ associated with the pseudo-Hermitian invariant $I_1^{ph}(t)$ is given by:

$$I_1^h(t) = \rho(t) I_1^{ph} \rho^{-1}(t) = p^2 + x + g(t)p + \frac{k^2(t)}{4} + s(t). \quad (35)$$

For the region $x \leq 0$, we take the non-Hermitian invariant I_2^{ph} as

$$I_2^{ph}(t) = \alpha_1(t)p^2 + \alpha_2(t)x + \alpha_3(t)p + \alpha_4(t), \quad (36)$$

where $\alpha_i(t)$ are arbitrary complex functions to be determined.

In the same way as the precedent case, inserting the Expressions (19) and (36) in Equation (17), where we take $\alpha_1(t) = 1$ and $\alpha_2(t) = -1$, so $\alpha_3(t)$ and $\alpha_4(t)$ are given by:

$$\alpha_3(t) = -g(t) - ik(t), \quad (37)$$

$$\alpha_4(t) = s(t) + iw(t). \quad (38)$$

Then, the final results of $I_2^{ph}(t)$ and $\eta_2(t)$ are:

$$I_2^{ph}(t) = p^2 - x - [g(t) + ik(t)]p + s(t) + iw(t), \quad (39)$$

$$\eta_2(t) = \exp[k(t)x - [2w(t) - g(t)k(t)]p]. \quad (40)$$

We take $\rho_2(t)$ as a Hermitian operator, then $\eta_2(t) = \rho_2^2$,

$$\rho_2(t) = \exp\left[\frac{k(t)}{2}x + \left[\frac{k(t)g(t)}{2} - w(t)\right]p\right], \quad (41)$$

and the related Hermitian invariant $I_2^h(t)$ is:

$$I_2^h(t) = p^2 - x - g(t)p + \frac{k^2(t)}{4} + s(t). \quad (42)$$

To derive the eigenvalues equations of the invariants $I_j^h(t)$ for the two regions ($j = 1, 2$), we introduce the unitary transformations $U_j(t)$:

$$|\phi_{n,j}(t)\rangle = U_j(t) |\varphi_n\rangle, \quad j = 1, 2, \quad (43)$$

where φ_n will be determined later and

$$U_1(t) = \exp\left[-i\frac{g(t)}{2}x + \frac{i}{4}[k^2(t) - g^2(t) + 4s(t)]p\right], \quad (44)$$

$$U_2(t) = \exp\left[i\frac{g(t)}{2}x - \frac{i}{4}[k^2(t) - g^2(t) + 4s(t)]p\right]. \quad (45)$$

According to these transformations, the invariants $I_1^h(t)$ and $I_2^h(t)$ turn into:

$$I_1 = U_1^\dagger(t)I_1^h(t)U_1(t) = p^2 + x, \quad (46)$$

$$I_2 = U_2^\dagger(t)I_2^h(t)U_2(t) = p^2 - x, \quad (47)$$

and they can be written in the following combined form:

$$I = p^2 + |x|. \quad (48)$$

We note here that I can be considered as the Hamiltonian of a particle of mass $m_0 = 1/2$ confined in the linear symmetric potential well $|x|$. Therefore, the eigenvalue equation of the invariant I :

$$\left[\frac{d^2}{dx^2} + (\lambda_n - |x|)\right]\varphi_n(x) = 0, \quad (49)$$

is a well-known problem in quantum mechanics. The bound states $\varphi_n(x)$ are given in terms of the Airy functions Ai and Bi [47, 48]:

$$\varphi_n(x) = N_n Ai(|x| - \lambda_n) + N'_n Bi(|x| - \lambda_n). \quad (50)$$

This solution is not relevant because $Bi(|x| - \lambda_n)$ tends to infinity for $(|x| - \lambda_n) > 0$. Thus, we take $N'_n = 0$ and the above solution reduces to:

$$\varphi_n(x) = N_n Ai(|x| - \lambda_n). \quad (51)$$

The eigenvalues λ_n are determined by matching the functions $\varphi_n(x)$ and their derivatives in the two regions at the point $x = 0$:

$$\varphi_n^{(1)}(0) = \varphi_n^{(2)}(0), \quad (52)$$

$$\varphi_n^{\prime(1)}(0) = \pm\varphi_n^{\prime(2)}(0), \quad (53)$$

from which there are two possibilities for λ_n and the normalisation constant N_n depending on whether n is even or odd:

- If n is even:

$$\lambda_n = -a'_{\frac{n}{2}+1}, \quad (54)$$

where a'_k is the k^{th} zero of the derivative Ai' of the Airy function, and all values of a'_k are negative numbers [49].

The normalisation constant is:

$$N_n = \frac{1}{\sqrt{-2a'_{\frac{n}{2}+1}Ai(a'_{\frac{n}{2}+1})}}, \quad (55)$$

and the corresponding eigenfunction of I is:

$$\varphi_n(x) = \frac{1}{\sqrt{-2a'_{\frac{n}{2}+1}Ai(a'_{\frac{n}{2}+1})}}Ai(|x| + a'_{\frac{n}{2}+1}), \quad (56)$$

- If n is odd:

$$\lambda_n = -a_{\frac{n+1}{2}}, \quad (57)$$

where a_k is the k^{th} zero of the Airy function Ai , and all values of a_k are negative numbers [49].

The normalisation constant is:

$$N_n = \frac{1}{\sqrt{2Ai'(a_{\frac{n+1}{2}})}}, \quad (58)$$

and the corresponding eigenfunction of I is:

$$\varphi_n(x) = \text{sgn}(x)\frac{1}{\sqrt{2Ai'(a_{\frac{n+1}{2}})}}Ai(|x| + a_{\frac{n+1}{2}}). \quad (59)$$

The eigenfunctions of the Hermitian invariants $I_j^h(t)$ are written for each region as:

$$|\phi_{n,j}(t)\rangle = U_j(t) |\varphi_n\rangle, \quad (60)$$

then, the eigenfunctions of the pseudo-Hermitian invariants $I_j^{ph}(t)$ are given by:

$$|\psi_{n,j}(t)\rangle = \rho_j^{-1}(t)U_j(t) |\varphi_n\rangle, \quad (61)$$

thus, the solutions of the time-dependent Schrödinger Equation (20) take the form:

$$|\Psi_{n,j}(t)\rangle = e^{i\epsilon_n^j(t)} |\psi_{n,j}(t)\rangle, \quad (62)$$

where $\epsilon_n^j(t)$ is the phase ($\epsilon_n^1(t)$ for $x \geq 0$ and $\epsilon_n^2(t)$ for $x \leq 0$), which is obtained from the following relation:

$$\begin{aligned} \dot{\epsilon}_n^j(t) &= \langle \psi_{n,j}(t) | \eta_j(t) \left[i \frac{\partial}{\partial t} - H(t) \right] | \psi_{n,j}(t) \rangle \\ &= \langle \phi_{n,j}(t) | i \rho_j(t) \dot{\rho}_j^{-1}(t) | \phi_{n,j}(t) \rangle \\ &\quad - \langle \phi_{n,j}(t) | \rho_j(t) H(t) \rho_j^{-1}(t) | \phi_{n,j}(t) \rangle \\ &\quad + \langle \phi_{n,j}(t) | i \frac{\partial}{\partial t} | \phi_{n,j}(t) \rangle \\ &= \theta(t) - \langle \phi_{n,j}(t) | \frac{p^2}{2m(t)} | \phi_{n,j}(t) \rangle \\ &\quad + \langle \phi_{n,j}(t) | i \frac{\partial}{\partial t} | \phi_{n,j}(t) \rangle, \end{aligned} \quad (63)$$

where

$$\theta(t) = \frac{1}{2} f(t) \left[\frac{k(t)}{2} g(t) - w(t) \right]. \quad (64)$$

Using the unitary transformations $U_j(t)$, we found:

$$\dot{\epsilon}_n^j(t) = \chi^j(t) - \frac{1}{2m(t)} \langle \varphi_n(t) | (p^2 \pm x) | \varphi_n(t) \rangle, \quad (65)$$

where

$$\chi^1(t) = \theta(t) - \frac{1}{16m(t)} [k^2(t) + 3g^2(t) + 4s(t)], \quad (66)$$

$$\chi^2(t) = \theta(t) + \frac{1}{16m(t)} [k^2(t) - g^2(t) + 4s(t)]. \quad (67)$$

From the eigenvalue equation of the invariant I , we have:

$$(p^2 \pm x) |\varphi_n(t)\rangle = \lambda_n |\varphi_n(t)\rangle, \quad (68)$$

then, the phases $\epsilon_n^j(t)$ take the form:

$$\epsilon_n^j(t) = \int (\chi^j(t) - \frac{\lambda_n}{2m(t)}) dt, \quad (69)$$

and the solution of the TD Schrödinger Equation (20) is given by:

$$|\Psi_{n,j}(t)\rangle = \exp [i\epsilon_n^j(t)] \rho_j(t)^{-1} |\phi_{n,j}(t)\rangle. \quad (70)$$

In position representation we have:

$$\begin{aligned} \langle x | \rho_j^{-1}(t) | \phi_j(t) \rangle &= \exp [i\zeta(t)] \exp \left[\pm \frac{k(t)}{2} x \right] \\ &\times \phi_j(x \pm i(\frac{g(t)k(t)}{2} - w(t)), t), \end{aligned} \quad (71)$$

where (+) is for the positive region while (-) is for the negative region, and

$$\zeta(t) = -\frac{k}{4} \left(\frac{g(t)k(t)}{2} - w(t) \right). \quad (72)$$

Then, the solution of the Schrödinger Equation for each region (70) can be written as:

$$\begin{aligned} \Psi_{n,j}(x, t) &= \exp [i(\epsilon_n^j(t) + \zeta(t))] \exp \left[\pm \frac{k(t)}{2} x \right] \\ &\times \phi_{n,j}(x \pm i(\frac{g(t)k(t)}{2} - w(t)), t), \end{aligned} \quad (73)$$

and the general solution of the Schrödinger Equation (20) is given by:

$$\Psi(x, t) = \begin{cases} \Psi_{n,1}(x, t) & \text{for } x \geq 0, \\ \Psi_{n,2}(x, t) & \text{for } x \leq 0. \end{cases} \quad (74)$$

According to the Equations (51), (60), (61) and (62), the probability density function is given by:

$$\begin{aligned} |\rho_1(t)\Psi_{n,1}|^2 + |\rho_2(t)\Psi_{n,2}|^2 &= \\ = |\phi_{n,1}|^2 + |\phi_{n,2}|^2 &= |\varphi_n|^2, \end{aligned} \quad (75)$$

and because $\varphi_n(x)$ is determined in terms of Airy function $Ai(x)$, which is a real function, and according to Equations (56) and (59), the probability density expression can be written as:

- For n is even

$$|\varphi_n(x)|^2 = \frac{1}{(-2a'_{\frac{n}{2}+1}) [Ai(a'_{\frac{n}{2}+1})]^2} [Ai(|x| + a'_{\frac{n}{2}+1})]^2, \quad (76)$$

and which is represented in Figure 1 for the first three even states ($n = 0, 2, 4$).

- For n is odd

$$|\varphi_n(x)|^2 = \frac{1}{2 [Ai'(a_{\frac{n+1}{2}})]^2} [Ai(|x| + a_{\frac{n+1}{2}})]^2, \quad (77)$$

and which is represented in Figure 2 for the first three odd states ($n = 1, 3, 5$).

We note here that the probability in the region $x \leq 0$ is

$$\begin{aligned} \langle \Psi_{n,2}(t) | \eta_2(t) | \Psi_{n,2}(t) \rangle &= \langle \varphi_n | \varphi_n \rangle_{x \leq 0} \\ &= \int_{-\infty}^0 \varphi_n^*(x) \varphi_n(x) dx = \frac{1}{2}, \end{aligned} \quad (78)$$

and the probability in the region $x \geq 0$ is

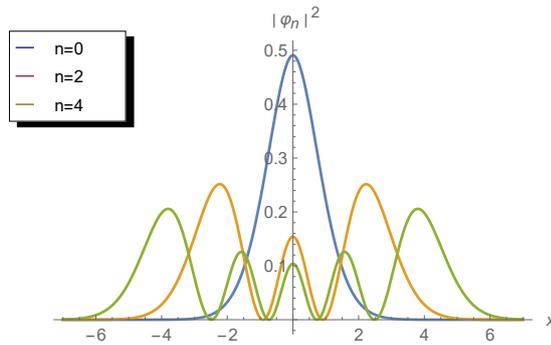


FIGURE 1. Probability density of Equation 76 for even values of $n = 0, 2, 4$.

$$\begin{aligned} \langle \Psi_{n,1}(t) | \eta_1(t) | \Psi_{n,1}(t) \rangle &= \langle \varphi_n | \varphi_n \rangle_{x \geq 0} \\ &= \int_0^{\infty} \varphi_n^*(x) \varphi_n(x) dx = \frac{1}{2}. \end{aligned} \quad (79)$$

So the two regions are equiprobable and the probability in all space is equal to one

$$\begin{aligned} \langle \Psi(t), \Psi(t) \rangle_{\eta} &= \langle \Psi_{n,1} | \eta_1(t) | \Psi_{n,1} \rangle + \langle \Psi_{n,2} | \eta_{n,2}(t) | \Psi_{n,2} \rangle \\ &= \int_{-\infty}^{\infty} \varphi_n^*(x) \varphi_n(x) dx = 1. \end{aligned} \quad (80)$$

5. CONCLUSION

The pseudo-invariant method has been used to obtain the exact analytical solutions of the time-dependent Schrödinger equation for a particle with time-dependent mass moving in a complex time-dependent symmetric potential well. We have shown that the problem can be reduced to solve a well-known eigenvalue equation for a time-independent Hermitian invariant. In fact, with a specific choice of the TD metric operators, $\eta_1(t)$ and $\eta_2(t)$, and the Dyson maps, $\rho_1(t)$ and $\rho_2(t)$, and using unitary transformations, the pseudo-invariants operators ($I_1^{ph}(t)$ for $x \geq 0$ and $I_2^{ph}(t)$ for $x \leq 0$) are mapped to two time-independent Hermitian invariants $I_1^h(t)$ and $I_2^h(t)$, which can be combined in a unique form $I = p^2 + |x|$. The latter can be considered as the Hamiltonian of a particle confined in a linear time-independent symmetric potential well, where its eigenfunctions are given in terms of the Airy function Ai . The phases have been calculated for the two regions and are real. Thus, the exact analytical solution of the problem has been deduced. Finally, let us highlight the fact that the probability density associated with the model in question is time-independent.

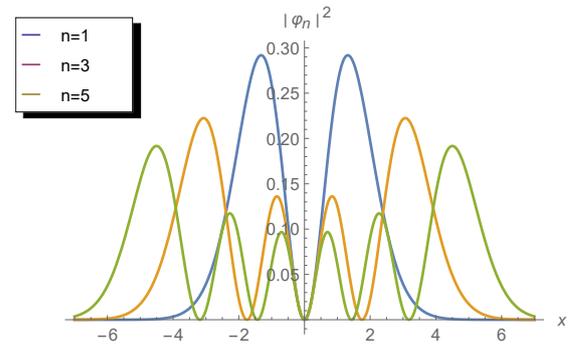


FIGURE 2. Probability density of Equation 77. for odd values of $n = 1, 3, 5$.

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