Structure of the Enveloping Algebras

Č. Burdík, O. Navrátil, S. Pošta

The adjoint representations of several small dimensional Lie algebras on their universal enveloping algebras are explicitly decomposed. It is shown that commutants of raising operators are generated as polynomials in several basic elements. The explicit form of these elements is given and the general method for obtaining these elements is described.

Keywords: adjoint representation, enveloping algebra, Lie algebra.

1 Introduction

Let g be a finite dimensional complex semisimple Lie algebra and $U(g) = U$ its corresponding enveloping algebra. If we want to study all two-sided ideals of the associative algebra U (see [1] chapters 6–10), i.e. to find all the vector spaces $I \subset U$ which satisfy

$$\text{for all } a, b \in U : aIb \in I$$

we can easily see that such a two-sided ideal is invariant with respect to adjoint action of U: if we take adjoint action $\rho : g \to L(U)$, i.e.

$$\rho(x)a = [x, a] \text{ for } x \in g \text{ and } a \in U$$

and extend this definition to all U by

$$\rho(u) = \rho(x_1) \dots \rho(x_n) = \left[x_1, \left[\dots, [x_n, a] \dots\right]\right] \quad \text{for all } u = x_1 \dots x_n,\ x_i \in g$$

we see that

$$\rho(U)I \subset I.$$

It is well known that such adjoint action ρ is completely reducible for g semisimple: it is possible to cut the enveloping algebra U into pieces U_{Ψ_k} such that

$$U = \bigoplus_{k=0}^{+\infty} U_{\Psi_k}$$

and each piece U_{Ψ_k} is an invariant subspace (corresponding to an irreducible representation (component) of ρ) generated by its highest weight vector Ψ_k:

$$\rho(U)\Psi_k = U_{\Psi_k}$$

Now if we take any ideal $I \subset U$ and find any highest weight vector $\Psi_k \in I$, we automatically know that

$$\rho(U)\Psi_k = U_{\Psi_k} \subset I,$$

i.e. we know a large set of vectors which *must* be contained *also* in the ideal I. This can help considerably in the classification of all ideals of U.

The main aim of this paper is to give an explicit decomposition $\bigoplus_{k=0}^{+\infty} U_{\Psi_k}$, i.e. to give a list of highest weight vectors Ψ_k, $k = 0, 1, \dots$ for the simplest cases of Lie algebras and to show that even in more complicated examples the decomposition does not look too weird and is relatively easyli obtained by a generally described procedure.

2 The simplest example – algebra *sl*(2)

Let us have algebra $sl(2)$ which is a complex linear span of three vectors

$$E_{12}, E_{21}, H_1 = E_{11} - E_{22},$$

where by E_{ij} we denote the matrix having 1 in position (i, j) and 0s elsewhere. If we compute the commutation relations of these elements we obtain

$$[E_{12}, E_{21}] = H_1, [H_1, E_{12}] = 2E_{12}, [H_1, E_{21}] = -2E_{21}.$$

The enveloping algebra $U = U(sl(2))$ fulfils the Poincare-Birkhoff-Witt theorem and consists of all complex linear combinations of monomials

$$E_{12}^{\alpha} E_{21}^{\beta} H_1^{\gamma}, \quad \alpha, \beta, \gamma \geq 0.$$

It has a natural filtration U_n, $n \geq 0$ given by the degree n of the elements in U_n. It is easily seen that the adjoint action has U_n as its invariant subspace (by applying a commutator we cannot obtain an element of higher degree). Therefore it is completely reducible on each U_n, i.e. we can see U_n as a direct sum of invariant subspaces generated by certain highest weight vectors.

The highest weight vector v of weight m satisfies the relations

$$[E_{12}, v] = 0, \quad [H_1, v] = 2mv.$$

From this we can directly find that all highest weight vectors of small degree:

degree 0	1
degree 1	E_{12}
degree 2	E_{12}^2, C_1
degree 3	$E_{12}^3, C_1 E_{12}$
…	…

Here

$$C_1 = H_1^2 + 4E_{12}E_{21} - 2H_1$$

is a Casimir element from the center of U. By commuting these vectors we see that

$$U_0 = \rho(U)1,$$
$$U_1 = \rho(U)E_{12} \oplus U_0,$$
$$U_2 = \rho(U)E_{12}^2 \oplus \rho(U)C_1 \oplus U_1,$$
$$\ldots$$

From this we can claim that for any $n \geq 0$

$$U_n = \bigoplus_{\substack{k,m\geq 0\\ 2k+m\leq n}} \rho(U)C_1^k E_{12}^m \tag{1}$$

The proof of this general formula is based on a dimensional check:

First we see that the sum $\rho(U)v_1 + \rho(U)v_2$, where v_1, v_2 are highest weight vectors, is direct if and only if v_1, v_2 are linear independent. Thus in our example we must check that the vectors $C_1^k E_{12}^m$ are linear independent for different k, m. But this is easily seen from the definition, since

$$C_1^k E_{12}^m = E_{12}^m H_1^{2k} + \text{lower terms}$$

where "lower terms" contain monomials with H_1^{2k-1} and lower.

Next it is well known from representation theory that the dimension of the irreducible representation with highest weight m is $2m+1$.

On the other hand the dimension of U_n is also easy to determine, because it is well known that it is isomorphic (as a vector space) to the direct sum $\bigoplus_{k=0}^{n} S^k(g)$ of the k-th symmetric powers of Lie algebra g. We can construct the symmetric algebra $S(g)$ from the tensor algebra $T(g)$ by taking the quotient algebra of $T(g)$ by the ideal generated by all differences of products $v \otimes w - w \otimes v$ for $v, w \in g$; then there is a direct sum decomposition of $S(g)$ as a graded algebra into the summands $S^k(g)$ which consist of the linear span of the monomials in vectors of g of degree k. The symmetric algebra $S(g)$ is in effect the same as the polynomial ring in the indeterminates that are basis vectors for g.

Therefore we see that

$$\dim S^k(g) = \binom{\dim g + k - 1}{k}$$

and from this we have

$$\dim U_n = \sum_{k=0}^{n} \binom{\dim g + k - 1}{k} = \binom{n + \dim g}{\dim g},$$

thus in our example

$$\dim U_n = \binom{n+3}{3}.$$

Now it is sufficent to prove that the dimensions of two spaces on both sides (1) are equal, i.e. that

$$\binom{n+3}{3} = \sum_{\substack{k,m\geq 0\\ 2k+m\leq n}} (2m+1)\,.$$

We have

$$\sum_{\substack{k,m\geq 0\\ 2k+m\leq n}} (2m+1) = \sum_{k=0}^{+\infty} \sum_{m=0}^{n-2k} (2m+1) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{m=0}^{n-2k} (2m+1) =$$
$$\sum_{k=0}^{\left[\frac{n}{2}\right]} (n-2k+1)^2 = \frac{1}{6}(n+1)(n+2((n+3) = \binom{n+3}{3}.$$

3 Moving to the commutative case

There is an alternative approach to obtain this decomposition – move to the commutative case:
It is a well-known fact (see [2]) that the mapping

$$e_i \to \sum_{j,k=1}^{n} c_{ij}^k \, x_k \frac{\partial}{\partial x_j}$$

is a representation of an n-dimensional Lie algebra with basis e_i, $i = 1, \ldots, n$ on a space of polynomials of the variables x_i, $i = 1, \ldots, n$; if we view $S(g)$ as the polynomial ring, this representation is equivalent as a representation of the enveloping algebra to the adjoint representation via canonical isomorphism $U(g) \to S(g)$.

Therefore if we return to our example and represent the basis vectors of the Lie algebra $sl(2)$ by operators

$$E_{12} \to \frac{\partial}{\partial e_{21}} - 2e_{12}\frac{\partial}{\partial h_1}$$
$$E_{21} \to \frac{\partial}{\partial e_{12}} - 2e_{21}\frac{\partial}{\partial h_1}$$
$$H_1 \to 2e_{12}\frac{\partial}{\partial e_{12}} - 2e_{21}\frac{\partial}{\partial e_{21}}$$

acting on the space of polynomials in three variables e_{12}, e_{21}, h_1, we can find all highest weight vectors by solving this system of differential equations:

$$h_1\frac{\partial f}{\partial e_{21}} - 2e_{12}\frac{\partial f}{\partial h_1} = 0,$$
$$2e_{12}\frac{\partial f}{\partial e_{12}} - 2e_{21}\frac{\partial f}{\partial e_{21}} = 2mf.$$

This system can be easily solved, and the general solution is

$$f(e_{12}, e_{21}, h_1) = e_{12}^m F(c_1),$$

where F is any differentiable function of the variable $c_1 = h_1^2 + 4e_{12}e_{21}$. We see that when $F(x) = x^k$, the solution is polynomial and we can transfer back to the enveloping algebra using the canonical isomorphism $S(g) \to U(g)$: if $a_1, \ldots, a_k \in g$, this isomorphism sends

$$a_1 \ldots a_k \to \frac{1}{k!}\sum_{\pi} A_{\pi(1)} \ldots A_{\pi(k)}$$

(we sum over all permutations of the indices 1, …, k). In our example

$$c_1 \to H_1^2 + 4\left(\frac{1}{2}E_{12}E_{21} + \frac{1}{2}E_{21}E_{12}\right)$$
$$= H_1^2 + 4E_{12}E_{21} - 2H_1 = C_1.$$

Doing the dimension check as before the we see that decomposition is complete.

4 The algebras $sl(3)$ and $gl(3)$

The case of algebra $sl(3)$ is more complicated. The decomposition of this case is done in [3]. Let us show where the complications in the decomposition arise, using the similar case of Lie algebra gl(3) with the basis Eij, $i, j - 1, 2, 3$ and commutation relations

$$[E_{ij}, E_{kl}] = E_{ij}\delta_{jk} - E_{kj}\delta_{il}.$$

We turn directly into the commutative case. First we set up, as before, the system of differential equations for the highest weight vectors of weight (N_1, N_2, N_3):

$$E_{12} \to x_{13}\frac{\partial f}{\partial x_{23}} - x_{12}\left(\frac{\partial f}{\partial x_{11}} - \frac{\partial f}{\partial x_{22}}\right)$$
$$+(x_{11} - x_{22})\frac{\partial f}{\partial x_{21}} - x_{32}\frac{\partial f}{\partial x_{31}} = 0$$
$$E_{23} \to -x_{13}\frac{\partial f}{\partial x_{12}} - x_{23}\left(\frac{\partial f}{\partial x_{22}} - \frac{\partial f}{\partial x_{33}}\right)$$
$$+(x_{22} - x_{33})\frac{\partial f}{\partial x_{32}} + x_{21}\frac{\partial f}{\partial x_{31}} = 0,$$
$$E_{13} \to -x_{13}\left(\frac{\partial f}{\partial x_{11}} - \frac{\partial f}{\partial x_{33}}\right) - x_{23}\frac{\partial f}{\partial x_{21}}$$
$$+x_{12}\frac{\partial f}{\partial x_{32}} + (x_{11} - x_{33})\frac{\partial f}{\partial x_{31}} = 0,$$
$$E_{11} \to x_{12}\frac{\partial f}{\partial x_{12}} + x_{13}\frac{\partial f}{\partial x_{13}} - x_{21}\frac{\partial f}{\partial x_{21}} - x_{31}\frac{\partial f}{\partial x_{31}} = N_1 f,$$
$$E_{22} \to -x_{12}\frac{\partial f}{\partial x_{12}} + x_{23}\frac{\partial f}{\partial x_{23}} + x_{21}\frac{\partial f}{\partial x_{21}} - x_{32}\frac{\partial f}{\partial x_{32}} = N_2 f,$$
$$E_{33} \to -x_{13}\frac{\partial f}{\partial x_{13}} - x_{23}\frac{\partial f}{\partial x_{23}} + x_{32}\frac{\partial f}{\partial x_{32}} + x_{31}\frac{\partial f}{\partial x_{31}} = N_3 f.$$

From this immediately follows $N_1 + N_2 + N_3 = 0$. If we define new variables

$$x_1 = x_{13},\ x_3 = x_{12}^2 x_{23} + x_{12}x_{13}(x_{33} - x_{22}) - x_{13}^2 x_{32}$$

then any solution f can be written as

$$f(x_{ij}) = x_1^{N_2 - N_3} x_3^{-N_2} F,$$

where the new unknown function F satisfies the following system of differential equations:

$$x_{13}\frac{\partial F}{\partial x_{23}} - x_{12}\left(\frac{\partial F}{\partial x_{11}} - \frac{\partial F}{\partial x_{22}}\right) + (x_{11} - x_{22})\frac{\partial F}{\partial x_{21}} - x_{32}\frac{\partial F}{\partial x_{31}} = 0,$$
$$x_{13}\frac{\partial F}{\partial x_{12}} + x_{23}\left(\frac{\partial F}{\partial x_{22}} - \frac{\partial F}{\partial x_{33}}\right) - (x_{22} - x_{33})\frac{\partial F}{\partial x_{32}} - x_{21}\frac{\partial F}{\partial x_{31}} = 0,$$
$$x_{13}\left(\frac{\partial F}{\partial x_{11}} - \frac{\partial F}{\partial x_{33}}\right) + x_{23}\frac{\partial F}{\partial x_{21}} - x_{12}\frac{\partial F}{\partial x_{32}} - (x_{11} - x_{33})\frac{\partial F}{\partial x_{31}} = 0,$$
$$x_{12}\frac{\partial F}{\partial x_{12}} + x_{13}\frac{\partial F}{\partial x_{13}} - x_{21}\frac{\partial F}{\partial x_{21}} - x_{31}\frac{\partial F}{\partial x_{31}} = 0,$$
$$x_{13}\frac{\partial F}{\partial x_{13}} + x_{23}\frac{\partial F}{\partial x_{23}} - x_{32}\frac{\partial F}{\partial x_{32}} - x_{31}\frac{\partial F}{\partial x_{31}} = 0.$$

Solving this system, we obtain that the general solution of the system has the form:

$$f(x_{ij}) = x_1^{N_2 - N_3} x_3^{-N_2} F\left(\frac{x_2}{x_1}, c_1, c_2, c_3\right),$$

where

$$x_2 = x_{1k}x_{k3},\ c_1 = x_{kk},\ c_2 = x_{ik}x_{ki},\ c_3 = x_{ik}x_{kl}x_{li}.$$

(we omit the summation of repeated indices going over 1, 2, 3).

We now encounter the first problem which arises in the process of decomposition. Although we know the general solution of this system of differential equations, an important question is how to obtain enough *polynomial* solutions in variables x_{ij} from this general solution. This can be generally difficult. For example, the expression

$$x_4 = \frac{x_1^3\left(\frac{x_2^3}{x_1^3} - 2c_1\frac{x_2^2}{x_1^2} + \frac{1}{2}(3c_1^2 - c_2)\frac{x_2}{x_1} - \frac{1}{3}(c_1^3 - c_3)\right)}{x_3}$$
$$= x_{12}x_{23}^2 + x_{13}x_{23}(x_{11} - x_{22}) - x_{13}^2 x_{21}$$

although rational in variables x_j, c_j is polynomial in variables x_{ij}!

The second question which arises is, in some sense, the opposite kind of problem. We can generate *too many* polynomial solutions and their polynomial combinations will no longer be linear independent (or, equivalently, the polynomial solutions will be functionally dependent). For example, in our case of algebra $gl(3)$ we have

$$x_3 x_4 = x_2^3 - 2c_1 x_2^2 x_1 + \frac{1}{2}(3c_1^2 - c_2)x_2 x_1^2 - \frac{1}{3}(c_1^3 - c_3)x_1^3.$$

This dependency signals that it is forbidden to have product $x_3 x_4$ in the decomposition.

Luckily, both above-mentioned problems are easily found out by a dimension check. If there are not enough solutions we will not have enough linear independent vectors to construct the decomposition, so that the number of linear independent vectors we generate is less than the dimension of the corresponding filtration part U_n. In the opposite case, if the number is greater, it clearly indicates that there is some linear dependency between the considered vectors.

Summing up these facts, if we now anti-symmetrize x_j, c_j into elements X_j, C_j of the enveloping algebra U, i.e.

$$x_1 \to X_1 = E_{13}$$

$$x_2 \to X_2 = E_{12}E_{23} + E_{13}(E_{11} + E_{33}) - \frac{1}{2}E_{13}$$

etc.

we can construct vectors

$$C_1^{k_1}C_2^{k_2}C_3^{k_3}X_1^{n_1}X_2^{n_2}X_3^{n_3}X_4^{n_4},\ k_j, n_j \geq 0,\ n_3 n_4 = 0$$

and compute their weights. The weight of the product of any elements is obtained by the sum of the weights of the corresponding elements; we have

element	weight
X_1	(1,0,–1)
X_2	(1,0,–1)
X_3	(2,–1,–1)
X_4	(1,1,–2)
$C_{1,2,3}$	(0,0,0)

therefore

$$\text{weight }(C_1^{k_1}C_2^{k_2}C_3^{k_3}X_1^{n_1}X_2^{n_2}X_3^{n_3}X_4^{n_4}) = (n_1 + n_2 + 2n_3 + n_4, n_4 - n_3, -n_1 - n_2 - n_3 - 2n_4).$$

The degree of the general element is

$$n = k_1 + 2k_2 + 3k_3 + n_1 + 2n_2 + 3n_3 + 3n_4.$$

Now we can perform a dimension check to see if our hypothesis is correct: from representation theory, the dimension of the representation with highest weight (N_1, N_2, N_3) is

$$d(N_1, N_2, N_3) = \frac{1}{2}(N_1 - N_2 + 1)(N_1 - N_3 + 2)(N_2 - N_3 + 1),$$

hence the dimension of the representation generated by our general element is

$$\tilde{d}(n_1, n_2, n_3, n_4) = \frac{1}{2}(n_1 + n_2 + 3n_3 + 1)(n_1 + n_2 + 3n_4 + 1)(2n_1 + 2n_2 + 3n_3 + 3n_4 + 2).$$

It follows that we must prove that

$$\sum_{\substack{k_1,k_2,k_3,n_1,n_2,n_3,n_4 \geq 0 \\ n_3 n_3 = 0 \\ k_1 + 2k_2 + 3k_3 + n_1 + 2n_2 + 3n_3 + 3n_4 \leq n}} \tilde{d}(n_1, n_2, n_3, n_4) = \binom{n+9}{n}.$$

The first look leads us of course to check if it is correct for small n: we make a table

n	sum on the left side	$\binom{n+9}{9}$
1	10	10
2	55	55
3	220	220
4	715	715
…	…	

from which we see that it should work. A general proof of the formula can be performed for example using generating functions: because

$$\binom{n+9}{9} - \binom{(n-1)+9}{9} = \binom{n+8}{8}$$

it is sufficient to show

$$\sum_{\substack{k_1,k_2,k_3,n_1,n_2,n_3,n_4 \geq 0 \\ n_3 n_4 = 0 \\ k_1 + 2k_2 + 3k_3 + n_1 + 2n_2 + 3n_3 + 3n_4 = n}} \tilde{d}(n_1, n_2, n_3, n_4) = \binom{n+8}{8}$$

or, equivalently,

$$\sum_{n=0}^{+\infty} \sum_{\substack{k_1,k_2,k_3,n_1,n_2,n_3,n_4 \geq 0 \\ n_3 n_4 = 0 \\ k_1 + 2k_2 + 3k_3 + n_1 + 2n_2 + 3n_3 + 3n_4 = n}} \tilde{d}(n_1, n_2, n_3, n_4) x^n = \sum_{n=0}^{+\infty} \binom{n+8}{8} x^n.$$

This is easily done with the help of the sum of the geometric series, namely

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$$

which can be differentiated several times to get

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{+\infty} n\, x^n,\ \frac{x(x+1)}{(1-x)^3} = \sum_{n=0}^{+\infty} n^2\, x^n, \ldots$$

etc. This way, after expanding the right hand side we get

$$\sum_{n=0}^{+\infty} \binom{n+8}{8} x^n = \frac{1}{(1-x)^9}.$$

To remove the unwanted condition $n_3 n_4 = 0$ we can rewrite the inner sum as

$$\sum_{\substack{k_1,k_2,k_3,n_1,n_2,n_3,n_4 \geq 0 \\ n_3 n_4 = 0 \\ k_1 + 2k_2 + 3k_3 + n_1 + 2n_2 + 3n_3 + 3n_4 = n}} \tilde{d}(n_1, n_2, n_3, n_4) =$$

$$\sum_{\substack{k_1,k_2,k_3,n_1,n_2,n_3,n_4 \geq 0 \\ k_1 + 2k_2 + 3k_3 + n_1 + 2n_2 + 3n_3 + 3n_4 = n}} \tilde{d}(n_1, n_2, n_3, 0) + \sum_{\substack{k_1,k_2,k_3,n_1,n_2,n_3,n_4 \geq 0 \\ k_1 + 2k_2 + 3k_3 + n_1 + 2n_2 + 3n_3 + 3n_4 = n}} \tilde{d}(n_1, n_2, 0, n_4) - \sum_{\substack{k_1,k_2,k_3,n_1,n_2,n_3,n_4 \geq 0 \\ k_1 + 2k_2 + 3k_3 + n_1 + 2n_2 + 3n_3 + 3n_4 = n}} \tilde{d}(n_1, n_2, 0, 0)$$

and sum each of the summands separately. For the first one we have

$$\sum_{n=0}^{+\infty} \sum_{\substack{k_1,k_2,k_3,n_1,n_2,n_3\geq 0\\ k_1+2k_2+3k_3+n_1+2n_2+3n_3=n}} \tilde{d}(n_1,n_2,n_3,0)\, x^n =$$

$$\sum_{k_1,k_2,k_3,n_1,n_2,n_3=0}^{+\infty} \frac{1}{2}(n_1+n_2+1)(n_1+n_2+3n_3+1)(2n_1+2n_2+3n_3+2)\, x^{k_1+2k_2+3k_3+n_1+2n_2+3n_3} = \frac{10x^2+5x+1}{(1-x)^9(x+1)^5}$$

The second term is obviously equal to the first term. For the third sum we have
and sum each of the summands separately. For the first one we have

$$\sum_{n=0}^{+\infty} \sum_{\substack{k_1,k_2,k_3,n_1,n_2\geq 0\\ k_1+2k_2+3k_3+n_1+2n_2=n}} \tilde{d}(n_1,n_2,0,0)\, x^n =$$

$$\sum_{k_1,k_2,k_3,n_1,n_2,n_3=0}^{+\infty} \frac{1}{2}(n_1+n_2+1)^2(2n_1+2n_2+2)\, x^{k_1+2k_2+3k_3+n_1+2n_2} = \frac{x^4+6x^3+16x^2+6x+1}{(x-1)^8(x+1)^5}.$$

Now

$$2\frac{10x^2+5x+1}{(1-x)^9(x+1)^5} - \frac{x^4+6x^3+16x^2+6x+1}{(x-1)^8(x+1)^5} = \frac{1}{(1-x)^9}$$

and the dimension check succeeds.

5 Lie algebra *gl*(4)

Finally let us demonstrate how the procedure goes in the case of Lie algebra *gl*(4). For the vectors with highest weights (N_1, N_2, N_3, N_4) we obtain the following equations

$$E_{11} \to x_{12}\frac{\partial f}{\partial x_{12}} + x_{13}\frac{\partial f}{\partial x_{13}} + x_{14}\frac{\partial f}{\partial x_{14}} - x_{21}\frac{\partial f}{\partial x_{21}} - x_{31}\frac{\partial f}{\partial x_{31}} - x_{41}\frac{\partial f}{\partial x_{41}} = N_1 f,$$

$$E_{22} \to x_{21}\frac{\partial f}{\partial x_{21}} + x_{23}\frac{\partial f}{\partial x_{23}} + x_{24}\frac{\partial f}{\partial x_{24}} - x_{12}\frac{\partial f}{\partial x_{12}} - x_{32}\frac{\partial f}{\partial x_{32}} - x_{42}\frac{\partial f}{\partial x_{42}} = N_2 f,$$

$$E_{33} \to x_{31}\frac{\partial f}{\partial x_{31}} + x_{32}\frac{\partial f}{\partial x_{32}} + x_{34}\frac{\partial f}{\partial x_{34}} - x_{13}\frac{\partial f}{\partial x_{13}} - x_{23}\frac{\partial f}{\partial x_{23}} - x_{43}\frac{\partial f}{\partial x_{43}} = N_3 f,$$

$$E_{44} \to x_{41}\frac{\partial f}{\partial x_{41}} + x_{42}\frac{\partial f}{\partial x_{42}} + x_{43}\frac{\partial f}{\partial x_{43}} - x_{14}\frac{\partial f}{\partial x_{14}} - x_{24}\frac{\partial f}{\partial x_{24}} - x_{34}\frac{\partial f}{\partial x_{34}} = N_4 f,$$

$$E_{12} \to (x_{11}-x_{22})\frac{\partial f}{\partial x_{21}} + x_{12}\frac{\partial f}{\partial x_{22}} + x_{13}\frac{\partial f}{\partial x_{23}} + x_{14}\frac{\partial f}{\partial x_{24}} - x_{12}\frac{\partial f}{\partial x_{11}} - x_{32}\frac{\partial f}{\partial x_{31}} - x_{42}\frac{\partial f}{\partial x_{41}} = 0,$$

$$E_{23} \to x_{21}\frac{\partial f}{\partial x_{31}} + (x_{22}-x_{33})\frac{\partial f}{\partial x_{32}} + x_{23}\frac{\partial f}{\partial x_{33}} + x_{24}\frac{\partial f}{\partial x_{34}} - x_{13}\frac{\partial f}{\partial x_{12}} - x_{23}\frac{\partial f}{\partial x_{22}} - x_{43}\frac{\partial f}{\partial x_{42}} = 0,$$

$$E_{34} \to x_{31}\frac{\partial f}{\partial x_{41}} + x_{32}\frac{\partial f}{\partial x_{42}} + (x_{33}-x_{44})\frac{\partial f}{\partial x_{43}} + x_{34}\frac{\partial f}{\partial x_{44}} - x_{14}\frac{\partial f}{\partial x_{13}} - x_{24}\frac{\partial f}{\partial x_{23}} - x_{34}\frac{\partial f}{\partial x_{33}} = 0,$$

$$E_{13} \to (x_{11}-x_{33})\frac{\partial f}{\partial x_{31}} + x_{12}\frac{\partial f}{\partial x_{32}} + x_{13}\frac{\partial f}{\partial x_{33}} + x_{14}\frac{\partial f}{\partial x_{34}} - x_{13}\frac{\partial f}{\partial x_{11}} - x_{23}\frac{\partial f}{\partial x_{21}} - x_{43}\frac{\partial f}{\partial x_{41}} = 0,$$

$$E_{24} \to x_{21}\frac{\partial f}{\partial x_{41}} + (x_{22}-x_{44})\frac{\partial f}{\partial x_{42}} + x_{23}\frac{\partial f}{\partial x_{43}} + x_{24}\frac{\partial f}{\partial x_{44}} - x_{14}\frac{\partial f}{\partial x_{12}} - x_{24}\frac{\partial f}{\partial x_{22}} - x_{34}\frac{\partial f}{\partial x_{32}} = 0,$$

$$E_{14} \to (x_{11}-x_{44})\frac{\partial f}{\partial x_{41}} + x_{12}\frac{\partial f}{\partial x_{42}} + x_{13}\frac{\partial f}{\partial x_{43}} + x_{14}\frac{\partial f}{\partial x_{44}} - x_{14}\frac{\partial f}{\partial x_{11}} - x_{24}\frac{\partial f}{\partial x_{21}} - x_{34}\frac{\partial f}{\partial x_{31}} = 0.$$

Now let us denote

$$e_{ik}^{(1)} = x_{ik},\ e_{ik}^{(2)} = x_{in}\, x_{nk},\ e_{ik}^{(3)} = x_{in}\, x_{nm}\, x_{mk}.$$

Using these elements we define the following solutions of the system (summation of repeated indices going over 1… 4):

$$c_1 = e_{kk}^{(1)},\ c_2 = e_{kk}^{(2)},\ c_3 = e_{kk}^{(3)},\ c_4 = e_{kl}^{(3)} E_{lk}^{(1)},$$

$$y_1 = \det\begin{pmatrix} e_{13}^{(1)} & e_{14}^{(1)} \\ e_{23}^{(1)} & e_{24}^{(1)} \end{pmatrix}, \qquad y_2 = \det\begin{pmatrix} e_{13}^{(2)} & e_{14}^{(2)} \\ e_{23}^{(2)} & e_{24}^{(2)} \end{pmatrix}, \qquad y_3 = \det\begin{pmatrix} e_{13}^{(3)} & e_{14}^{(3)} \\ e_{23}^{(3)} & e_{24}^{(3)} \end{pmatrix},$$

$$z_1^+ = \det\begin{pmatrix} e_{13}^{(1)} & e_{14}^{(1)} \\ e_{13}^{(2)} & e_{14}^{(2)} \end{pmatrix}, \qquad z_2^+ = \det\begin{pmatrix} e_{13}^{(1)} & e_{14}^{(1)} \\ e_{13}^{(3)} & e_{14}^{(3)} \end{pmatrix}, \qquad z_3^+ = \det\begin{pmatrix} e_{13}^{(2)} & e_{14}^{(2)} \\ e_{13}^{(3)} & e_{14}^{(3)} \end{pmatrix},$$

$$z_1^- = \det\begin{pmatrix} e_{14}^{(2)} & e_{24}^{(2)} \\ e_{14}^{(1)} & e_{24}^{(1)} \end{pmatrix}, \qquad z_2^- = \det\begin{pmatrix} e_{14}^{(3)} & e_{24}^{(3)} \\ e_{14}^{(1)} & e_{24}^{(1)} \end{pmatrix}, \qquad z_3^- = \det\begin{pmatrix} e_{14}^{(3)} & e_{24}^{(3)} \\ e_{14}^{(2)} & e_{24}^{(2)} \end{pmatrix},$$

$$w_+ = \det\begin{pmatrix} e_{12}^{(1)} & e_{13}^{(1)} & e_{14}^{(1)} \\ e_{12}^{(2)} & e_{13}^{(2)} & e_{14}^{(2)} \\ e_{12}^{(3)} & e_{13}^{(3)} & e_{14}^{(3)} \end{pmatrix}, \qquad w_- = \det\begin{pmatrix} e_{14}^{(3)} & e_{24}^{(3)} & e_{34}^{(3)} \\ e_{14}^{(2)} & e_{24}^{(2)} & e_{34}^{(2)} \\ e_{14}^{(1)} & e_{24}^{(1)} & e_{34}^{(1)} \end{pmatrix},$$

$$t = 2e_{24}^{(2)}(x_{13}e_{14}^{(3)} - x_{14}e_{13}^{(3)}) - e_{14}^{(2)}(x_{23}e_{14}^{(3)} + x_{13}e_{24}^{(3)} - x_{14}e_{23}^{(3)} - x_{24}e_{13}^{(3)}).$$

The highest weights of the corresponding elements of the enveloping algebra are

element	weight
C_k	(0,0,0,0)
X_k	(1,0,0,–1)
Y_k	(1,1,–1,–1)
Z_k^+	(2,0,–1,–1)
Z_k^-	(1,1,0,–2)
W_+	(3,–1,–1,–1)
W_-	(1,1,1,–3)
T	(2,1,–1,–2)

The dimension of the representation with highest weight (N_1, N_2, N_3, N_4) is

$$d = \frac{1}{12}(N_1 - N_2 + 1)(N_1 - N_3 + 2)(N_1 - N_4 + 3) \times (N_2 - N_3 + 1)(N_2 - N_4 + 2)(N_3 - N_4 + 1).$$

Now we start as in the case of $gl(3)$ to count the dimensions of the filtration subspaces to see the conditions which arise as the degree of the elements grows.

Elements of degree 1 are

element	dimension
C_1	1
X_1	15
total = 16	

Elements of degree 2 are

element	dimension
C_1^2	1
C_1X_1	15
X_1^2	84
C_2	1
X_2	15
Y_1	20
total = 136	

… etc. Analyzing degree 6, we encounter the first linear dependency which signals some polynomial relations (syzygies) between the generators. Namely we have

$$Z_1^{(+)}Z_1^{(-)} = X_1^2Y_2 + X_2^2Y_1 - C_1X_1X_2Y_1,$$
$$X_1Z_3^{(+)} - X_2Z_2^{(+)} + X_3Z_1^{(+)} = 0$$
$$X_1Z_3^{(-)} - X_2Z_2^{(-)} + X_3Z_1^{(-)} = 0$$
$$8Y_3 + 4(C_2 - C_1^2)Y_2 + (C_1^4 - 2C_1^2C_2 + 2C_4 - C_2^2)Y_1 = 0.$$

Higher degrees give us many other relations of this type which lead to the system of conditions that must be fulfilled to keep the set of highest weight vectors linear independent. Going up to degree 12 finishes the analysis, and we no longer encounter new relations. Thus we are ready to formulate the hypothesis and after a successful dimension check we can state the final result, which is contained in the following theorem:

Theorem

The set of elements

$$C_2^{k_2}C_3^{k_3}C_4^{k_4}X_1^{n_1}X_2^{n_2}X_3^{n_3}Y_1^{m_1}Y_2^{m_2}T^p \times (Z_1^{(+)})^{r_1}(Z_2^{(+)})^{r_2}(Z_3^{(+)})^{r_3}(Z_1^{(-)})^{s_1}(Z_2^{(-)})^{s_2}(Z_3^{(-)})^{s_3}W_+^{t_1}W_-^{t_2}$$

with conditions

$$r_1n_3 = r_1s_1 = r_1s_2 = r_1s_3 = r_1p = r_1t_2 = s_1n_3 = s_1r_2 = s_1r_3 = s_1p = 0$$
$$r_2s_2 = r_2s_3 = r_2t_2 = r_2p = s_2r_3 = s_2t_1 = r_3s_3 = s_2p = s_1t_1 = 0,$$
$$r_3t_2 = r_3p = s_3t_1 = s_3p = t_1p = t_2p = t_1t_2 = 0,$$
$$r_2 = 0,1, \quad s_2 = 0,1, \quad r_3 = 0,1, \quad s_3 = 0,1, \quad p = 0,1$$

form a desired decomposition of the enveloping algebra $U(gl(4))$.

6 Conclusion

We have studied the structure of enveloping algebras $U(g)$ where $g = sl(2), gl(3), gl(4)$ and we have decomposed the adjoint representation into its irreducible components. We have found explicitly the highest weight vector in any such component. Our result can be useful for a further study of the tensor products of representations and ideals of enveloping algebras. The method can also be used for other simple Lie algebras (see [4]).

Acknowledgment

The authors are grateful to M. Havlíček for valuable and useful discussions. This work was partially supported by GAČR 201/05/0857 and by research plan MSM6840770039.

References

[1] Dixmier, J.: *Algebres Enveloppantes*. Gauthier-Villars Editeur, Paris 1974.

[2] Kirillov, A.: *Elementy Teorii Predstavlenij*. Nauka, Moscow 1972.

[3] Flath, D.: Decomposition of the Enveloping Algebra of sl_3. *J. Math. Phys*. Vol. **31** (1990), No. 5, p. 1076–1077.

[4] Burdík, Č., Navrátil, O.: Decomposition of the Enveloping Algebra $so(5)$, submitted to *J. Gener. Lie Theory and App*.

Prof. RNDr. Čestmír Burdík, DrSc.
phone:+420 224 358 549
e-mail: burdik@kmlinux.fjfi.cvut.cz

Ing. Severin Pošta, Ph.D.
phone: +420 224 358 562
e-mail: severin@km1.fjfi.cvut.cz

Department of Mathematics

Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering
Trojanova 13
120 00 Prague 2, Czech Republic

Doc. RNDr. Ondřej Navrátil, Ph.D.
phone: +420 224 890 714
e-mail: navratil@fd.cvut.cz

Department of Mathematics

Czech Technical University in Prague
Faculty of Transportation Sciences
Na Florenci 25
110 00 Prague 1, Czech Republic