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The Tangent Medial Circles Inside the Region Defined by Hermite Curve Tangent to Unit Circle

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Abstract

The design of curves, surfaces, and solids are important in computer aided geometric design (CAGD). Images, surround by boundary curves, are also investigated by many researchers. One way to describe an image is using the medial axis transform. Under this consideration, the properties of the boundary curves tangent to circles become important to design for 2D images. In this paper, we want to find the medial axis transform (MAT) for a special class of region, which is bounded by unit circle and the curves whose end points is on a the circle, and endpoints tangent vectors are parallel to the tangent of circle at the end points. During the process, we want find the medial circle tangent to other medial circle, until we reach the medial circle whose center is the center of the osculating circle for the point with local maximum curvature. There are 4 cases, symmetric/non-symmetric region with singular point/local maximum curvature point, and proposed algorithm for these 4 cases. We introduced algorithm for this 4 cases in this paper.

Keywords: medial axis transform, hermite curve, computer geometric modeling

1. Introduction

The 2D image can be stored by many different ways, including its boundary curves, a medial axis curves with radius function, union of many primitive figures, and so on. There are many researchers use different curves to simulate different images. For example, Cinque, Levialdi and Malizia [1] uses cubic Bezier curve to do the shape description. Yang, Lu and Lee[2] use Bezier curve to approach the shape description for Chinese calligraphy characters. Chang and Yan[3] derived an algorithm to approach the hand-drawn image by using cubic Bezier curve. Cao and Kot[4] derived an algorithm to do data embedding in electronic inks without losing data. The boundary curves can be also used to derived the offset curves and medial axis of an images[5], and also to simulate the nature objects, such as flowers[6]. In this paper, we would like to investigate the cubic Hermite curves, where the end points are on a unit circle, and its end points tangent line is parallel to the tangent line of circles. Using these properties, with more constraints on the singular point or maximum curvature at specified parameter, we survey the couture of the curves, so that the design of the curve may be simplify by giving constraint.

2. Definition and Theorem

Let introduce the Hermite curve first. Given two points P_0 and P_1 , with two associated tangent vectors   and V_1 , the Hermite curve C(t) defined as (See Fig. 1):

$$C(t) = [t^3, t^2, t, 1]\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ V_0 \\ V_1 \end{bmatrix}$$

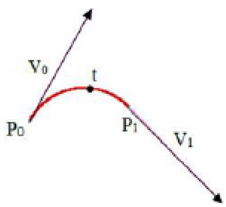

Fig. 1 Hermite Curve

To simplify our problem, we assume that the end points P_0 and P_1 are on the unit circle, the first point P_0 =(-1,0), and the second point P_1 =(-cosθ , sinθ). It's associated tangent vectors parallel to (0,1) and (sinθ , cosθ). We have the following definition for this curve.

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Definition 1: Two points P_0 =(-1, 0), P_1 =(-cosθ , sinθ) on the unit circle with its associated tangent vector $V_0=\alpha_0(0,1)$, $V_1=\alpha 1(\sin\theta , \cos\theta)$ produced a Hermite curve, we call it unit circle Hermite curve, denoted H1(t;θ,α_0,α_1), where $\alpha_0>0$, $\alpha_1>0$, 0<t<1, 0<θ<2π.

Notice that for the general case, we can always convert the design problem into the design problem for $H_1(t;\theta,\alpha_0,\alpha_1)$. We can always translate the center of current medial axis circle to the origin, scale the radius into one and rotate the circle so that the first boundary point is on (-1,0). We call this process the standardization of the problem. After we solve the problem, produce the curve we want, we can always inverse the rotate, scale, and translate process, to see the designed curves for the original design problem.

In [7], we have the following two theorems:

Theorm 1: Let $C(t)=H_1(t;\theta,\alpha_0,\alpha_1)$, then $C'(t)=(x'(t),y'(t)) =(0, 0) \leftrightarrow \alpha_0B(t)=\alpha_1D(t) =A(t) \tan(\frac{\theta}{2})$, where A(t) =6t(t-1), B(t)=(3t-1)(t-1), D(t)=t(3t-2).

We would like to find the MA and MAT for the region bounded by the Hermite curve with the circular arc from (-cosθ, sinθ) to the point (-1,0) rotate about the origin clockwise, we call this region R(t;θ,α).

Theorem 2: Let the Hermite curve $C(t)=(x(t),y(t))= H_1(t;\theta,\alpha_0,\alpha_1)$, if the local maximum curvature is at t_0, then $(x'(t_0)y'''(t_0)-x'''(t_0)y'(t_0))(x'(t_0)2+y'(t_0)2)-3(x'(t_0)y''(t_0)-x''(t_0)y'(t_0))(x'(t_0)x''(t_0)+y'(t_0)y''(t_0))=0$.

Notice that this equation has degree 5 in t, which means there are 5 local maximum/minimum curvatures on the curve.

The MA is very sensitive to its boundary, so many researcher use approximated MA instead. In pattern recognition, the important information for the region is easier to capture by using approximated MA. Although it could have more than one local maximum curvature in our region, which means more than one end point of the MAT in the region, we would like to find one endpoint in the region, so that the MA of the region has only two end points (one is on the singular point, or the center of the osculating circle associated with the boundary curve with local maximum curvature, the other one is on the origin). In this paper, we use MA to represent the approximated MA.

We start simple case here. Let's start from the case $\alpha_0=\alpha_1=\alpha$. Let's also assume that all points on the curve is outside the unit circle and the curve has no self-intersection. The MA of a region is easy to find if we know the region is symmetric to a line, so we have the following theorem:

Theorem 3: The curve $C(t)=H_1(t;\theta,\alpha,\alpha)$ is symmetric to the line $y= (-\tan\frac{\theta}{2})x$. Furthermore, C(t) and C(1-t) are symmetric points.

Proof: Let $T = -\tan\frac{\theta}{2}$, we have $\frac{C(t)+C(1-t)}{2}=\frac{\alpha Tt(t-1)-1}{1+T^2}$. Obviously, this point is on the line y=Tx.

Notice that this line y=Tx passing through C(1/2). Notice further that because C(t) and C(1-t) are symmetric points, so the point C(1/2) is a singular point, or C(t) is a vertex of the curve, which means C(1/2) is a singular point, or has constant curvature, or has local maximum/minimum curvature.

With the symmetric property, we know the MA of the region is on the line $(-\tan\frac{\theta}{2})x$. We would like to find the MA point from its footpoint C(t) or C(1-t), we have the following theorem:

Theorem 4: The medial axis (xm, ym) associated with C(t)=(x(t),y(t)), where 0<t<1/2 is (to simplify the representation of the equation, we eliminate the parameter t for x, y, x′, y′, x_m, y_m and r_m):

$$(x_m, y_m) = \frac{(x,y)\cdot(x',y')}{(1,T)\cdot(x',y')}(1,T) \text{ and } r_m = \left|\frac{(1,T)\cdot(-y,x)}{(1,T)\cdot(x',y')}\right| \|(x',y')\|$$

Where · is the inner product of two vectors, |a| is the absolute value of a, and ||(x,y)|| is the norm of the vector, $T=-\tan\frac{\theta}{2}$.

With the MA point and its distance to its footpoint, we find MAT of the region. We also call MAT the medial axis circles in this paper. Notice that $(x_m(0), y_m(0)) = (x_m(1), y_m(1))=(0,0)$.

3. Algorithm and Experimental Result

In this paper, we want find the medial axis circles inside the region, and all medial axis circles tangent with its neighbors. In our assumption, we can always find one end points of the MAT. In some case, there is singular point exist, the singular point is the end point of the MAT branch (the other one is on the origin). On the other case, we can always find a point with local maximum curvature; we use the center of its osculating circle as the end point of the MAT branch.

How to find normal point of the MAT? Consider the end MA point we find, either singular point or points with local maximum curvature. Assume it has parameter t_{end}, $0<t_{end}<1$, $C(t)=(x(t),y(t))$ and we want find t_1 and t_2, $0<t_0<t_{end}$ and $t_{end}<t1<1$, so that $C(t_1)$ and $C(t_2)$ are footpoints of a MA point (x,y), from the property of MA point, we have:

$$(x-x(t_0),y-y(t_0)) \bullet (x'(t_0),y'(t_0))=0$$
$$(x-x(t_1),y-y(t_1)) \bullet (x'(t_1),y'(t_1))=0$$
$$((x-x(t_0))^2+(y-y(t_0))^2=(x-x(t_1))^2+(y-y(t_1))^2$$

The first two equations indicates that the line through MA point and its footpoints perpendicular to tangent of the footpoint on the curve. The third equation indicates that the distances from MA point to its two footpoints are equal.

Now we have 3 equations, and 4 variables, which are x, y, t_0 and t_1. Given t_0, we can find the MA point (x, y) and t_1. The idea to solve the system of equation is:

Algorithm 1: Finding one MA point.

1. Solve (x,y) by Cramer's rule from the first 2 equations, and x, y has two variables t_0 and t_1.
2. Substitute the value x, y into the third equations, and leave one equation with 2 variables t_0 and t_1.
3. Given t_0, we have one equation with one variable t_1, so we can find solutions for t_1. With careful selection of t_1, we find the MA points with its two footpoints.

The above idea solves the MAT (MA with radius function) for one specified point. As long as we know the parameter for the end point, we can always find all MAT for parameter from 0 to t_{end}.

We would like to find not only one MA point, but also its MA circle tangent to current MA circles. We proposed two methods to find next MA circle. One method solve the problem algebraically, and the other method using an algorithm with binary search to approach the result. The first method only suitable for the symmetric case, which means the case with $\alpha_0=\alpha_1$.

On the case that $\alpha_0=\alpha_1=\alpha$, we want find the next circle tangent to the first circle, which centered at (0,0) with radius 1 in the very beginning. From theorem 4, we know the curve whose parameter t associate a MAT circle (x_m, y_m, r_m), we assume the next circle whose associated boundary parameter is t, so we want to find the circle tangent to the unit circle centered at the origin. The constraints we give is the distance between the centers of two circle is equal to the sum of the radius of two circles. That is, $\|(x_m, y_m)\| = r_m + 1$. To find the medial axis circle (x_m, y_m, r_m) associated with $C(t)=(x(t),y(t))$, where $0<t<1/2$, tangents to the unit circle centered at the origin, we have to solve a one parameter equation with degree 10. Because the solutions for this equation are sensitive to its coefficient, we did not implement this method. We use the binary search strategy instead.

Algorithm 2: Finding the next MA point (eps is a small number):

Current MA circle is (x,y,r) with parameter t. The return value is next MA circle (x_1, y_1, r_1) and t_1

1. Set the t_{min}=t, the current parameter value, and t_{max} is the parameter value for the end MA point.
2. Let dist (distance between two centers) be 2, and srad (the sum of the radius of two circles) be 1.
3. While |dist – srad| > eps and $|t_{max}-t_{min}|$ < eps :

   tt = $(t_{min}+t_{max})/2$

   Find the MAT (x_1, y_1, r_1) for the parameter tt

   # (from Algorithm 1).

   Find dist, srad from these two circles.

   If dist - srad > eps, then t_{max} = tt

   If dist - srad < -eps, then t_{min} = tt
4. The value x_1, y_1, r_1 is the MA circle we find.

From the above algorithm, we can find the next MA circles for the parameter t. However, when the parameter t value close to the parameter associate with the local maximum curvature, it can only produce circles intersect with previous circle, which means it cannot find more circle tangent to current circle, and "inside" the Hermite curve. From the theorem describes in the last section, with the idea mentioned above, we give 4 examples in this section. The first two examples is for the symmetric boundary curve where $\alpha_0=\alpha_1=\alpha$, for boundary curve with singular point and with local maximum curvature respectively. The third and fourth cases are for the non-symmetric boundary curve, also for boundary curve with singular point and with local maximum curvature respectively.

Example 1: (Symmetric curve with singular point): Given $\theta=5/9$, $t=0.5$, we can find $\alpha_0=\alpha_1=7.15025$. The associated MAT is shown in Fig. 2.

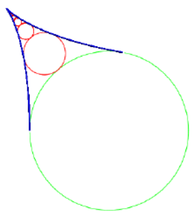

Fig. 2 Symmetric region with singular point

Example 2: (Symmetric Curve with local maximum curvature): Given $\theta=5/6$, $\alpha_0=\alpha_1=8$, the associated MAT is shown in Fig. 3.

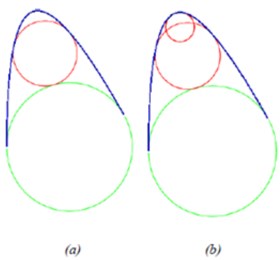

Fig. 3 Symm. region with localmaximum curvature

In Fig. 3(a), we find one MA circle tangent to the original one. We can find the next circle tangent to the red curve and the Hermite cuve, however, this circle is not totally “inside” the Hermit curve. On this case, we draw the osculating circle associated with the Hermite curve at local maximum curvature (See Figure 3(b)).

Notice that the above two examples has the properties that the MA points are on one straight line.The design requirements and design constraints are summarized based on the characteristics of the mechanism.

Example 3: (Non-symmetric Curve with singular point): Given $\theta=5/6$, $t=0.4$, we find $\alpha_0=44.78461$, $\alpha_1=16.79423$. The associated MAT is shown in Fig. 4.

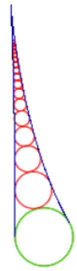

Fig. 4 Non-Symm. region with Singular point

Example 4: (Non-symmetric with local maximum curvature): Given $\theta=7/6$, $\alpha_0=18$, $\alpha_1=6$, the associated MAT is shown in Fig. 5.

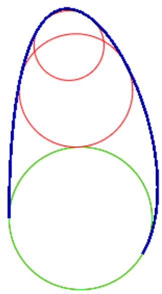

Fig. 5 Non-Symm. region with maximum curvature

In Fig. 5, the same as Fig 3, the last circle is associated with the Hermite curve at local maximum curvature.

4. Conclusions

The unit circle Hermite curve, $H_1(t;\theta,\alpha_0,\alpha_1)$, can be used to design curves and images. On many cases, the singular point may be needed to be considered. For example, the Chinese characters may have cusp on the boundary. Before the design of the character, the relationship between the parameter value and the contour of the boundary curve is important. When we give two values of the parameters for $H_1(t;\theta,\alpha_0,\alpha_1)$ curve, the can find the other two values with simple computation. When we define a curve, the only memory we need is 7 locations to store these 4 parameters, and other 3 parameters for the standardization process. So, the design process saved not only the time, but also the memory. When we find the MA circles inside the region bounded by unit circle and $H_1(t;\theta,\alpha_0,\alpha_1)$, it is possible that the last circle intersects the osculating circle at the local maximum curvature. The osculating circle associated with end MAT point of the region, so it is better to show this circle, so that the MAT of the region also contains the end MAT point. There are more constraints we can used to design the curves. For example, the maximum curvature happened at t_0, the curve passing through a point p_0 , the curve tangent to a line L_0 , and so on. We believe the result will as simple as the case we introduced here. We can also consider the inverse process, that is, from circles tangents to each other, and find its boundary Hermite curve. We will leave this for further research.

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