Linear models of dissipation whose Q is almost frequency independent (*)

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Ricevuto il 31 Maggio 1966

Summary. — Laboratory experiments and field observations indicate that the Q of many non ferromagnetic inorganic solids is almost frequency independent in the range 10^7 *to* 10^{-2} *cps;* although no single substance has been investigated over the entire frequency spectrum. One of the purposes of this investigation is to find the analytic expression of a linear dissipative mechanism whose Q is almost frequency independent over large frequency ranges. This will be obtained by introducing fractional derivatives in the stress strain relation.

Since the aim of this research is to also contribute to elucidating the dissipating mechanism in the earth free modes, we shall treat the cases of dissipation in the free purely torsional modes of a shell and the purely radial vibration of a solid sphere.

The theory is checked with the new values determined for the Q of the spheroidal free modes of the earth in the range between 10 and 5 minutes integrated with the Q of the Railegh waves in the range between 5 and 0.6 minutes.

Another check of the theory is made with the experimental values of the Q of the longitudinal waves in an alluminum rod, in the range between 10^{-5} and 10^{-3} seconds.

In both cheks the theory represents the observed phenomena very satisfactorly.

Riassunto. — I risultati delle ricerche di laboratorio e delle osservazioni in fenomeni naturali indicano che il Q di parecchi solidi inorganici non ferromagnetici è indipendente dalle frequenze nell'intervallo 10^{-2}, 10^7 cicli al secondo; per quanto nessuna sostanza sia stata studiata in tutto

(*) This paper was presented at the 1966 annual meeting of AGU in Washington DC.

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questo intervallo di frequenze. Uno degli scopi della presente ricerca è quello di trovare l'espressione analitica di un modello di dissipazione lineare in cui Q sia indipendente dalla frequenza in un vasto intervallo di frequenze. Questo sarà ottenuto introducendo derivate di ordine frazionario nelle relazioni fra sforzo e deformazione.

Poiché uno degli scopi di questa ricerca è anche di contribuire ad una miglior comprensione dei meccanismi di dissipazione dell'energia nelle oscillazioni libere della Terra, in questa nota si applicherà la legge di dissipazione citata al caso delle oscillazioni torsionali libere di uno strato sferico.

La teoria esposta viene poi applicata allo studio dei valori di Q osservati nelle onde di Rayleigh e nelle oscillazioni sferoidali della Terra.

Un'altra applicazione della teoria è fatta allo studio dei valori di Q osservati nelle onde longitudinali di una sbarra di alluminio.

In entrambe le applicazioni la teoria rappresenta in maniera soddisfacente i fenomeni osservati.

INTRODUCTION

In a homogeneous isotropic elastic field the elastic properties of the substance are specified by a description of the strain and stre sses in a limited portion of the field since the strains and stresses are linearly related by two parameters which describe the elastic properties of the field. If the elastic field is not homogeneous nor isotropic the properties of the field are specified in a similar manner by a larger number of parameters which also depend on the position.

These perfectly elastic fields are insufficient models for the description of many physical phenomena because they do not allow to explain the dissipation of energy. A more complete description of the actual elastic fields is obtained by introducing stress-strain relations which include also linear combinations of time derivatives of the strain and the stress. The numerical coefficients appearing in the general linear combinations of higher order derivatives are called visco-elastic constants of higher order.

Elastic fields described by elastic constants of higher order have been discussed by many authors, [e.g. see Knopoff, 1954; Caputo, 1966]. Knopoff studied the case in which the stress strain relations are of the type

$$\tau_{rs} = \lambda g^{ki} g_{rs} e_{ki} + 2\mu e_{rs} + \frac{d^m}{d t^m} (\lambda_m g^{ki} g_{rs} e_{ki} + 2\mu_m e_{rs}) \qquad [1]$$

where λ_m and μ_m are constant, he obtained a condition for these visco-elastic constants of higher order analogous to those existing for the perfectly elastic fields and also proved that in order to have a dissipative elastic field the stress-strain relations should contain time derivatives of odd order.

A generalization of the relation [1] is

$$\tau_{rs} = \sum_{o}^{p}{}_{m} \frac{d^m}{d\,t^m} [\lambda_m\, g^{hi}\, g_{rs}\, e_{hi} + 2\mu_m\, e_{rs}] \qquad [2]$$

where one can also consider λ_m and μ_m functions of position.

We can generalize [2] to the case when the operation $\frac{d^m}{d\,t^m}$ is performed with m as a real number z (see appendix) and also further by substituting the summation with an integral as follows

$$\tau_{rs} = \int_{a_1}^{b_1} f_1\,(r, z)\,\frac{d^z}{d\,t^z}\, g^{hi}\, g_{rs}\, e_{hi}\, dz + 2 \int_{a_2}^{b_2} f_2\,(r, z)\,\frac{d^z}{d\,t^z}\, e_{rs}\, dz\,. \qquad [3]$$

μ is the radial coordinate in a spherical coordinate system.

Relations [1] and [2] are a special case of [3] — they are obtained by setting

$$f_1\,(r, z) = \sum_{q}^{p}{}_{m}\, \delta\,(z - m)\,\lambda_m$$
$$f_2\,(r, z) = \sum_{q}^{p}{}_{m}\, \delta\,(z - m)\,\mu_m \qquad [4]$$

where $\delta(z - m)$ are unitary delta functions.

If $a_i = q = p = 0$, then we have the case of a perfectly elastic field: if $q = p = 1$ then we have a perfectly viscous field; if $q = a_i = 0$ and $p = 1$ then we have a viscoelastic field.

Dissipation in a plane wave

In the simple case of a plane wave, assuming $f = \eta\delta(z - z_o)$, the stress-strain relation [3] gives the following equation of motion

$$\varrho\,\frac{\partial^2 u}{\partial\, t^2} + \mu\frac{\partial^2 u}{\partial\, x^2} + \eta\,\frac{\partial^{z_o}}{\partial\, t^{z_o}}\,\frac{\partial^2 u}{\partial\, x^2} = 0\;. \qquad [5]$$

By taking the Laplace transform of [5] we have

$$\varrho\, p^2\, U + \mu\,\frac{\partial^2 U}{\partial\, x^2} + \eta\, p^{z_o}\frac{\partial^2 U}{\partial\, x^2} = 0$$

and the nature of the motion depends on the roots of the following equation

$$\eta\, a^2\, p^{z_0} + \varrho\, p^2 + \mu\, a^2 = 0\ . \qquad [6]$$

The approximate solution of [6], neglecting the term in η, which we assume to be small with respect to μ, is

$$p_{\cdot}^2 = -\frac{\mu\, a^2}{\varrho}$$

the solution which takes into account the dissipation is

$$p = i\,|\,p_o\,|\; 1 + \left\{ \frac{\eta\,|p_o|^{z_o}\, a^2}{2\,|p_o|^2\,\varrho} \left(\cos\frac{\pi}{2}\,z_0 + i\,\sin\frac{\pi}{2}\,z_o\right)\right\} \qquad [7]$$

and the specific dissipation is

$$Q^{-1} = \frac{\eta\,|p_o|^{z_o}}{\mu}\,\sin\frac{\pi}{2}\,z_o\,. \qquad [8]$$

Solution of the equations of motion in spherical coordinates

We shall follow the method described in Caputo [1966]; the operator o_i introduced in that paper is

$$o_i = \nu_{ij}\,(r)\,\frac{\partial^j}{\partial\, t^j} \qquad [9]$$

here, according to the definition [3] of the stress-strain relation, these operators will be

$$o_i = \int_{a_1}^{b_1} f_1\,(r, z)\,\frac{\partial^z}{\partial\, t^z}\,d\,z + 2\int_{a_1}^{b_1} f_2\,(r\,z)\,\frac{\partial^z}{\partial\, t^z}\,d\,z$$

$$o_2 = \int_{a_1}^{b_1} f_2\,(r, z)\,\frac{\partial^z}{\partial\, t^z}\,d\,z\,. \qquad [10]$$

One can see that the method of solving the equations of equilibrium resulting from the definition [1] of the stress-strain relation (see Caputo 1966) can be applied also to the case when the estress-strain relation is [3].

The Laplace transform $S\,(S_r,\ S_\vartheta,\ S_\varphi)$ of the displacement vector $s(s_r, s_\vartheta, s_\varphi)$ — where r, ϑ, φ are spherical coordinates, ϑ colatitude, φ longitude — is:

$$\begin{aligned} S_r &= \sum_n \sum_k R_1\ Y_n^k \\ S_\vartheta &= \sum_n \sum_k \left(R_2 \frac{\partial Y_n^k}{\partial \vartheta} + R_3 \frac{1}{\sin\vartheta} \frac{\partial Y_n^k}{\partial \varphi}\right) \\ S_\varphi &= \sum_n \sum_k \left(R_2 \frac{1}{\sin\vartheta} \frac{\partial Y_n^k}{\partial \varphi} - R_3 \frac{\partial Y_n^k}{\partial \vartheta}\right) \end{aligned} \qquad [11]$$

where Y_n^k are spherical harmonics

$$Y_n^k = \begin{cases} \left(\dfrac{2n+1}{4\pi}\right)^{1/2} P_n(\cos\vartheta) & \text{if } k=0 \\ \left(\dfrac{2n+1}{2\pi}\ \dfrac{(n-K)!}{(n+K)}\right)^{1/2} P_n^{(k)}(\cos\vartheta)\cos K\varphi & \text{if } k=1,2,\ldots,n \\ \left(\dfrac{2n+1}{2\pi}\ \dfrac{(2n-K)!}{K!}\right)^{1/2} P_n^{(k-n)}(\cos\vartheta)\cos(k-n)\varphi, & \text{if } k=n+1,\ldots,2n \end{cases} \qquad [12]$$

and $R_{1,n},\ R_{2,n},\ R_{3,n}$ are solutions of the system

$$\frac{d}{dr}\left[O_1 \overline{\overline{\nabla}}\right] + 2\,O_2 \frac{d\overline{\nabla}}{dr} - O_2\left[\frac{R_1}{r} n(n+1) - \frac{1}{r^2}\ \frac{d(rR_2)}{dr} n(n+1)\right] +$$

$$2\,\dot{O}_1 \frac{dR_1}{dr} - \varrho \frac{dP_o}{dr} \overline{\overline{\nabla}} + \frac{d(P-P_o)}{dr} + \varrho \frac{d}{dr}\left(R_1 \frac{dP_o}{dr}\right) - p^2 \varrho R_1 = 0$$

$$r^{-1}\left\{O_1 \overline{\overline{\nabla}} - O_2\left[\frac{dR_1}{dr} - \frac{d^2(rR_2)}{dr^2}\right] + \dot{O}_2 r\left[\frac{R_1}{r} + r\frac{dR_2/r}{dr}\right]\right\}$$

$$+ \frac{d}{dr}(P - P_o) + \varrho \frac{R_1}{r}\ \frac{dP_o}{dr} - p^2 \varrho R_2 = 0 \qquad [13]$$

$$r^{-2}\frac{d}{dr}\left[r^2 \frac{d(P-P_o)}{dr}\right] - 4\pi\varrho G \overline{\overline{\nabla}} + \frac{n(n+1)}{r}\ \frac{P-P_o}{r} - 4\pi G\varrho \frac{dR_1}{dr} = 0$$

$$O_2\left[\frac{1}{r}\ \frac{d^2 rR_2}{dr^2} - \frac{n(n+1)}{r^2} R_3\right] + \dot{O}_2 r \frac{dR_3/r}{dr} + O_3 R_3 = 0$$

$$\overline{\overline{\nabla}} = -\frac{n(n+1)}{r} R_2 + \frac{1}{r^2}\ \frac{d}{dr}(r^2 R_1)$$

with

$$P - P_o = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} (V - V_o)\, e^{-pt} \sin\vartheta\, d\varphi\, d\vartheta\, dt\,.$$

$V - V_o$ is the perturbation of the gravitational potential arising from the perturbation of the density field and from the attraction of the density perturbation at the deformed interfaces.

We assume also that the integration in [3] can be interchanged with the integration of the Laplace transformation and also that we can use formula [38] (see appendix at the end of the paper). Then e.g.

$$O_2 = \int_0^\infty o_2 \, s_i \, e^{-pt} \, dt = \left[\int_{a_2}^{b_2} p^z f_2(r, z) \, dz \right] \int_0^\infty s_i \, e^{-pt} \, dt \, . \qquad [14]$$

Dissipation in the torsional oscillation of a shell

We shall discuss some models of dissipation, obtained by specializing the functions $f_i(r, z)$ appearing in [3], in the free purely torsional and radial vibrations of a shell.

The equation which governs the motions of the torsional modes in a non perfectly elastic shell of radiae r_1 and r_2, assuming a stress-strain relation of the type [3], is:

$$\int_{a_2}^{b_2} f_2(r, z) \frac{d^z}{d\,t^z} \left[\frac{1}{r} \frac{\partial^2 rs}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial \vartheta} \left(\frac{1}{\sin \vartheta} \frac{\partial s \sin \vartheta}{\partial \vartheta} \right) \right] dz + \\ + \int_{a_2}^{b_2} \left[\frac{\partial}{\partial r} f_2(r, z) \right] \frac{d^z}{d\,t^z} \left[r \frac{\partial}{\partial r} \left(\frac{s}{r} \right) \right] d\,z = \varrho \frac{\partial^2 s}{\partial t^2} \, . \qquad [15]$$

We assume an Earth model defined by a liquid core and a homogeneous mantle, and assume also that the dissipation of energy due to the viscous interaction between the core and the mantle, is negligible [Caputo, 1966] and that $\frac{\partial}{\partial r} f_2(r, z) = 0$, then the boundary condition is

$$\begin{vmatrix} F_n(r_2) & F_{-n}(r_2) \\ F_n(r_1) & F_{-n}(r_1) \end{vmatrix} = 0 \qquad [16]$$

$$F_{\pm n}(r) = \frac{d}{d\,r} r^{-3/2} J_{\pm n + 1/2}(ra)$$

where

$$S_3 = \left\{ A_n J_{n+1/2}(\alpha r) + (-1)^{n+1} A_{-n} J_{-n-1/2}(\alpha r) \right\} r^{-1/2} \frac{d P_n}{d \vartheta} \qquad [17]$$

$$\alpha^{-2} = \varrho^{-1} \int_{b_2}^{a_2} f_2(z)\, p^z\, dz$$

is a solution of the Laplace transform of [15]. $J_{-n-1/2}(\alpha r)$ and $J_{n+1/2}(\alpha r)$ are Bessel functions, and P_n (cos ϑ) are Legendre polinomials.

The solutions of [16] determine the periods of free oscillation and also the Q. Without loss of generality [10] can be written

$$f_2(z) = \mu + \int_{a_2}^{b_2} \bar{f}_2(z)\, p^z\, dz\,. \qquad [18]$$

An interesting case arises when

$$\bar{f}_2(z) = \mu_1\, \delta\,(z - z_o + \varepsilon) \qquad [19]$$

equation [16] is then

$$-\frac{\alpha^2}{\varrho}\left[\mu + \mu_1 p^{z_o-\varepsilon}\right] = p^2 \qquad [20]$$

which gives

$$p = i\,|\,p_o\,|\left\{1 + \frac{\mu_1}{\mu}\, p_o^{z_o-\varepsilon}\right\}^{1/2}, \quad |\,p_o\,| = \alpha\sqrt{\frac{\mu}{\varrho}} \qquad [21]$$

If $z_o = 2\,m$ (m integer) then we have

$$p = i\,|\,p_o\,|\left\{1 + \frac{\mu_1\,|\,p_o\,|^{2m-\varepsilon}}{\mu}(-1)^m\left[\cos\frac{\pi\,\varepsilon}{2} - i\sin\frac{\pi\,\varepsilon}{2}\right]\right\}^{1/2} \qquad [22]$$

and if $m = 0$

$$p = i\,|\,p_o\,|\left\{1 + \frac{\mu_1\,|\,p_o\,|^{-\varepsilon}}{2\,\mu}\left[\cos\frac{\pi\,\varepsilon}{2} - i\sin\frac{\pi\,\varepsilon}{2}\right]\right\} \qquad [23]$$

The specific dissipation function is therefore

$$Q^{-1} = \frac{\mu_1}{\mu}\,|\,p_o\,|^{-\varepsilon}\left|\sin\frac{\pi}{2}\,\varepsilon\right|\,. \qquad [24]$$

Dissipation in the purely radial modes

The study of the dissipation of energy in the purely radial modes, can be done as that of the torsional modes. The only difference is that in this case the forces which govern the periods of the modes are the elastic forces and also the gravitational forces. Assuming a homogeneous earth model of density ϱ_o and $\frac{\partial f_1}{\partial r} = 0$ one can see that the periods of the free modes and the dissipation are given by

$$\varrho_o r_2^2 (p^2 - 4A) = -x^2 \int_{a_1}^{b_1} f_1(z)\, p^z\, dz$$
$$A = \frac{4\pi G \varrho_o}{3} \qquad [25]$$

and x is a solution of

$$\frac{\tan x}{x} = \frac{1}{1 - \frac{x^2}{4}\left(2 + \frac{\lambda}{\mu}\right)} . \qquad [26]$$

Without loss of generality f_1 can be written

$$f_1 = \lambda + 2\mu + \int_{a_1}^{b_1} \bar{f}_1(z)\, p^z\, dz . \qquad [27]$$

An interesting case arisen when

$$\bar{f}_1(z) = \lambda_1 \delta(z - z_o + \varepsilon) \qquad [28]$$

equation [25] is then

$$p^2 = 4A - \frac{x^2}{\varrho r_2^2}(\lambda + 2\mu + \lambda_1 p^{z_o - \varepsilon}) = -|p_o|^2 - \frac{x^2 \lambda_1}{\varrho r_2^2} p_o^{z_o - \varepsilon}$$
$$= -|p_o|^2 - \left[\frac{x^2}{\varrho r_2^2}(\lambda + 2\mu) - 4A\right] \frac{\lambda_1 p_o^{z_o - \varepsilon}}{\lambda + 2\mu} + \frac{4A\lambda_1}{\lambda + 2\mu} p_o^{z_o - \varepsilon}$$
$$p_o^2 = 4A - \frac{x^2}{\varrho r_2^2}(\lambda + 2\mu) . \qquad [29]$$

Its solutions are

$$p = \pm i \, | p_o | \left\{ 1 + \frac{\lambda_1}{\lambda + 2\mu} p_o^{z_o - \varepsilon} - \frac{4 A \lambda_1 p_o^{z_o - \varepsilon}}{(\lambda + 2\mu) | p_o |^2} \right\}^{1/2} \qquad [30]$$

which becomes for $z_o = 0$

$$p = \pm i \, | p_o | \left\{ 1 + \frac{\lambda_1}{\lambda + 2\mu} p_o^{-\varepsilon} - \frac{4 A \lambda_1 p_o^{-\varepsilon}}{(\lambda + 2\mu) | p_o |^2} \right\}^{1/2} . \qquad [31]$$

Equation [31] can also be written

$$p = \pm i \, | p_o | \left\{ 1 + \frac{\lambda_1 | p_o |^{-\varepsilon}}{2(\lambda + 2\mu)} \left\{ 1 - \frac{4 A}{| p_o |^2} \right\} \left[\cos \frac{\pi}{2} \varepsilon - i \sin \frac{\pi}{2} \varepsilon \right] \right\} \qquad [32]$$

and the specific dissipation is

$$Q^{-1} = \frac{\lambda_1 | p_o |^{-\varepsilon}}{\lambda + 2\mu} \left| \sin \frac{\pi}{2} \varepsilon \right| \left\{ 1 - \frac{4 A}{| p_o |^2} \right\} . \qquad [33]$$

The ratio of the observed Q's gives ε

$$\frac{Q_o^{-1}}{Q_1^{-1}} = \left(\frac{| p_{oo} |}{| p_{o1} |} \right)^{-\varepsilon} , \qquad \varepsilon \ln \frac{| p_{oo} |}{| p_{o1} |} = \ln \frac{Q_o}{Q_1} , \qquad [34]$$

λ_1 is therefore obtained from [34] by substitution

$$\lambda_1 = \frac{(\lambda + 2\mu) | p_o |^{\varepsilon}}{Q \left| \sin \frac{\varepsilon \pi}{2} \right|} . \qquad [35]$$

Checks with the observations

A very extensive analysis of the attenuation of the Rayleigh waves has been made by Ben-Menahem (1964) who measured it from four great earthquakes from observation of multiple circuits around the earth past one station. By Fourier analysis he obtained the specific attenuation factor in the period range between 300 and 40 seconds; from the attenuation factor he computed then the specific dissipation.

The discrete spectrum of the spheroidal oscillations of the earth, where the matter is both compressed and sheared, approaches the continuous Rayleigh waves spectrum. In order to extend the range where the Q is known, we computed it also from the free spheroidal oscillations of the Earth of period between 5 and 18 minutes, using the 110 hours record fo the 1961 Chilean quake recorded at UCLA. The record was subdivided in several intervals whose initial points where 4.5 and 13.5 hours apart; from the power spectral analysis of these intervals we obtained the decrease in energy and subsequently the *Q's* by means of formula [24] of Caputo (1966). From a preliminary analysis of these *Q's* we can see that the agreement with the mathematical model of dissipation proposed in this paper, with $\varepsilon = -0.15$, is satisfactory.

Another check of this theory can be made using the *Q's* obtained by Zemaneck and Rudnik (1961) for longitudinal waves in the period range between 10^{-3} and 10^{-5} seconds. Here again the agreement of the observed *Q's* with the model of dissipation proposed in the present paper, with $\varepsilon = -0.15$, is satisfactory. We plan to complete and publish soon the discussion of the above mentioned experimental results.

Appendix

A generalization of the operation of differentiation with real order of differentiation, for a wide class of analytic functions, can be made as follows.

Let

$$\frac{d^z}{d\,t^z}\,(ct^x) = \begin{cases} c\,\dfrac{\Gamma(x+1)}{\Gamma(x+1-z)}\,t^{x-z} & \text{if } x>0 \\ 0 & \text{if } x=0 \end{cases} \quad \begin{matrix} 0<z<1 \\ x \text{ integer} \end{matrix} \qquad [36]$$

It $f(t)$ is an analytic function, the operator [36] can be applied to the terms of its power series expansion; the resulting series, if convergent, can be assumed to be the z order derivative of $f(t)$.

We shall need to evaluate the Laplace transform of derivatives of order z of a class of analytic functions. Let the power series expansion of $f(t)$, convergent in the interval $(0, \infty)$, be

$$\begin{aligned} f(t) &= \Sigma_i\, a_i\, t^i \\ a_i &= 0 \quad \text{if } i \leqslant 0 \end{aligned} \qquad [37]$$

and, differentiating,

$$\frac{d^z f(t)}{d\,t^z} = \Sigma_i\, a_i\,\frac{\Gamma(i+1)}{\Gamma(i+1-z)}\,t^{i-z}. \qquad [38]$$

We want to prove that for $0 < z < 1$

$$\int_0^\infty \frac{d^z f(t)}{d t^z} e^{-pt} d t = p^z \int_0^\infty f(t) e^{-pt} d t. \qquad [38]$$

To obtain [38] we shall assume that the Laplace transform of $\frac{d^z f(t)}{d t^z}$ exists, than we have

$$\int_0^\infty \frac{d^z f(t)}{d t^z} e^{-pt} d t = \int_0^\infty \sum_i \frac{\Gamma(i+1)}{\Gamma(i+1-z)} t^{i-z} e^{-pt} d t$$

and, assuming that it is possible to interchange the sum with the integral

$$\sum_i a_i \frac{\Gamma(i+1)}{\Gamma(i+1-z)} \int_0^\infty t^{i-z} e^{-pt} dt = \sum a_i \Gamma(i+1) p^{z-1-i} =$$

$$= p^z \sum_i a_i \Gamma(i+1) p^{-p-1} = p^z \sum a_i \int_0^\infty t^i e^{-pt} d t =$$

$$= p^z \int_0^\infty \sum a_i t^i e^{-pt} d t = p^z \int_0^\infty f(t) e^{-pt} d t$$

which proves [38].

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