A study on the displacement components of Rayleigh wave (*)

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Summary. — The study has been carried out to observe the effects of Poisson ratio on the displacement components of Rayleigh waves in elastic media. In general the problem of surface waves have been studied by many authors. The authors have adopted arbitrary values of Poisson ratio in order to study the variations in the ratio of the displacement components on the surface and inside the media. The variations of the horizontal and vertical amplitudes of surface wave displacement with depth is a matter of great interest both in Seismology and Earthquake Engineering.

Riassunto. — Questo studio è stato eseguito allo scopo di osservare gli effetti del rapporto di Poisson sulle componenti degli spostamenti delle onde di Rayleigh in mezzi elastici. In generale il problema delle onde superficiali è stato studiato da molti autori. Gli autori hanno adottato valori arbitrari per il rapporto di Poisson in modo da studiarne le variazioni sulle componenti degli spostamenti rispetto alla superficie e nell'interno dei mezzi. Le variazioni delle ampiezze orizzontali e verticali dello spostamento dell'onda superficiale rispetto alla profondità sono di notevole interesse sia per la Sismologia che per l'Ingegneria sismica.

1. – Introduction.

It is well known that in an isotropic homogeneous and unbounded elastic media only two types of waves are propagated. Rayleigh in

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1887 showed that the free surface of an elastic homogeneous material can support a disturbance involving displacements in a vertical plane along the direction of propagation in the form of a retrograde ellipse and propagate with a velocity which was about 0.9 times the shear wave velocity in that medium.

Jeffreys and Stoneley and later Ewing and Press greatly expanded the theory to cover cases of layered semi-infinite media. The results indicate that in all but the simplest case assumed by Lord Rayleigh these waves suffered dispersion. The surface waves composing the Rayleigh and Love waves are distinguished from P and S body waves in being more or less confined to the surface. Thus their amplitude decreases very rapidly with depth below the surface.

2. – Formulation of the problem.

Assume a simple harmonic plane wave train travelling in the x direction such that

I) The disturbance is independent of y coordinate

II) It increases rapidly with distance z from the free surface.

Notation

The following symbols have been adopted for use in this paper:

E = elasticity
μ = coefficient of rigidity
ϱ = density of media
λ = Lame's constant
Φ = Scalar potential of displacement
$\Psi(\psi_1, \psi_2, \psi_3)$ = vector potential of displacement
L = wave length
a = compressional wave velocity
b = shear wave velocity
c = phase velocity
θ = cubical dilatation
σ = Poisson's ratio
ω = angular frequency
k = parameter for wave equation
$F_i(F_x, F_y, F_z)$ = body forces
U, V, W = displacement in x, y and z directions respectively
V_R = Rayleigh wave velocity
τ = rate of decrease of amplitude
η = ratio of shear wave velocity to compressional wave velocity
ξ = ratio of phase velocity to shear wave velocity.

Consider the x-y plane to be as boundary and the z axis positive toward the interior of the solid. Let U, V, and W be the displacements in x, y and z directions respectively.

The displacement function $D = f(U, V, W)$ is defined (3) as

$$D = \nabla \Phi + \nabla x \Psi (\psi_1, \psi_2, \psi_3) . \quad [1]$$

Since displacement is independent of y we have

$$\begin{cases} U = \dfrac{\partial \Phi}{\partial x} - \dfrac{\partial \Psi_2}{\partial z} \\ V = \dfrac{\partial \Psi_1}{\partial z} - \dfrac{\partial \Psi_3}{\partial x} \\ W = \dfrac{\partial \Phi}{\partial z} + \dfrac{\partial \Psi_2}{\partial x} \end{cases} \quad [2]$$

Stress at a point given as (2)

$$P_{ij} = \lambda \theta \delta_{ij} + 2 \mu e_{ij} \quad [3]$$

where λ and μ are the Lame's constants and are given

as $$\lambda = \frac{\sigma E}{(1+\sigma)(1-\sigma)} \quad \text{and } \mu = \frac{E}{2(1+\sigma)}$$

θ is the cubical dilatation and δ_{ij} is Cronecker delta i.e.

$$\delta_{ij} \begin{cases} = 1 \text{ for } i = j \\ = 0 \text{ for } i \neq j \end{cases}$$

σ is the Poisson ratio and E is the modulus of elasticity.

Equation of motion is given as

$$\varrho \ddot{D}_i = \varrho F_i + P_{ij,j} \quad [4]$$

where $\ddot{D}_i$ is the second derivative of D_i with respect to time, ϱ is the density and

$$P_{ij,j} = \frac{\partial (P_{ij})}{\partial j} .$$

From [1], [2], [3] and [4]

$$\varrho \ddot{D}_i = (\lambda + \mu) \frac{\partial \theta}{\partial i} + \mu \nabla^2 D_i$$

or generalizing

$$\varrho \ddot{D} = (\lambda + \mu) \nabla \theta + \mu \nabla^2 D. \qquad [5]$$

But
$$\theta = \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right) = \nabla \cdot D = \nabla^2 \Phi$$

and we obtain

$$\varrho \ddot{D} = (\lambda + \mu) \nabla (\nabla \cdot \Phi) + \mu \nabla^2 (D). \qquad [6]$$

Substituting for D and separating real and imaginary parts we obtain

$$\left\{\begin{aligned} \nabla^2 \Phi &= \frac{1}{a^2} \frac{\partial^2 \Phi}{\partial t^2} \\ \nabla^2 \Psi_1 &= \frac{1}{b^2} \frac{\partial^2 \Psi_1}{\partial t^2} \\ \nabla^2 V &= \frac{1}{b^2} \frac{\partial^2 V}{\partial t^2} \end{aligned}\right. \qquad [7]$$

where

$$a = \sqrt{\frac{\lambda + 2\mu}{\varrho}} \qquad \text{and} \qquad b = \sqrt{\frac{\mu}{\varrho}} . \qquad [8]$$

Solution of [7] can be written as

$$\left\{\begin{aligned} \Phi &= A \exp [\pm q z + i (\omega t - kx)] \\ \Psi &= B \exp [\pm sz + i (\omega t - kx)] \\ V &= C \exp [\pm sz + i (\omega t - kx)] \end{aligned}\right. \qquad [9]$$

where

$$\left\{\begin{aligned} &q = \sqrt{k^2 - k^2_a} \qquad \text{and} \qquad s = \sqrt{k^2 - k^2_b} \\ &k_a = \frac{\omega}{a} \qquad\qquad \text{and} \qquad k_b = \frac{\omega}{b} \end{aligned}\right. \qquad [10]$$

Since for surface waves the motion becomes negligible at a distance of a few wave lengths from the free surface solution [9] takes the form

$$\left\{\begin{aligned} \Phi &= A \exp [- qz + i (\omega t - kx)] \\ \Psi &= B \exp [- sz + i (\omega t - kx)] \\ V &= C \exp [- sz + i (\omega t - kx)] \end{aligned}\right. \qquad [11]$$

provided $c < b < a$.

The sign has been so chosen so that potential becomes zero as z tends to infinity.

To obtain constants A, B, C, the boundary conditions are: all stresses must vanish at free surface i.e. $z = 0$

$$P_{zz} = P_{zx} = P_{zy} = 0 \qquad \text{at} \qquad z = 0 . \tag{12}$$

From [3], [11] and [12]

$$\begin{cases} C = 0 \\ A\,(2\,iqk) - B\,(k^2 + s^2) = 0 \\ A\,[(\lambda + 2\,\mu)\,q^2 - \lambda k^2] + B\,(2\,\mu is) = 0 \,. \end{cases} \tag{13}$$

Eliminating A and B we get

$$\xi^2\,[\xi^6 - 8\,\xi^4 + \xi^2\,(24 - 16\,\eta^2) + 16\,(\eta^2 - 1) = 0 \tag{14a}$$

where

$$\xi = \frac{c}{b} \qquad \text{and} \qquad \eta = \frac{b}{a} \,.$$

From this we see that one root of ξ is zero but this gives phase velocity equal to zero and hence is of not use.

Therefore

$$\xi^6 - 8\,\xi^4 + \xi^2\,(24 - 16\,\eta^2) + 16\,(\eta^2 - 1) = 0 \,. \tag{14}$$

Also from eq. [14] it is obvious that the equation holds good when

$$0 < c < b < a \,.$$

Rewriting

$$\eta = \frac{b}{a} = \sqrt{\frac{1 - 2\,\sigma}{2\,(1 - \sigma)}} \,.$$

Displacements U and W are

$$U = \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial z} \qquad \text{and} \qquad W = \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial z}$$

ψ_2 has been replaced by Ψ because other two components of Ψ do not come in to play.

From [11] and [15] we get (1)

$$U = -A\,[Akie^{-qz} - Bse^{-sz}]\,\exp\,[i\,(\omega t - kx)]\,.$$

Taking only real part

$$U = Ak \left| e^{-qz} - \frac{2qs}{k^2 + s^2} e^{-sz} \right| \sin(\omega t - kx)$$

and similarly [16]

$$W = -Aq \left| e^{-qz} - \frac{2\,k^2}{k^2 + s^2} e^{-sz} \right| \cos(\omega t - kx) \quad .$$

The rate at which the amplitude of the displacement along the direction of propagation decreases with depth depends upon the factor

$$\tau_x = \left| e^{-qz} - \frac{2\,qs}{k^2 + s^2} e^{-sz} \right| . \qquad [17]$$

Also the rate at which the amplitude of the motion in a direction normal to the surface decreases with depth depends upon the factor

$$\tau_z = \left| e^{-qz} - \frac{2\,k^2}{k^2 + s^2} e^{-sz} \right| . \qquad [18]$$

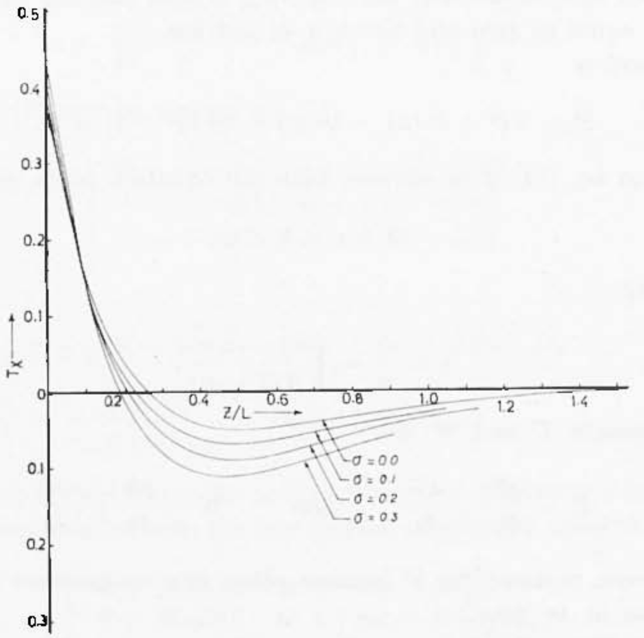


Fig. 1 – Coefficient of decay of amplitude in the direction of propagation of Rayleigh wave in elastic media for different values of Poisson's ratio.

Now

$$q^2 = k^2 - k_a^2 = k^2 (1 - \eta^2 \xi^2)$$

and

$$s^2 = k^2 - k_b^2 = k^2 (1 - \xi^2)$$

$k = \frac{\omega}{C} = \frac{2\pi}{L}$ where L is the wave length of Rayleigh wave.

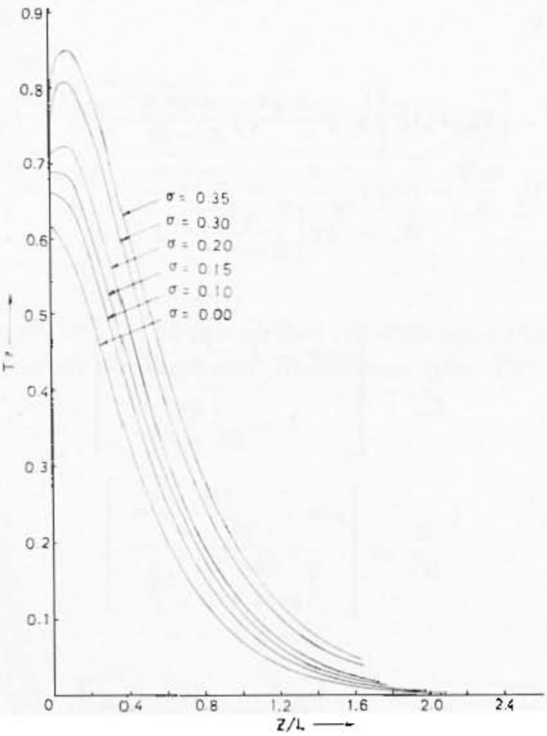


Fig. 2 – Coefficient of decay of amplitude in the direction perpendicular to propagation of Rayleigh wave in elastic media for different values of Poisson's ratio

Substituting [3] in [17] and [18] we get

$$\tau_x = \exp \left| - 2\pi \sqrt{1 - \eta^2} \quad \frac{2\sqrt{1 - \eta^2 \xi^2}\sqrt{1 - }}{2 - \xi^2} \right| \quad \left| 2\pi \quad \sqrt{1 - \xi^2} \right|$$

and

$$\tau_z = -\exp\left[-2\pi\sqrt{1-\eta^2\xi^2}\cdot\frac{z}{L}\right] + \\ + \frac{2}{2-\xi^2}\exp\left[-2\pi\frac{z}{L}\sqrt{1-\xi^2}\right] \qquad [20]$$

(see Fig. 1 and 2).

Again let U_o and W_o be the displacement components in the and perpendicular to the direction of propagation respectively at the free surface $z = 0$.

Then

$$U_o = Ak\left[1 - \frac{2\sqrt{1-\eta^2\xi^2}\sqrt{1-\xi^2}}{2-\xi^2}\right] \qquad [21]$$

and

$$W_o = Aq\left[\frac{2}{2-\xi^2} - 1\right]. \qquad [22]$$

Then

$$\frac{U}{U_o} = \left[\frac{e^{-qz} - 2\dfrac{qs}{k^2+s^2}e^{-sz}}{1 - \dfrac{2qs}{k^2+s^2}}\right] \qquad [23]$$

$$\frac{W}{W_o} = \left[\frac{e^{-qz} - \dfrac{2k^2}{k^2+s^2}e^{-sz}}{\left(\dfrac{2k^2}{k^2+s^2} - 1\right)}\right] \qquad [24]$$

and

$$\frac{U}{W} = -\frac{k}{q}\left[\frac{e^{-qz} - \dfrac{2qs}{k^2+s^2}e^{-sz}}{e^{-qz} - \dfrac{2k^2}{k^2+s^2}e^{-sz}}\right]. \qquad [25]$$

Substituting for q, s and k we obtain

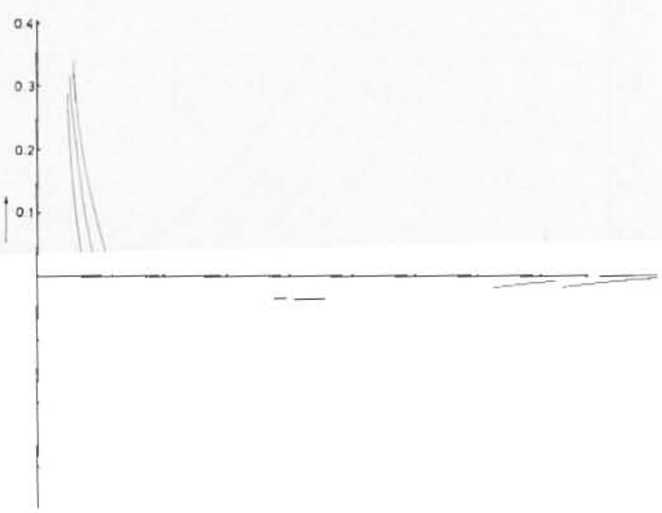
$$\frac{U}{U_o} = \left[\frac{\exp\left[-\dfrac{2\pi^2}{L}\sqrt{1-\eta^2\xi^2} - \dfrac{2\sqrt{1-\eta^2\xi^2}\sqrt{1-\xi^2}}{2-\xi^2}\exp\left[-2\pi\dfrac{z}{L}\sqrt{1-\xi^2}\right]\right.}{\dfrac{2\sqrt{1-\eta^2\xi^2}\sqrt{1-\xi^2}}{2-\xi^2}}\right] \qquad [26]$$

$$\frac{W}{W_o} = \left[\frac{-\exp\left[-2\pi \frac{z}{L}\sqrt{1-\eta^2\xi^2}\right] + \frac{2}{2-\xi^2}\exp\left[-2\pi\frac{z}{L}\sqrt{1-\xi^2}\right]}{\frac{\xi^2}{2-\xi^2}} \right] \quad [27]$$

and

$$\frac{U}{W} = -\frac{1}{\sqrt{1-\eta^2\xi^2}} \cdot \left[\frac{\exp\left[-\frac{2\pi z}{L}\sqrt{1-\eta^2\xi^2}\right] - \frac{2}{2-\xi^2}\sqrt{1-\eta^2\xi^2}\sqrt{1-\xi^2}\exp\left[-\frac{2\pi z}{L}\sqrt{1-\xi^2}\right]}{-\exp\left[-\frac{2\pi z}{L}\sqrt{1-\eta^2\xi^2}\right] + \frac{2}{2-\xi^2}\exp\left[-\frac{2\pi z}{L}\sqrt{1-\xi^2}\right]} \right] \quad [28]$$

Equations 19, 20, 26, 27, 28 are plotted for different values of Poisson ratio. The plots are obtained non dimensional. See Fig. (1, 2, 3, 4,5).



ig. 3 – Ratio of amplitude in the direction of propagation at the surface (U_o) and in the media (U) for different values of Poisson's ratio.

3. – Conclusions.

Fig. 1 and Fig. 2 represent the coefficient of decay of amplitude in the direction and perpendicular to the direction of propagation. Fig. 6 indicates a comparative study of these two coefficients and

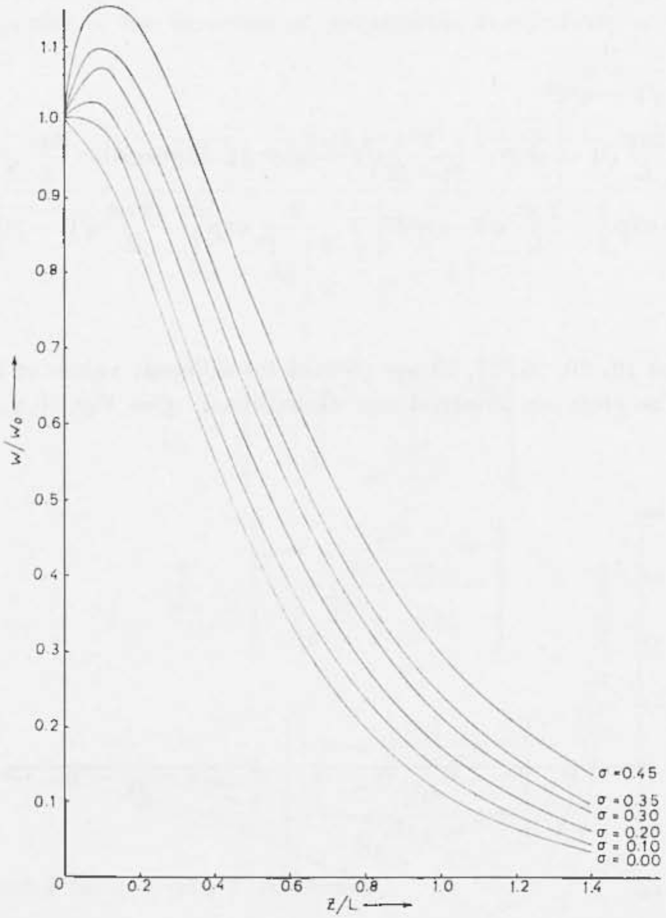


Fig. 4 – Ratio of amplitude in the direction perpendicular to propagation at the surface (W_o) and in the media (W) for different values of Poisson's ratio.

reveals that the coefficient corresponding to the displacement component in the direction of propagation show a change in the phase near to

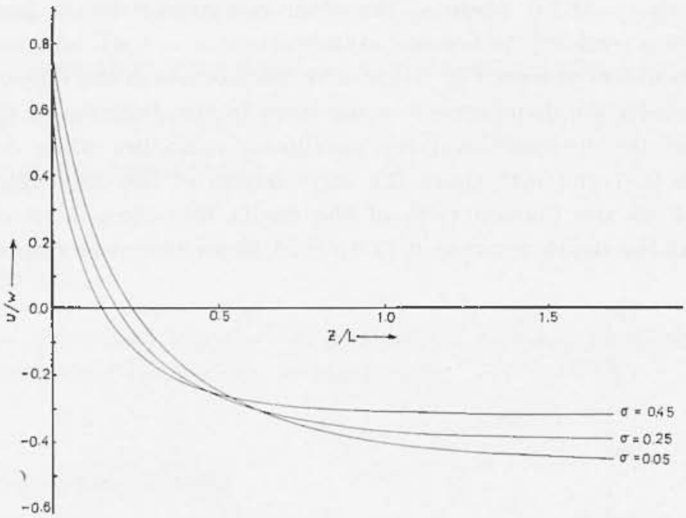


Fig. 5 – Ratio of amplitude in the direction (U) and perpendicular to the direction (W) of propagation for different values of Poisson's ratio.

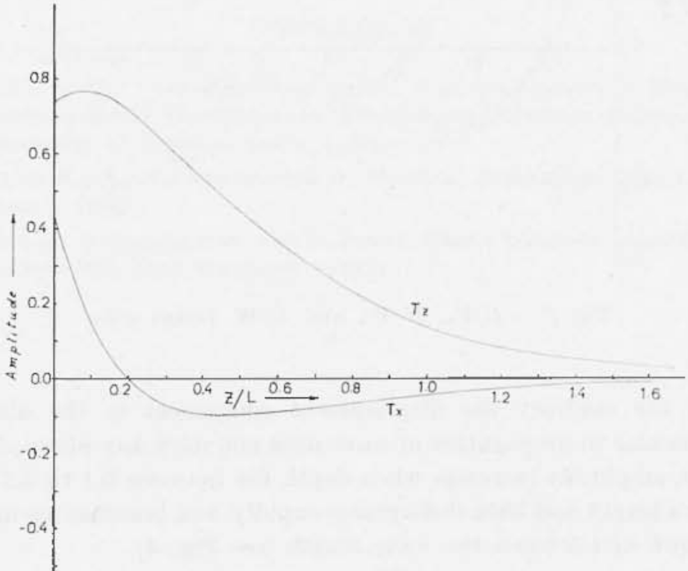


Fig. 6 – Coefficient of decay of displacement in the direction of propagation (τ_x) and in the direction perpendicular to propagation (τ_z).

the depth $z = 0.2L$ where as the other component decays gradually and have a tendency to become asymptotic at $z = 1.6L$ and onwards.

It is observed from Fig. 3 that with the increase in the Poisson ratio of the media the displacement component in the direction of the propagation also increases and the maximum value lies when depth is between 0.37 and 0.45 times the wave length of the Rayleigh wave. Also for all the Poisson ratio of the media this component changes phase at the depth between 0.15 to 0.25 times the wave length.

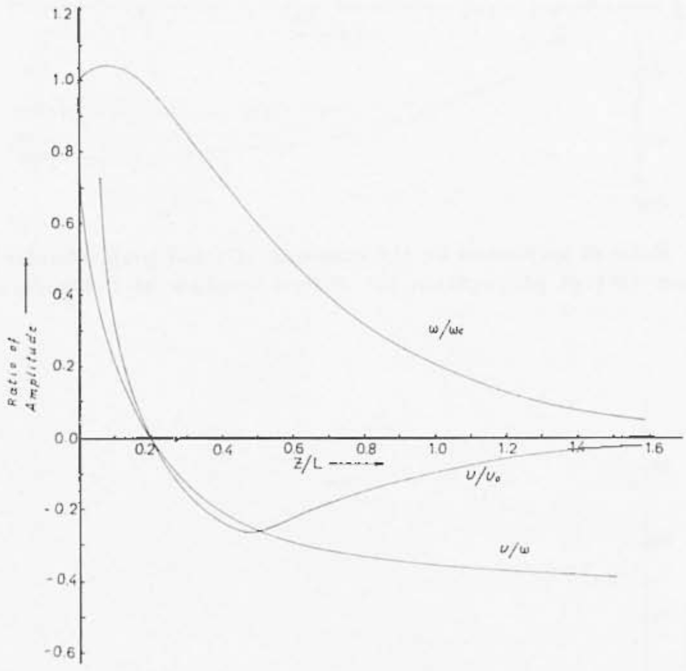


Fig. 7 – U/U_o, W/W_o and U/W versus z/L.

On the contrary the displacement component in the direction perpendicular to propagation of wave does not show any phase change. Also the amplitude increases when depth lies between 0.1 to 0.2 times the wave length and then it decreases rapidly and becomes asymptotic at a depth of 1.5 times the wave length (see Fig. 4).

The gradient of decay of $\frac{U}{W}$ (see Fig. 5) increases with increase in the value of Poisson ratio and up to depth $= 0.5L$. Then decreases with increase in Poisson ratio. Also the value $\frac{U}{W}$ is independent of the Poisson ratio when depth is near to 0.5 times the wave length.

Fig. 7 shows comparative study of $\frac{U}{W}$, $\frac{U}{U_o}$ and $\frac{W}{W_o}$ components of the displacement.

4. – Remark.

The study can further be extended for the variation in the modulus of elasticity and the density of the media.

5. – Acknowledgements.

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