APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 20, no. 1 (2019), 135-153
doi:10.4995/agt.2019.10474

Remarks on fixed point assertions in digital topology

Laurence Boxer [a] and P. Christopher Staecker [b]

[a] Department of Computer and Information Sciences, Niagara University, NY 14109, USA; and Department of Computer Science and Engineering, State University of New York at Buffalo. (boxer@niagara.edu)
[b] Department of Mathematics, Fairfield University, Fairfield, CT 06823-5195, USA. (cstaecker@fairfield.edu)

Communicated by S. Romaguera

Abstract

Several recent papers in digital topology have sought to obtain fixed point results by mimicking the use of tools from classical topology, such as complete metric spaces and homotopy invariant fixed point theory. We show that some of the published assertions based on these tools are incorrect or trivial; we offer improvements on others.

2010 MSC: *54H25.*

Keywords: *digital topology; fixed point; metric space.*

1. Introduction

Recent papers have attempted to apply to digital images ideas from Euclidean topology and real analysis concerning metrics and fixed points. While the underlying motivation of digital topology comes from Euclidean topology and real analysis, some applications recently featured in the literature seem of doubtful worth. Although papers including [30, 7] have valid and interesting results for fixed points and for "almost" or "approximate" fixed points in digital topology, other published assertions concerning fixed points in digital topology are incorrect or trivial (e.g., applicable only to singletons, or functions studied forced to be constant), as we will discuss in the current paper.

Received 02 July 2018 – Accepted 19 November 2018

2. Preliminaries

We let $\mathbb{Z}$ denote the set of integers, and $\mathbb{R}$, the real line.

We consider a digital image as a graph (X, κ), where $X \subset \mathbb{Z}^n$ for some positive integer n and κ is an adjacency relation on X.

A digital metric space is [12] a triple (X, d, κ) where (X, κ) is a digital image and d is a metric for X. In [12], d was taken to be the Euclidean metric, but we will not limit our discussion to the Euclidean metric.

The *diameter* of a metric space (X, d) is

$$diam\, X = \sup\{d(x, y) \mid x, y \in X\}.$$

2.1. **Adjacencies.** The most commonly used adjacencies for digital images are the c_u-adjacencies, defined as follows.

Definition 2.1. Let $p, q \in \mathbb{Z}^n$, $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$, $p \neq q$. Let $1 \leq u \leq n$. We say p and q are c_u-adjacent, denoted $p \leftrightarrow_{c_u} q$ or $p \leftrightarrow q$ when the adjacency is understood, if

- for at most u distinct indices i, $|p_i - q_i| = 1$, and
- for all other indices j, $p_j = q_j$.

Often, a c_u-adjacency is denoted by the number of points in $\mathbb{Z}^n$ that are c_u-adjacent to a given point. E.g.,

- in $\mathbb{Z}^1$, c_1-adjacency is 2-adjacency;
- in $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency;
- in $\mathbb{Z}^3$, c_1-adjacency is 8-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

An adjacency often used for Cartesian products of digital images is the *normal product adjacency*, denoted in the following by κ_* and defined [2] as follows. Given digital images (X, κ) and (Y, λ) and points $x, x' \in X$, $y, y' \in Y$, we have $(x, y) \leftrightarrow_{\kappa_*} (x', y')$ in $X \times Y$ if and only if one of the following holds.

- $x \leftrightarrow_\kappa x'$ and $y = y'$, or
- $x = x'$ and $y \leftrightarrow_\lambda y'$, or
- $x \leftrightarrow_\kappa x'$ and $y \leftrightarrow_\lambda y'$.

Other adjacencies for digital images are discussed in papers such as [16, 5, 6].

A *digital interval* is a digital image of the form $([a, b]_Z, 2)$, where $a < b$ and $[a, b]_Z = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$.

Digital connectedness is defined in terms of adjacency.

Definition 2.2 ([30])**.** A digital image (X, κ) is *κ-connected* (or *connected* when κ is understood) if given distinct $x, y \in X$ there is a sequence $\{x_i\}_{i=0}^n \subset X$ such that $x = x_0$, $x_n = y$, and $x_i \leftrightarrow_\kappa x_{i+1}$ for $0 \leq i < n$.

2.2. ℓ_p **metric.** Let $X \subset \mathbb{R}^n$ and let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be points of X. Let $1 \leq p \leq \infty$. The ℓ_p metric d for X is defined by

$$d(x, y) = \begin{cases} \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p} & \text{for } 1 \leq p < \infty; \\ \max\{|x_i - y_i|\}_{i=1}^n & \text{for } p = \infty. \end{cases}$$

For $p = 1$, this gives us the *Manhattan metric* $d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$; for $p = 2$, we have the *Euclidean metric* $d(x,y) = (\sum_{i=1}^{n} |x_i - y_i|^2)^{1/2}$.

Notice that for any ℓ_p metric d, if $x, y \in \mathbb{Z}^n$ and $d(x,y) < 1$, then $x = y$.

We note that digital images in the literature are commonly, although not exclusively, finite; also, the c_u adjacencies or adjacencies based on them (e.g., the normal product adjacency for Cartesian products with c_u adjacencies) are common. Many papers simply state an assumption that every digital image is a finite subset of $\mathbb{Z}^n$ with some c_u adjacency. Digital metric spaces typically use the Euclidean or some other ℓ_p metric. The development of classical topology has often placed emphasis on "wild" counterexamples, and in that setting finiteness and focus on ℓ_p metrics is very restrictive.

But digital topology is motivated primarily by "real-world" digital images, which are represented in the real world as subsets of $\mathbb{Z}^2$ with either the c_1 or c_2 adjacency. When a metric is used in a real world digital image, it's usually ℓ_1 or ℓ_2. Thus if in some results we assume that our images are finite and use the ℓ_p metric, these should be regarded as light assumptions.

2.3. **Digital continuity and homotopy.**

Definition 2.3 ([30, 4])**.** A function $f : (X, \kappa) \to (Y, \lambda)$ between digital images is (κ, λ)-*continuous* (or just *continuous* when κ and λ are understood) if for every κ-connected subset X' of X, $f(X')$ is a λ-connected subset of Y.

Theorem 2.4 ([4])**.** *A function $f : (X, \kappa) \to (Y, \lambda)$ between digital images is* (κ, λ)-continuous *if and only if $x \leftrightarrow_\kappa x'$ in X implies either $f(x) = f(x')$ or $f(x) \leftrightarrow_\lambda f(x')$ in Y.*

As in topology, the digital topology notion of homotopy can be understood as one function deforming in a continuous fashion into another. Precisely, we have the following.

Definition 2.5 ([4])**.** Let $f, g : (X, \kappa) \to (Y, \lambda)$. We say f and g are homotopic, denoted $f \simeq_{(\kappa,\lambda)} g$ or $f \simeq g$ when κ and λ are understood, if there is a function $F : X \times [0, m]_{\mathbb{Z}} \to Y$ for some $m \in \mathbb{N}$ such that

- $F(x, 0) = f(x)$ and $F(x, m) = g(x)$ for all $x \in X$.
- The induced function $F_t : X \to Y$ defined by $F_t(x) = F(x, t)$ is (κ, λ)-continuous for all $t \in [0, m]_{\mathbb{Z}}$.
- The induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$ defined by $F_x(t) = F(x, t)$ is $(2, \lambda)$-continuous for all $x \in X$.

2.4. **Cauchy sequences and complete metric spaces.** The papers [12, 18, 20, 21, 23, 24, 26, 27] apply to digital images the notions of Cauchy sequence and complete metric space. Since for common metrics such as an ℓ_p metric, a digital metric space is discrete, the digital versions of these notions are quite limited.

Recall that a sequence of points $\{x_n\}$ in a metric space (X, d) is a *Cauchy sequence* if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $m, n > n_0$ implies

$d(x_m, x_n) < \varepsilon$. If every Cauchy sequence in X has a limit, then (X, d) is a *complete metric space*.

It has been shown that under a mild additional assumption, a digital Cauchy sequence is eventually constant. The following is an easy generalization of Proposition 3.6 of [18], where only the Euclidean metric was considered. The proof given in [18] is easily modified to give the following.

Theorem 2.6. *Let $a > 0$. If d is a metric on a digital image (X, κ) such that for all distinct $x, y \in X$ we have $d(x, y) > a$, then for any Cauchy sequence $\{x_i\}_{i=1}^{\infty} \subset X$ there exists $n_0 \in \mathbb{N}$ such that $m, n > n_0$ implies $x_m = x_n$.*

An immediate consequence of Theorem 2.6 is the following.

Corollary 2.7 ([18])**.** *Let (X, d, κ) be a digital metric space. If d is a metric on (X, κ) such that for all distinct $x, y \in X$ we have $d(x, y) > a$ for some constant $a > 0$, then any Cauchy sequence in X is eventually constant, and (X, d) is a complete metric space.*

Remark 2.8. It is easily seen that the hypotheses of Theorem 2.6 and Corollary 2.7 are satisfied for any finite digital metric space, or for a digital metric space (X, d, κ) for which the metric d is any ℓ_p metric. Thus, a Cauchy sequence that is not eventually constant can only occur in an infinite digital metric space with an unusual metric. Such an example is given below.

Example 2.9. Let d be the metric on $(\mathbb{N}, c_1)$ defined by $d(i, j) = |1/i - 1/j|$. Then $\{i\}_{i=1}^{\infty}$ is a Cauchy sequence for this metric that does not have a limit.

2.5. **Digital fixed points and approximate fixed points.** The study of the fixed points of continuous self-maps is prominent in many areas of mathematics. We say a topological space X or a digital image (X, κ) has the *fixed point property* if every continuous (respectively, (κ, κ)-continuous) $f : X \to X$ has a fixed point, i.e., a point $p \in X$ such that $f(p) = p$.

A version of Theorem 2.10 below was proved by Rosenfeld in [30] for the case when X is a *digital picture*, that is, a digital image of the form $\Pi_{i=1}^{n}[a_i, b_i]_{\mathbb{Z}} \subset \mathbb{Z}^n$ with the c_n-adjacency. For general digital images, Theorem 2.10 was proved in [7].

Theorem 2.10. *A digital image (X, κ) has the fixed point property if and only if X is a singleton.*

This theorem led to the study in [7] of the *approximate fixed point property*, an idea suggested by results of [30]. An *approximate fixed point* [7], called an *almost fixed point* in [30], of a (κ, κ)-continuous function $f : (X, \kappa) \to (X, \kappa)$ is a point $p \in X$ such that $f(p) = p$ or $f(p) \leftrightarrow_{\kappa} p$. A digital image (X, κ) has the *approximate fixed point property (AFPP)* [7] if for every continuous $f : X \to X$ there is an approximate fixed point of f.

We have rephrased the following to conform with terminology used in this paper.

Theorem 2.11 (Theorem 4.1 of [30])**.** *Every digital picture $(\Pi_{i=1}^{n}[a_i, b_i]_{\mathbb{Z}}, c_n)$ has the AFPP.*

In Remark 6.2 (2) of [19], the author incorrectly attributes to [30] the claim that "Every digital image $(Y, 8)$ has the AFPP." The attribution is incorrect, since, as we have shown above, the citation should be about digital pictures, not the more general digital images. Further, the claim is false, as the following example shows.

Example 2.12. Let $n \geq 4$ and let $Y = (\{y_i\}_{i=0}^{n-1}, 8) \subset \mathbb{Z}^2$ be a digital simple closed curve with the points y_i indexed circularly. Then $(Y, 8)$ does not have the AFPP.

Proof. The function $f : Y \to Y$ defined by $f(y_i) = y_{(i+2) \mod n}$ is easily seen to be $(8, 8)$-continuous and free of approximate fixed points. □

2.6. **Contraction and expansion functions.** We cite several fixed point theorems for digital topology that are modeled on analogs for the topology of $\mathbb{R}^n$. In section 4 below, we explore limitations on many of the types of functions introduced in this section; in many cases, their inspirations in topology are not similarly limited.

In the following definitions, we assume (X, d, κ) is a digital metric space and $f : X \to X$ is a function. Several of these definitions are unmodified from their inspirations in the topology of metric spaces.

Definition 2.13 ([12])**.** If for some $\alpha \in (0, 1)$ and all $x, y \in X$, $d(f(x), f(y)) < \alpha d(x, y)$, then f is a *digital contraction map*. We say α is the *multiplier*.

Note such a function should not be confused with a *digital contraction* [3], a homotopy between an identity map and a constant function.

Definition 2.14. If

$$d(f(x), f(y)) \leq \alpha[d(x, f(x)) + d(y, f(y))]$$

for all $x, y \in X$, where $0 < \alpha < 1/2$, we say f is a *Kannan contraction map*.

Definition 2.15. If

$$d(f(x), f(y)) \leq \alpha[d(x, f(y)) + d(y, f(x))]$$

for all $x, y \in X$, where $0 < \alpha < 1/2$, we say f is a *Chatterjea contraction map*.

Definition 2.16. If

$$d(f(x), f(y)) \leq ad(x, f(x)) + bd(y, f(y)) + cd(x, y)$$

for all $x, y \in X$ and all nonnegative a, b, c such that $a + b + c < 1$, then f is a *Reich contraction map*.

Proposition 2.17. *A Reich contraction map is a digital contraction map and is a Kannan contraction map.*

Proof. Let f be a Reich contraction map. That f is a digital contraction map follows from the observation of [28] that in Definition 2.16, we can take $a = b = 0$ and obtain the conclusion from Definition 2.13. That f is a Kannan contraction map follows from the observation that in Definition 2.16, we can take $a = b \in (0, 1/2)$ and $c = 0$ to obtain the conclusion from Definition 2.14. □

Definition 2.18 ([26])**.** Let (X, d, κ) be a digital metric space and let $f : X \to X$ be a function. If there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$ we have

$$d(f(x), f(y)) \le \alpha \max\{d(x,y), \frac{d(x, f(x)) + d(y, f(y))}{2}, \frac{d(x, f(y)) + d(y, f(x))}{2}\}$$

then f is called a *Zamfirescu digital contraction*.

Definition 2.19 ([26])**.** Let (X, d, κ) be a digital metric space and let $f : X \to X$ be a function. If there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$ we have

$$d(f(x), f(y)) \le \alpha \max\{d(x,y), \frac{d(x, f(x)) + d(y, f(y))}{2}, d(x, f(x)), d(y, f(y))\}$$

then f is called a *Rhoades digital contraction*.

Proposition 2.20. *Let (X, d, κ) be a digital metric space and let $f : X \to X$ be a function. If f is a digital contraction map, then f is a Zamfirescu digital contraction and a Rhoades digital contraction.*

Proof. If f is a digital contraction map, then $d(f(x), f(y)) \le \alpha d(x, y)$ for all $x, y \in X$. The assertion follows from Definitions 2.18 and 2.19,. □

The following is a minor generalization of a definition in [20]. Therefore, results we derive in this paper for digitally (α, κ)-uniformly locally contractive functions apply to the version in [20].

Definition 2.21. Suppose $0 \le \alpha < 1$. Let (X, d, κ) be a digital metric space. Let $f : X \to X$ be a function such that $d(x, y) \le 1$ implies $d(f(x), f(y)) \le \alpha d(x, y)$. Then f is called *digitally (α, κ)-uniformly locally contractive*.

Proposition 2.22. *A digital contraction map with multiplier α is a digitally (α, κ)-uniformly locally contractive map.*

Proof. This is obvious from Definition 2.13 and Definition 2.21. □

Below, we define a set of functions Ψ that will be used in the following.

Definition 2.23 ([24])**.** Let Ψ be a set of functions $\psi : [0, \infty) \to [0, \infty)$ such that for each $\psi \in \Psi$ we have

- ψ is nondecreasing, and
- there exists $k_0 \in \mathbb{N}$, $a \in (0, 1)$, and a convergent series $\sum_{k=1}^{\infty} v_k$ of non-negative terms such that $k \ge k_0$ implies $\psi^{k+1}(t) \le a\psi^k(t) + v_k$ for all $t \in [0, \infty)$, where ψ^k represents the k-fold composition of ψ.

The following will be used later in the paper.

Example 2.24. The constant function with value 0 is a member of Ψ.

Definition 2.25 ([31])**.** Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say T is *α-admissible* if $\alpha(x, y) \ge 1$ implies $\alpha(T(x), T(y)) \ge 1$.

Definition 2.26 ([31])**.** Let (X,d) be a metric space, $T : X \to X$, $\alpha : X \to X$, and $\psi \in \Psi$. We say T is an $\alpha-\psi$-*contractive mapping* if $\alpha(x,y)d(T(x),T(y)) \leq \psi(d(x,y))$ for all $x,y \in X$.

Remark 2.27 ([24]). A digital contraction map $f : (X,d,\kappa) \to (X,d,\kappa)$ is an $\alpha - \psi$-contractive mapping for $\alpha(x,y) = 1$ and $\psi(t) = \lambda t$ for $\lambda \in (0,1)$.

Definition 2.28 ([24])**.** Let (X,d,κ) be a digital metric space, $T : X \to X$, $\beta : X \times X \to [0,\infty)$, and $\psi,\phi \in \Psi$ such that

$$\psi(d(T(x),T(y))) \geq \beta(x,y)\psi(d(x,y)) + \phi(d(x,y))$$

for all $x,y \in X$. Then T is a $\beta - \psi - \phi$-*expansive mapping*.

Depending on the choice of functions β,ϕ in Definition 2.28, the definition may not be very discriminating, as we see in the following.

Remark 2.29. Every function $T : X \to X$ is a $\beta - \psi - \phi$-expansive mapping if we take β and ϕ to be constant functions with value 0.

Proof. The assertion follows from Example 2.24 and Definition 2.28. □

Definition 2.30 ([9])**.** Let (X,d,κ) be a digital metric space. Let $T : X \to X$. Then T is a *weakly uniformly strict digital contraction* if given $\varepsilon > 0$ there exists $\delta > 0$ such that $\varepsilon < d(x,y) < \varepsilon + \delta$ implies $d(T(x),T(y)) < \varepsilon$ for all $x,y \in X$.

Definition 2.31 ([24])**.** Let (X,d,κ) be a complete digital metric space. Let $T : X \to X$. If T satisfies the condition $d(T(x),T(y)) \geq kd(x,y)$ for all $x,y \in X$ and some $k > 1$, then T is a *digital expansive mapping*.

Example 2.32. The function $T : \mathbb{N} \to \mathbb{N}$ defined by $T(n) = 2n$ is a digital expansive mapping, using the usual Euclidean metric. This map is not c_1-continuous [30].

The literature contains the following theorems concerning fixed points for such functions.

The following is a digital version of the Banach contraction principle [1].

Theorem 2.33 ([12])**.** *Let (X,d,κ) be a complete digital metric space, where d is the Euclidean metric in $\mathbb{Z}^n$. Let $f : X \to X$ be a digital contraction map. Then f has a unique fixed point.*

The following is a digital version of the Kannan fixed point theorem [25].

Theorem 2.34 ([27])**.** *Let $f : X \to X$ be a Kannan contraction map on a digital metric space (X,d,κ). Then f has a unique fixed point in X.*

The following is a digital version of the Chatterjea fixed point theorem [8].

Theorem 2.35 ([27])**.** *If $f : X \to X$ is a Chatterjea contraction map on a digital metric space (X,d,κ), then f has a unique fixed point.*

The following gives digital versions of Zamfirescu [32] and Rhoades [29] fixed point theorems.

Theorem 2.36 ([26])**.** *Let d be the Euclidean metric on $\mathbb{Z}^n$ and let (X, d, κ) be a digital metric space. If $f : X \to X$ is a Zamfirescu digital contraction or a Rhoades contraction map, then f has a unique fixed point.*

The following is a digital version of the Reich fixed point theorem [28]. We give a simpler proof of the digital version than appeared in [27].

Theorem 2.37. *Let $f : (X, d, \kappa) \to (X, d, \kappa)$ be a Reich contraction map on a digital metric space. Then f has a unique fixed point in X.*

Proof. This follows immediately from Proposition 2.17 and Theorem 2.33. □

The following is a version of the Edelstein fixed point theorem [10] for digital images.

Theorem 2.38 ([20])**.** *A digitally (α, κ)-uniformly locally contractive function on a connected complete digital metric space has a unique fixed point.*

3. Digital homotopy fixed point theory

The paper [13] defines a *digital fixed point property* as follows. The digital image (X, κ) has the digital fixed point property with respect to the digital interval $[0, m]_{\mathbb{Z}}$ if for all (κ_*, κ)-continuous functions $f : (X \times [0, m]_{\mathbb{Z}}, \kappa_*) \to (X, \kappa)$, where κ_* is the normal product adjacency for $(X, \kappa) \times ([0, m]_{\mathbb{Z}}, 2)$, there is a κ-path $p : [0, m]_{\mathbb{Z}} \to X$ of fixed points, i.e., $p(t)$ is a fixed point of the induced function $f_t : X \to X$ defined by $f_t(x) = f(x, t)$.

Also, [13] defines a *digital homotopy fixed point property* and states that this is equivalent to the following: A digital image (X, κ) has the digital homotopy fixed point property if for each digital homotopy $f : X \times [0, m]_{\mathbb{Z}} \to X$ there is a κ-path $p : [0, m]_{\mathbb{Z}} \to X$ such that for all $t \in [0, m]_{\mathbb{Z}}$, $p(t)$ is a fixed point of the induced function f_t.

These imply triviality, as follows.

Theorem 3.1. *A digital image (X, κ) has the digital fixed point property and the digital homotopy fixed point property if and only if X is a singleton.*

Proof. Clearly a singleton has the digital homotopy fixed point property and the digital homotopy fixed point property. Conversely, if X is not a singleton, then by Theorem 2.10 there is a continuous function $f : X \to X$ that does not have a fixed point. Let $F : X \times [0, m]_{\mathbb{Z}} \to X$ be defined by $F(x, t) = f(x)$. Then F is both (κ_*, κ)-continuous (where $\kappa_* = \kappa_*(\kappa, 2)$ is the normal product adjacency) and a homotopy, and fails to have a fixed point for any of the induced functions $f_t = f$. Thus, (X, κ) does not have the digital fixed point property or the digital homotopy fixed point property. □

4. Results for various contraction and expansion maps

4.1. Digital contraction maps.

In both of the papers [12, 20], arguments are given for the incorrect assertion that every digital contraction map is digitally continuous. Both papers present an error of confusing (topological) continuity of a map between metric spaces with (digital) continuity of a map between digital images. We present a counterexample to this assertion; our example is also used to show that Kannan, Chatterjea, Zamfirescu, and Rhoades contraction maps need not be digitally continuous. We use the Manhattan metric for its ease of computation, but the Euclidean or other ℓ_p metrics could be used to obtain similar conclusions.

Example 4.1. Let

$$X = \{p_1 = (0,0,0,0,0), \quad p_2 = (2,0,0,0,0), \quad p_3 = (1,1,1,1,1)\} \subset \mathbb{Z}^5.$$

Let $f : X \to X$ be defined by $f(p_1) = f(p_2) = p_1$, $f(p_3) = p_2$. Then f is not (c_5, c_5)-continuous. However, with respect to the Manhattan metric d, f is

- a digital contraction map,
- a Kannan contraction map,
- a Chatterjea contraction map,
- a Zamfirescu contraction map,
- a Rhoades contraction map,
- a $(0.45, c_5)$-uniformly locally contractive function,
- an $\alpha - \psi$-contractive mapping for $\alpha(x,y) = 1$ and $\psi(t) = \lambda t$ for $\lambda \in (0,1)$,
- a $\beta - \psi - \phi$-expansive mapping, where ψ and ϕ are constant functions with the value 0,
- a weakly uniformly strict digital contraction.

Proof. Note f is not (c_5, c_5)-continuous, since $p_1 \leftrightarrow_{c_5} p_3 \leftrightarrow_{c_5} p_2$, but $f(X) = \{p_1, p_2\}$ is not c_5-connected.

Observe that

$$\begin{aligned} d(p_1,p_2) &= 2, \quad d(f(p_1), f(p_2)) = 0, \\ d(p_1,p_3) &= 5, \quad d(f(p_1), f(p_3)) = 2, \\ d(p_2,p_3) &= 5, \quad d(f(p_2), f(p_3)) = 2. \end{aligned}$$

Therefore, we have, for all $x, y \in X$ such that $x \neq y$, $d(f(x), f(y)) \leq 2/5\, d(x,y) < 0.45 d(x,y)$. Therefore, f is a digital contraction map, a Zamfirescu contraction map, a Rhoades contraction map, and a $(0.45, c_5)$-uniformly locally contractive function.

Since

$$d(f(x), f(y)) \leq 2/5[d(x, f(x)) + d(y, f(y))] < 0.45[d(x, f(x)) + d(y, f(y))]$$

for all $x, y \in X$ such that $x \neq y$, f is a Kannan contraction map.

Note

$$\begin{aligned} d(f(p_1), f(p_2)) &= 0 < 0.45[d(p_1, f(p_2)) + d(p_2, f(p_1))], \\ d(f(p_1, f(p_3))) &= 2 < 0.45(2+5) = 0.45[d(p_1, f(p_3)) + d(p_3, f(p_1)),] \\ d(f(p_2), f(p_3)) &= 2 < 0.45(0+5) = 0.45(d(p_2, f(p_3)) + d(p_3, f(p_2))). \end{aligned}$$

Therefore, f is a Chatterjea contraction map.

That f is an $\alpha - \psi$-contractive mapping for $\alpha(x,y) = 1$ and $\psi(t) = \lambda t$ for $\lambda \in (0,1)$ follows from Remark 2.27.

Since Example 2.24 notes that the constant function with value 0 is in Ψ, it follows from Definition 2.28 that f is a $\beta-\psi-\phi$-expansive mapping.

It follows easily from Definition 2.30 that f is a weakly uniformly strict digital contraction. □

The following generalizes Theorem 4.7(1) of [18]. We give a proof, essentially that of [18], so we can refer to it below.

Theorem 4.2. *Let (X, d, c_1) be a digital metric space that is c_1-connected, where d is any ℓ_p metric in $\mathbb{Z}^n$. Let $f : X \to X$ be a digital contraction map. Then f is a constant function.*

Proof. Let $\alpha \in (0,1)$ satisfy $d(f(x), f(y)) \leq \alpha d(x,y)$ for all $x, y \in X$. If $x \leftrightarrow_{c_1} y$ in X, then $d(x,y) = 1$, so $d(f(x), f(y)) \leq \alpha$, which implies $f(x) = f(y)$, since every distinct pair of points in $\mathbb{Z}^n$ has distance of at least 1.

Given $x_0 \in X$, for any $x \in X$ there is a path $P = \{x_0, x_1, \ldots, x_m = x\} \subset X$ from x_0 to x such that $x_i \leftrightarrow_{c_1} x_{i+1}$, $0 \leq i < m$. It follows from the above that f is the constant function with value $f(x_0)$. □

Theorem 4.7(2) of [18] gives examples of c_2-connected images with digital contraction maps that are continuous and not constant. However, modification of Theorem 4.2 yields the following.

Theorem 4.3. *Let (X, d, κ) be a digital metric space that is κ-connected, where, for some $M_1 \geq M_2 > 0$ we have that $x \neq y$ implies $d(x,y) \geq M_2$ and $x \leftrightarrow_\kappa y$ implies $d(x,y) \leq M_1$ Let $f : X \to X$ be a digital contraction map with multiplier α such that $\alpha < M_2/M_1$. Then f is a constant function.*

Proof. Let $x, y \in X$ such that $x \leftrightarrow_\kappa y$. Then $d(x,y) \leq M_1$ and

$$d(f(x), f(y)) < \alpha d(x,y) < (M_2/M_1)M_1 = M_2.$$

By choice of M_2, $f(x) = f(y)$. It follows as in the proof of Theorem 4.2 that f is constant. □

Remark 4.4. Notice that Theorem 4.3 applies to all connected digital images (X, d, c_u), where $X \subset \mathbb{Z}^n$, $1 \leq u \leq n$, and d is any ℓ_p metric.

4.2. **Kannan and Chatterjea contractions.** Example 4.1 shows that neither a Kannan contraction map nor a Chatterjea contraction map must be constant. However, we have the following.

Theorem 4.5. *Let (X, d, κ) be a digital metric space of finite diameter, where d is any ℓ_p metric. Let $f : X \to X$ be a function. If f is a Kannan contraction map or a Chatterjea contraction map with α as in Definition 2.14 or Definition 2.15, respectively, satisfying $0 < \alpha < \frac{1}{2\, diam X}$, then f is a constant function.*

Proof. We have $d(f(x), f(y)) < 1$ for all $x, y \in X$, by Definition 2.14 in the case of a Kannan contraction map, and by Definition 2.15 in the case of a Chatterjea contraction map. Since d is an ℓ_p metric, it follows that $f(x) = f(y)$ for all $x, y \in X$. □

4.3. **Reich contractions.**

Theorem 4.6. *Let (X, d, κ) be a digital metric space of finite diameter, where d is any ℓ_p metric. Let $f : X \to X$ be a function. If f is a Reich contraction map with a, b, c as in Definition 2.16 satisfying $a, b, c \in (0, \frac{1}{3\,diam X})$, then f is a constant function.*

Proof. We have $d(f(x), f(y)) < 1$ for all $x, y \in X$. Since d is an ℓ_p metric, it follows that $f(x) = f(y)$ for all $x, y \in X$. □

Also, it follows from Proposition 2.17 that Theorem 4.2 and Theorem 4.3 apply to a Reich contraction map.

4.4. (α, κ)**-uniformly locally contractive functions.** Theorem 2.38 turns out to be a trivial result for connected digital metric spaces that use an ℓ_p metric, as we see below.

Theorem 4.7. *Let (X, d, κ) be a κ-connected digital metric space, where d is any ℓ_p metric. Let $f : X \to X$ be an (α, κ)-uniformly locally contractive function. Then f is a constant function.*

Proof. Let $x_0, x \in X$. Since X is connected, there is a κ-path in X, $\{x_i\}_{i=0}^m$, from x_0 to x such that $x_m = x$ and $x_i \leftrightarrow_{c_1} x_{i+1}$ for $0 \le i < m$. If $d(x_i, x_{i+1}) \le 1$, then

$$d(f(x_i), f(x_{i+1})) \le \alpha d(x_i, x_{i+1}) < 1.$$

Since d is an ℓ_p metric, $f(x_i) = f(x_{i+1})$. It follows that f is a constant function. □

4.5. **Digital expansive mappings.** We saw in Example 2.32 that a digital expansive mapping need not be digitally continuous.

Theorem 3.2 and Corollary 3.3 of [23] hypothesize a digital expansive mapping $T : X \to X$ that is onto. But in "real world" image processing, a digital image is finite, and therefore cannot support such a map, as shown by the following Theorems 4.8 and 4.9.

Theorem 4.8. *Let (X, d, κ) be a digital metric space. If X has points x_0, y_0 such that $d(x_0, y_0) = diam X > 0$, then there is no self-map $T : X \to X$ that is onto and a digital expansive mapping.*

Proof. Suppose there is a digital expansive mapping $T : X \to X$. Let $x_0, y_0 \in X$ be such that $d(x_0, y_0) = diam X > 0$. Then for some $k > 1$,

$$d(T(x_0), T(y_0)) \ge k d(x_0, y_0) = k\, diam X > diam X. \tag{4.1}$$

Since statement (4.1) is contradictory, the assertion follows. □

Theorem 4.9. *Let (X, d, κ) be a digital metric space of more than one point. If there exist $x_0, y_0 \in X$ such that*

$$d(x_0, y_0) = \min\{d(x, y) \mid x, y \in X, x \neq y\} \tag{4.2}$$

then there is no self-map $T : X \to X$ that is onto and a digital expansive mapping.

Proof. Suppose there exists a map $T : X \to X$ that is a digital expansive mapping. Let $x_0, y_0 \in X$ be as in equation (4.2). Let k be the expansive constant of T. Since T is onto, there exist distinct $x', y' \in X$ such that $T(x') = x_0$ and $T(y') = y_0$. Then

$$d(x_0, y_0) = d(T(x'), T(y')) \geq k\, d(x', y'),$$

which contradicts our choice of x_0, y_0. The assertion follows. □

Note Theorem 4.9 is applicable when X is finite or when d is any ℓ_p metric.

Remark 4.10. Example 3.8 of [23] claims that the self-map $T : X \to X$ of some subset of $\mathbb{Z}$ given by $T(n) = 2n - 1$ is onto. It is easy to see that this claim is only true for $X = \{1\}$.

4.6. **$\beta - \psi - \phi$-expansive mappings.** An analog of Theorem 2.1 of [31] is asserted as Theorem 3.2 of [24]:

> *Let (X, d, κ) be a complete digital metric space, $\beta : X \times X \to [0, \infty)$, and let $T : X \to X$ be a $\beta - \psi - \phi$-expansive mapping for some $\psi, \phi \in \Psi$. If there exist functions and such that*
> - *T^{-1} is β-admissible;*
> - *there exists $x_0 \in X$ such that $\beta(x_0, T^{-1}(x_0)) \geq 1$; and*
> - *T is digitally continuous,*
>
> *then T has a fixed point.*

However, this assertion is false, as we see in the following.

Example 4.11. Let $X = [-1, 1]_{\mathbb{Z}}^2 \setminus \{(0, 0)\} \subset \mathbb{Z}^2$. Let $\beta = d$ be the Manhattan metric on $\mathbb{Z}^2$. Let ϕ and ψ be constant functions with value 0. Let $T : X \to X$ be the map defined by $T(x, y) = (-x, -y)$. By Remark 2.29, T is a $\beta - \psi - \phi$-expansive mapping. Clearly T is β-admissible. For every $p \in X$ we have $\beta(p, T^{-1}(p)) = d(p, -p) > 1$. Also, T is both c_1-continuous and c_2-continuous. However, T has no fixed point.

Proof. It was observed in Example 2.24 that the constant function with value 0 is a member of Ψ. It is easy to see that the assertion follows. □

The following is given as Theorem 3.3 (and, with another hypothesis, as Theorem 3.7) of [24].

Theorem 4.12. *Let (X, d, κ) be a complete digital metric space and let $T : X \to X$ be a $\beta - \psi - \phi$-expansive mapping such that for some sequence $\{x_n\}_{n=1}^{\infty} \in X$ we have $\beta(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to y \in X$ as $n \to \infty$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\beta(x_{n_k}, y) \geq 1$ for all k. Then T has a fixed point.*

However, we have the following.

Example 4.13. If X is finite or d is an ℓ_p metric, and $\beta = d$, then Theorem 4.12 is vacuously true, as no such sequence $\{x_n\}_{n=1}^{\infty} \in X$ exists.

Proof. By Corollary 2.7, $x_n \to y$ implies that for some k_0, $k > k_0$ implies $x_n = y$ and therefore $\beta(x_n, y) = d(x_n, y) = 0$. Thus, no sequence $\{x_n\}_{n=1}^{\infty}$ satisfies the hypotheses of Theorem 4.12. □

Remark 4.14. The following assertion is stated as Theorem 3.6 of [24].

Let (X, d, κ) be a complete digital metric space. Let $T : X \to X$ be a $\beta - \psi - \phi$-expansive mapping such that

$$\psi(d(T(x), T(y))) \geq \beta(x, y)\psi(M(x, y)) + \phi(M(x, y)) \tag{4.3}$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(x, y), d(x, T(x)), d(y, T(y)), \frac{d(x, T(y)) + d(y, T(x))}{2}\}.$$

Then T has a fixed point.

But this assertion is false.

Proof. Consider the choices of $X, d, \kappa, T, \beta, \psi, \phi$ of Example 4.11, where we saw that T is a $\beta - \psi - \phi$-expansive mapping. Since ψ and ϕ are constant functions with value 0, (4.3) is satisfied. However, as noted at Example 4.11, T has no fixed point. □

4.7. **Remarks on [9].** The publisher of [9] identified one of the authors of the current paper, L. Boxer, as a reviewer. In fact, errors and other shortcomings mentioned in Boxer's review remain in the published version of [9].

The assertion stated as Theorem 3.1 of [9] is the following.

> *Let (X, d, κ) be a complete metric space such that $T : X \to X$ satisfies $d(T(x), T(y)) \leq \psi(d(x, y))$ for all $x, y \in X$, where $\psi : [0, \infty) \to [0, \infty)$ is monotone nondecreasing and $\psi^n(t) \to 0$ as $n \to \infty$. Then T has a unique fixed point.*

The argument offered in proof of this assertion confuses topological continuity (the "$\varepsilon - \delta$ definition") and digital continuity (preservation of connectedness) in order to conclude that T is continuous. However, in Example 4.1, using $\psi(t) = t/2$, we have a function that satisfies the hypotheses above and is not digitally continuous.

Further, if we add hypotheses that are often satisfied to Theorem 3.1 of [9], then T is forced to be a constant function, as seen in the following.

Proposition 4.15. *Let (X, d, c_u) be a digital metric space, $X \subset \mathbb{Z}^n$, such that $T : X \to X$ is (c_u, c_u)-continuous and satisfies $d(T(x), T(y)) \leq \psi(d(x, y))$ for all $x, y \in X$, where $\psi : [0, \infty) \to [0, \infty)$ is monotone nondecreasing and $\psi^n(t) \to 0$ as $n \to \infty$. If X is c_u-connected, d is an ℓ_p metric, and $\psi(t) < 1/u^{1/p}$ for all $t \in [0, \infty)$, then T is a constant function.*

Proof. This follows easily from Theorem 4.3. □

The argument given in proof of the assertion stated as Theorem 3.3 of [9] is similarly flawed. The assertion is the following.

> *Let (X, d, κ) be a complete digital metric space and $T : X \to X$ a weakly uniformly strict digital contraction mapping. Then T has a unique fixed point z. Moreover, for any $x \in X$, $lim_{n\to\infty} T^n(x) = z$.*

As above, Example 4.1 provides a counterexample to the claim appearing in the "proof" of this assertion that a weakly uniformly strict digital contraction mapping is digitally continuous.

Therefore, we must regard the assertions stated as Theorems 3.1 and 3.3 of [9] as unproven. Since these and assertions dependent on these make up all of the new assertions of the paper, we conclude that nothing new is correctly established in [9].

5. Common fixed points of intimate maps

The paper [21] obtains a result for common fixed points of intimate maps. We show in this section that the characterization of intimate maps given in [21] can be simplified, and that the primary result of [21] is rather limited.

Definition 5.1 ([21])**.** Let (X, d, κ) be a digital metric space. Let $f, g : X \to X$. Let α be either the *lim inf* or the *lim sup* operation. If for every $\{x_n\}_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t \tag{5.1}$$

for some $t \in X$ we have, for n sufficiently large,

$$\alpha\, d(g(f(x_n)), g(x_n)) \leq \alpha\, d(f(f(x_n)), f(x_n)) \tag{5.2}$$

then we say f is *g-intimate*.

Proposition 5.2. *Let (X, d, κ) be a digital metric space, where d is any ℓ_p metric. Let $f, g : X \to X$. Then f is g-intimate if and only if for every sequence $\{x_n\}_{n=1}^{\infty} \subset X$ satisfying statement (5.1) we have*

$$d(g(t), t) \leq d(f(t), t).$$

Proof. From Theorem 2.6, a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ satisfying statement (5.1) has, for n sufficiently large, $f(x_n) = g(x_n) = t$. The assertion follows easily. □

Theorem 5.3 ([21])**.** *If (X, d, κ) is a digital metric space and $A, B, S, T : X \to X$ are such that*

a) $S(X) \subset B(X)$ and $T(X) \subset A(X)$;

b) for some $\alpha \in (0, 1)$ and all $x, y \in X$,

$$d(S(x), T(y)) \leq \alpha\, F(x, y) \tag{5.3}$$

where

$$F(x, y) = \max\{d(A(x), B(y)), d(A(x), S(x)), d(B(y), T(y)), d(S(x), B(y)), d(A(x), T(y))\};$$

c) $A(X)$ is complete;

d) S is A-intimate and T is B-intimate, *then A, B, S, and T have a unique common fixed point.*

But Theorem 5.3 is limited, as shown by the following.

Proposition 5.4. *Suppose we assume the hypotheses of Theorem 5.3, with d being an ℓ_p metric. Suppose*

$$0 < \alpha < \inf\{1/F(x,y) \mid x,y \in X\} \tag{5.4}$$

or

$$diam(S(X) \cup T(X)) = diam\, X < \infty. \tag{5.5}$$

Then S and T are constant functions that have the same value, and, in case (5.5), X is a singleton.

Proof. Inequality (5.4) and hypothesis b) of Theorem 5.3 imply that $d(S(x),T(y)) < 1$, hence $S(x) = T(y)$ for all $x,y \in X$. Since d is an ℓ_p metric, we have, for some $x_0,y_0 \in X$, $S(x_0) = T(y)$ and $S(x) = T(y_0)$ for all $x,y \in X$. Hence S and T are constant functions with the same values.

Statement (5.5) implies there exist $x_0,y_0 \in X$ such that $d(S(x_0),T(y_0)) = diam\, X$. Since $F(x_0,y_0) \leq diam\, X$, inequality (5.3) becomes $diam\, X \leq \alpha\, diam\, X$, which implies $diam\, X = 0$. Thus, X is a singleton, so S and T are constant functions with the same values. □

6. Homotopy invariant fixed point theory

It is natural in topology to consider the behavior of the fixed point set when a continuous function is changed by homotopy. In classical topology for nice spaces (for example the geometric realization of any finite simplicial complex), when fixed points of a certain function exist, then there is a standard construction to change the function by homotopy to increase the number of fixed points. The more interesting question is whether or not the number of fixed points can be decreased by homotopy.

The following Proposition 6.1 was the key to the proof of Theorem 2.10.

Proposition 6.1 ([7])**.** *Let (X,κ) be a connected digital image of more than one point. Let $x_0 \leftrightarrow_\kappa x_1$ in X. Then the function $g : X \to X$ defined by*

$$g(x) = \begin{cases} x_0 & \text{if } x \neq x_0; \\ x_1 & \text{if } x = x_0, \end{cases}$$

is continuous and has no fixed points.

Proposition 6.2. *Let (X,κ) be a connected digital image of more than one point. Then any constant map $f : X \to X$ is homotopic to a map without fixed points.*

Proof. Let $x_0,x_1 \in X$ with $x_0 \leftrightarrow_\kappa x_1$. Let $f : X \to X$ be the constant map with image $\{x_0\}$. Let $H : X \times [0,1]_{\mathbb{Z}} \to X$ be defined by $H(x,0) = x_0$; $H(x,1) = x_0$ for $x \neq x_0$; $H(x_0,1) = x_1$. It is easy to see that H is a homotopy from f to a function g as in Proposition 6.1 without fixed points. □

Let $\mathrm{MF}(f)$ be the minimal number of fixed points among all continuous functions homotopic to f. For example, if X has only 1 point then clearly $\mathrm{MF}(f) = 1$. If X has more than 1 point and (X, κ) is contractible, then any continuous function $f : X \to X$ is homotopic to a constant function, and it follows from Proposition 6.2 that $\mathrm{MF}(f) = 0$.

Several examples are given in [15] of digital images (X, κ) for which no function on X is homotopic to the identity except for the identity itself. Such images are called *rigid*. For example, a wedge product of two loops, each having at least 5 points, is rigid. Clearly if X is a rigid digital image having n points and id denotes the identity function, then $\mathrm{MF}(\mathrm{id}) = |X|$.

In classical topology, $\mathrm{MF}(f)$ can often be computed by Nielsen fixed point theory; see [22]. Each fixed point is assigned an integer-valued *fixed point index*, which can be computed homologically. When x is an isolated fixed point of f, the fixed point index of x is denoted $\mathrm{ind}(f, x)$. The general definition of the index is complicated, but if the space is a smooth manifold, then f can be smoothed by homotopy so that $\mathrm{ind}(f, x) = \mathrm{sign}(\det |I - df_x|)$ where df_x is the derivative map and I is the identity matrix.

This index is a sort of multiplicity count for the fixed point: when $\mathrm{ind}(f, x) = 0$ then the fixed point at x can be removed by a homotopy. The fixed point index is homotopy invariant in the following sense: if, during some homotopy $f \simeq g$, the fixed point x of f moves into a fixed point y of g, then $\mathrm{ind}(f, x) = \mathrm{ind}(g, y)$. Furthermore, when the fixed point set of f is finite, then the sum of all the fixed point indices equals the Lefschetz number $L(f)$, which is the alternating sum of traces of the induced maps of f in the homology groups:

$$L(f) = \sum_{i=1}^{\infty} (-1)^i \,\mathrm{trace}(f_{*i} : H_i(X, \mathbb{Q}) \to H_i(X, \mathbb{Q})).$$

Since $\mathrm{ind}(f, x)$ sums to $L(f)$, it is often said that $\mathrm{ind}(f, x)$ is localized version of the Lefschetz number.

In Nielsen fixed point theory, the fixed points are grouped into *Nielsen classes*, and the number of such classes having nonzero index sum is the *Nielsen number* $N(f)$. This number is a homotopy invariant satisfying $N(f) \le \mathrm{MF}(f)$, and in many cases (for example when X is a manifold of dimension different from 2), $N(f) = \mathrm{MF}(f)$.

The 2012 paper [11] by Ege & Karaca attempts to develop a Lefschetz fixed point theorem for digital images, but the main result is incorrect, and was retracted in the 2016 paper [7]. The same authors attempted to develop a Nielsen theory in the 2017 paper [14] based on their faulty Lefschetz theory. The theory developed in [14] is also incorrect.

The main problem in [14] is inherited from problems in [11], and concerns the definition of the fixed point index. Definition 3.2 of [14] states the following.

> *Let (X, κ) be a digital image, $A \subset X$, and $f : A \to X$ a digital map. We define the* fixed point index of f *as $ind(f) = deg(F)$ where $F(x) = x - f(x)$ and $x \in X$.*

This does not give a satisfactory definition of the function F. It seems motivated by the classical fact that $\text{ind}(f, x) = \text{sign}(\det |I - df_x|)$, but the domain and range of F are never specified. The subtraction $x - f(x)$ is apparently performed in X, but $x - f(x)$ need not be a member of X.

Furthermore, the degree $deg(F)$ used is inadequate because the appropriate homology groups, as defined earlier in [14], are not necessarily isomorphic to $\mathbb{Z}$, which is a requirement for the definition of the degree of a function.

Ege & Karaca claim to define an integer valued Nielsen number $N(f)$ which is a homotopy invariant (Theorem 3.6) and a lower bound for $\text{MF}(f)$ (Theorem 3.7). Their Example 3.4, claiming that if f is a constant then $N(f) = 1$, yields a contradiction for connected images X such that $|X| > 1$, since our Proposition 6.2 implies that $MF(f) = 0$ for such functions f.

The errors in this work are not merely mistakes but indicate fundamental flaws in the theory. Anything resembling the standard homological definitions of $L(f)$ and the fixed point index will require that the Lefschetz number and fixed point index of the constant map equal 1. This cannot be reconciled with the fact that, when X has more than 1 point, the constant map can be changed by homotopy to have no fixed points.

The authors believe that any successful theory for computing $\text{MF}(f)$ will involve techniques very different from classical Lefschetz and Nielsen theory. The setting of digital images also allows the study of the quantity $\text{XF}(f)$, the maximum number of fixed points among all functions homotopic to f. In classical topological fixed point theory this number is typically infinite, but for a digital image with n points, clearly $\text{XF}(f) \leq n$ for any f. In fact our definition of $\text{XF}(f)$ implies that f is homotopic to the identity if and only if $\text{XF}(f) = n$. When f is a constant, then $1 \leq \text{XF}(f) \leq n$, and for many choices of the image X we will have $\text{XF}(f) < n$. We do not know if it is possible for $\text{XF}(f) = 0$ for any function on a connected digital image.

Although many of the concepts discussed in this paper turn out to be trivial or otherwise uninteresting, the questions of computing $\text{MF}(f)$ and $\text{XF}(f)$ seem to be difficult and interesting, and present opportunities for further work. Variations that count approximate fixed points would also be interesting objects of study.

7. Concluding remarks

Although the study of fixed points, or approximate fixed points, is important in digital topology as in other branches of mathematics, it does not appear that the use of metric spaces yields useful knowledge in this area. We have seen that metric space functions introduced to study fixed points in digital topology - digital contraction maps, Kannan contraction maps, Chatterjea contraction maps, Zamfirescu contraction maps, Rhoades contraction maps, Reich contraction maps, uniformly locally contractive functions, intimate functions - often turn out to be either discontinuous or constant - hence, arguably uninteresting - when the image considered is finite or when common metrics are used.

It appears to us that the most natural metric function to use for a connected digital image (X, κ) is the path length metric [17]: $d(x, y)$ is the length of a shortest κ-path from x to y. Since this metric reflects κ, it seems far superior to an ℓ_p metric on a digital image. However, even this metric gives us little new information. Since it is integer-valued, its Cauchy sequences are also eventually constant.

We have also corrected errors and pointed out trivialities in other papers concerned with fixed points or approximate fixed points of continuous self-maps of digital images.

Acknowledgements. *The authors are grateful to the anonymous reviewer for many suggestions and corrections.*

References

[1] S. Banach, Sur les operations dans les ensembles abstraits et leurs applications aux equations integrales, Fundamenta Mathematicae 3 (1922), 133–181.

[2] C. Berge, Graphs and Hypergraphs, 2nd edition, North-Holland, Amsterdam, 1976.

[3] L. Boxer, Digitally continuous functions, Pattern Recognition Letters 15 (1994), 833–839.

[4] L. Boxer, A classical construction for the digital fundamental group, Journal of Mathematical Imaging and Vision 10 (1999), 51–62.

[5] L. Boxer, Generalized normal product adjacency in digital topology, Applied General Topology 18, no. 2 (2017), 401–427.

[6] L. Boxer, Alternate product adjacencies in digital topology, Applied General Topology 19, no. 1 (2018), 21–53.

[7] L. Boxer, O. Ege, I. Karaca, J. Lopez and J. Louwsma, Digital fixed points, approximate fixed points, and universal functions, Applied General Topology 17, no. 2 (2016), 159–172.

[8] S. K. Chatterjea, Fixed point theorems, Comptes rendus de l'Académie bulgare des Sciences 25 (1972), 727–730.

[9] U. P. Dolhare and V. V. Nalawade, Fixed point theorems in digital images and applications to fractal image compression, Asian Journal of Mathematics and Computer Research 25, no. 1 (2018), 18–37.

[10] M. Edelstein, An extension of Banach's contraction principle, Proceedings of the American Mathematical Society 12, no. 1 (1961), 7–10.

[11] O. Ege and I. Karaca, The Lefschetz fixed point theorem for digital images, Fixed Point Theory and Applications 2013:253, 2013.

[12] O. Ege and I. Karaca, Banach fixed point theorem for digital images, Journal of Nonlinear Sciences and Applications, 8 (2015), 237–245.

[13] O. Ege and I. Karaca, Digital homotopy fixed point theory, Comptes Rendus Mathematique 353, no. 11 (2015), 1029–1033.

[14] O. Ege and I. Karaca, Nielsen fixed point theory for digital images, Journal of Computational Analysis and Applications 22, no. 5 (2017), 874–880.

[15] J. Haarmann, M. P. Murphy, C. S. Peters and P. C. Staecker, Homotopy equivalence of finite digital images, Journal of Mathematical Imaging and Vision 53, no. 3 (2015), 288–302.

[16] G. Herman, Oriented surfaces in digital spaces, CVGIP: Graphical Models and Image Processing 55 (1993), 381–396.
[17] S.-E. Han, Non-product property of the digital fundamental group, Information Sciences 171 (2005), 73û-91.
[18] S.-E. Han, Banach fixed point theorem from the viewpoint of digital topology, Journal of Nonlinear Science and Applications 9 (2016), 895–905.
[19] S.-E. Han, The fixed point property of an M-retract and its applications, Topology and its Applications 230 (2017), 139–153.
[20] A. Hossain, R. Ferdausi, S. Mondal, and H. Rashid, Banach and Edelstein fixed point theorems for digital images, Journal of Mathematical Sciences and Applications 5, no. 2 (2017), 36–39.
[21] D. Jain, Common fixed point theorem for intimate mappings in digital metric spaces, International Journal of Mathematics Trends and Technology 56, no. 2 (2018), 91–94.
[22] B. Jiang, Lectures on Nielsen fixed point theory, Contemporary Mathematics 18 (1983).
[23] K. Jyoti and A. Rani, Digital expansions endowed with fixed point theory, Turkish Journal of Analysis and Number Theory 5, no. 5 (2017), 146–152.
[24] K. Jyoti and A. Rani, Fixed point theorems for $\beta - \psi - \phi$-expansive type mappings in digital metric spaces, Asian Journal of Mathematics and Computer Research 24, no. 2 (2018), 56–66.
[25] R. Kannan, Some results on fixed points, Bulletin of the Calcutta Mathematical Society 60 (1968), 71–76.
[26] L. N. Mishra, K. Jyoti, A. Rani and Vandana, Fixed point theorems with digital contractions image processing, Nonlinear Science Letters A 9, no. 2 (2018), 104–115.
[27] C. Park, O. Ege, S. Kumar, D. Jain and J. R. Lee, Fixed point theorems for various contraction conditions in digital metric spaces, Journal of Computational Analysis and Applications 26, no. 8 (2019), 1451–1458.
[28] S. Reich, Some remarks concerning contraction mappings, Canadian Mathematical Bulletin 14 (1971), 121–124.
[29] B. E. Rhoades, Fixed point theorems and stability results for fixed point iteration procedures, II, Indian Journal of Pure and Applied Mathematics 24, no. 11 (1993), 691–703.
[30] A. Rosenfeld, 'Continuous' functions on digital images, Pattern Recognition Letters 4 (1986), 177–184.
[31] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for $\alpha - \psi$-contractive mappings, Nonlinear Analysis: Theory, Methods & Applications 75, no. 4 (2012), 2154–2165.
[32] T. Zamfirescu, Fixed point theorems in metric spaces, Archiv der Mathematik 23 (1972), 292–298.