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Orbitally discrete coarse spaces

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Abstract

Given a coarse space $(X, \mathcal{E})$, we endow X with the discrete topology and denote $X^\sharp = \{p \in \beta G$: each member $P \in p$ is unbounded $\}$. For $p, q \in X^\sharp$, $p||q$ means that there exists an entourage $E \in \mathcal{E}$ such that $E[P] \in q$ for each $P \in p$. We say that $(X, \mathcal{E})$ is orbitally discrete if, for every $p \in X^\sharp$, the orbit $\overline{\overline{p}} = \{q \in X^\sharp : p||q\}$ is discrete in βG. We prove that every orbitally discrete space is almost finitary and scattered.

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1. Introduction and preiminaries

Given a set X, a family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* on X if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X = \{(x, x) \in X : x \in X\}$;
- if E, $E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x, y) : \exists z((x, z) \in E, \ (z, y) \in E')\}$, $E^{-1} = \{(y, x) : (x, y) \in E\}$;
- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$;
- $\bigcup \mathcal{E} = X \times X$.

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A subfamily $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* for $\mathcal{E}$ if, for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $E \subseteq E'$. For $x \in X$, $A \subseteq X$ and $E \in \mathcal{E}$, we denote

$$E[x] = \{y \in X : (x,y) \in E\},\ E[A] = \bigcup_{a \in A} E[a], \quad E_A[x] = E[x] \cap A$$

and say that $E[x]$ and $E[A]$ are *balls of radius E around x* and A.

The pair $(X, \mathcal{E})$ is called a *coarse space* [19] or a ballean [12], [18].

For a coarse space $(X, \mathcal{E})$, a subset $B \subseteq X$ is called *bounded* if $B \subseteq E[x]$ for some $E \in \mathcal{E}$ and $x \in X$. The family $\mathcal{B}_{(X,\mathcal{E})}$ of all bounded subsets of $(X, \mathcal{E})$ is called the bornology of $(X, \mathcal{E})$. We recall that a family $\mathcal{B}$ of subsets of a set X is a *bornology* if $\mathcal{B}$ is closed under taking subsets and finite unions, and $\mathcal{B}$ contains all finite subsets of X.

A coarse space $(X, \mathcal{E})$ is called *finitary,* if for each $E \in \mathcal{E}$ there exists a natural number n such that $|E[x]| < n$ for each $x \in X$.

Let G be a transitive group of permutations of a set X. We denote by X_G the set X endowed with the coarse structure with the base

$$\{\{(x, gx) : g \in F\}\} : F \in [G]^{<\omega},\ id \in F\}.$$

By [8, Theorem 1], for every finitary coarse structure $(X, \mathcal{E})$, there exists a transitive group G of permutations of X such that $(X, \mathcal{E}) = X_G$. For more general results, see [10].

Let X be a discrete space and let βX denote the *Stone-Čech compactification* of X. We take the points of βX to be the ultrafilters on X, with the points of X identified with the principal ultrafilters, so $X^* = \beta X \setminus X$ is the set of all free ultrafilters. The topology of βX is generated by the base consisting of the sets $\bar{A} = \{p \in \beta X : A \in p\}$, where $A \subseteq X$. The universal property of βX states that every mapping $f : X \longrightarrow Y$ to a compact Hausdorff space Y can be extended to a continuous mapping $f^\beta : \beta X \longrightarrow X$.

Given a coarse space $(X, \mathcal{E})$, we endow X with the discrete topology and denote by $X^\sharp$ the set of all ultrafilters p on X such that each member $P \in p$ is unbounded. Clearly, $X^\sharp$ is a closed subset of X^* and $X^\sharp = X^*$ if $(X, \mathcal{E})$ is finitary.

Following [7], we say that two ultrafilters $p, q \in X^\sharp$ are *parallel* (and write $p||q$) if there exists $E \in \mathcal{E}$ such that $E[P] \in q$ for each $P \in p$. Then $||$ is an equivalence on $X^\sharp$. We denote

$$\overline{\overline{p}} = \{q \in X^\sharp : q||p\}$$

and say that $\overline{\overline{p}}$ is the *orbit* of p. If $(X, \mathcal{E})$ is finitary and $(X, \mathcal{E}) = X_G$ then $\overline{\overline{p}} = Gp$.

A coarse space $(X, \mathcal{E})$ is called *orbitally discrete* if, for every $p \in X^\sharp$, the orbit $\overline{\overline{p}}$ is discrete. Every discrete coarse space is orbitally discrete. We recall that $(X, \mathcal{E})$ is *discrete* if, for each $E \in \mathcal{E}$, there exists a bounded subset B such that $E[x] = \{x\}$ for each $x \in X \setminus B$. In this case, $\overline{\overline{p}} = \{p\}$ for each $p \in X^\sharp$.

Every bornology $\mathcal{B}$ on a set X defines the discrete coarse structure on X with the base $\{E_B : B \in \mathcal{B}\}$, $E_B[x] = B$ if $x \in B$, and $E_B[x] = \{x\}$ if $x \in X \setminus B$.

By [15, Theorem 5.4], for a finitary coarse space $(X, \mathcal{E})$, the following conditions are equivalent: X_G is orbitally discrete, X_G is scattered, X_G has no piecewise shifted FP-sets.

A coarse space $(X, \mathcal{E})$ is called *scattered* if, for every unbounded subset A of X, there exists $E \in \mathcal{E}$ such that A has asymptotically E-isolated balls: for each $E' \in \mathcal{E}$, there is $a \in A$ such that $E'_A[a] \setminus E_A[a] = \varnothing$.

This notion arouse in the characterization of the Cantor macrocube [3] and, in the case of finitary coarse groups, was explored in [2].

Let G be a group of permutations of a set X. Let $(g_n)_{n\in\omega}$ be a sequence in G and let $(x_n)_{n\in\omega}$ be a sequence in X such that

(1) $\{g_0^{\epsilon_0} \dots g_n^{\epsilon_n}\ x_n : (\epsilon_i)_{i=0}^n \in \{0,1\}^{n+1}\} \cap \{g_0^{\epsilon_0} \dots g_m^{\epsilon_m}\ x_m : (\epsilon_i)_{i=0}^n \in \{0,1\}^{n+1}\} = \varnothing$ for all distinct $n, m \in \omega$;
(2) $\{g_0^{\epsilon_0} \dots g_n^{\epsilon_n}\ x_n : (\epsilon_i)_{i=0}^n \in \{0,1\}^{n+1}\}| = 2^{n+1}$ for every $n \in \omega$.

Following [15], we say that a subset Y of X is a *piecewise shifted FP-set* if there exist $(g_n)_{n\in\omega}$, $(x_n)_{n\in\omega}$ satisfying (1), (2) and such that

$$Y = \{g_0^{\epsilon_0} \dots g_n^{\epsilon_n}\ x_n : \epsilon_i \in \{0,1\}\},\ n \in \omega\}.$$

After exposition of results in Section 2, we survey some known classes of orbitally discrete spaces in Section 3.

2. Resuts

A coarse space $(X, \mathcal{E})$ is called *almost finitary* if, for every $E \in \mathcal{E}$, there exists a bounded subset B and a natural number n such that $|E[x]| < n$ for each $x \in X \setminus B$. Every discrete space and every finitary space are almost finitary.

Theorem 2.1. *Every orbitally discrete coarse space is almost finitary.*

Proof. We suppose the contrary and choose $E \in \mathcal{E}$, $E = E^{-1}$ such that, for any bounded subset B and a natural number n, there exists $x \in X \setminus B$ such that $|E[x]| > n$.

We claim that there exists $p \in X^\sharp$ such that, for every $P \in p$, $\{x \in P : |E^2[x] \cap P| > 1\} \in p$. Otherwise, for every $p \in X^\sharp$, there exists $Q_p \in p$ such that $\{x \in Q_p : E^2[x] \cap Q_p| = 1\} \in p$. We consider the open covering $\{Q_p^\sharp : p \in X^\sharp\}$ of $X^\sharp$ and choose its finite subcovering $Q_{p_1}^\sharp, \dots, Q_{p_m}^\sharp$. Then the set $B = X \setminus (Q_{p_1} \cup, \dots, \cup Q_{p_m})$ is bounded and $|E[x]| \leq m$ for each $x \in X \setminus E[B]$, but this contradicts the choice of E.

We show that the orbit $\overline{\overline{p}}$ is not discrete. Given any $P \in p$, we choose $Q \in p$, $Q \subseteq P$ such that $|E^2[x] \cap P| > 1$ for each $x \in Q$. For every $x \in Q$, we take $f(x) \in E^2[x] \cap P$ such that $x \neq f(x)$. Then we extend the mapping $x \mapsto f(x)$

from Q to X by $f(x) = x$ for each $x \in X \setminus Q$. Clearly, $f^\beta(p) \neq p$, $P \in f^\beta(p)$ and $f^\beta(p)||p$ because $(x, f(x)) \in E^2$ for each $x \in X$. □

To clarify the structure of an almost finitary coarse space, we use the following construction from [6]. A bornology $\mathcal{B}$ on a coarse space $(X, \mathcal{E})$ is called *$\mathcal{E}$-compatible* if $E[B] \in \mathcal{B}$ for all $B \in \mathcal{B}$, $E \in \mathcal{E}$. Every $\mathcal{E}$-compatible bornology $\mathcal{B}$ defines the *$\mathcal{B}$-strengthening* $(X, \mathcal{H})$ of $(X, \mathcal{E})$, where $\mathcal{H}$ has the base

$$\{H_{B,E} : B \in \mathcal{B},\ E \in \mathcal{E}\},$$

$$H_{B,E}[x] = \begin{cases} E[B], & \text{if } x \in B, \\ E[x], & \text{if } x \in X \setminus B. \end{cases}$$

For description of the upper bound $\mathcal{E} \vee \mathcal{E}'$ of coarse structures, see [13].

Theorem 2.2. *For a coarse space $(X, \mathcal{E})$, the following statements are equivalent*

- (i) $(X, \mathcal{E})$ *is almost finitary;*
- (ii) $(X, \mathcal{E})$ *is the $\mathcal{B}$-stregthening of some finitary coarse space $(X, \mathcal{E}')$ by the bornology $\mathcal{B}$ of bounded subspaces of $(X, \mathcal{E})$;*
- (iii) $\mathcal{E}$ *is the upper bound of a discrete and a finitary coarse structures on X.*

Proof. $(i) \Longrightarrow (ii)$. For $B \in \mathcal{B}$ and $E \in \mathcal{E}$, we pick $B'_{B,E} \in \mathcal{B}$ and a natural number n such that $B \subseteq B'_{B,E}$ and $|E[x]| < n$ for each $x \in X \setminus B'_{B,E}$. We note that $\{B'_{B,E} : B \in \mathcal{B},\ E \in \mathcal{E}\}$ is a base for $\mathcal{B}$. For $B \in \mathcal{B}$, $E \in \mathcal{E}$ we put

$$E'_{B,E} = \begin{cases} x & \text{if } x \in B'_{B,E}, \\ E[x] & \text{if } x \in X \setminus B'_{B,E}, \end{cases}$$

denote by $\mathcal{E}'$ the smallest coarse structure on X containing all entourages $\{H_{B,E} : B \in \mathcal{B},\ E \in \mathcal{E}\}$, observe that $\mathcal{E}'$ is finitary and $(X, \mathcal{E})$ is the $\mathcal{B}$-strengthening of $(X, \mathcal{E}')$.

$(ii) \Longrightarrow (iii)$. If $(X, \mathcal{E})$ is the $\mathcal{B}$-strengthening of $(X, \mathcal{E}')$ then $\mathcal{E}$ is the upper bounded of $\mathcal{E}'$ and the discrete coarse structure on X defined by the bornology $\mathcal{B}$.

$(iii) \Longrightarrow (i)$. We assume that $\mathcal{E}$ is the upper bound of finitary coarse structure $\mathcal{E}'$ and discrete coarse structure on X defined by some bornology $\mathcal{B}$. We choose the smallest bornology $\mathcal{B}'$ on X such that $\mathcal{B} \subseteq \mathcal{B}'$ and $E'(B') \in \mathcal{B}'$ for all $E' \in \mathcal{E}'$. Then $\mathcal{B}'$ is the bornology of bounded subsets of $(X, \mathcal{E})$ and $(X, \mathcal{E})$ is the $\mathcal{B}'$-strengthening of $(X, \mathcal{E}')$, so $(X, \mathcal{E})$ is almost finitary. □

Remark. Let $(X, \mathcal{E})$ be the $\mathcal{B}$-strengthening of a finitary coarse space $(X, \mathcal{E}')$. If $(X, \mathcal{E}')$ is orbitally discrete then $(X, \mathcal{E})$ is orbitally discrete, but the converse statement needs not to be true. Let X be the disjoint union of two infinite subsets Y, Z. We endow Y with the finitary coarse structure $\mathcal{E}_Y$ such that $(Y, \mathcal{E}_Y)$ is not orbitaly discrete, and denote by $\mathcal{E}_Z$ the discrete coarse structure on Z defined by the bornology of finite subset. We take the smallest coarse structure $\mathcal{E}'$ on X such that $\mathcal{E}'|_Y = \mathcal{E}_Y$, $\mathcal{E}'|_Z = \mathcal{E}_Z$. Clearly, $\mathcal{E}'$ is finitary but not orbitally discrete. We denote by $\mathcal{B}$ the smallest bornology on X such that $Y \in \mathcal{B}$. Then the $\mathcal{B}$-strengthening of $(X, \mathcal{E}')$ is discrete.

Theorem 2.3. *For almost finitary coarse space $(X, \mathcal{E})$ and $p, q \in X^\sharp$, we have $p||q$ if and only if there exist $E \in \mathcal{E}$ and a permutation g of X such that $gp = q$, $gp = \{gP : P \in p\}$ and $(x, gx) \in E$ for each $x \in X$.*

Proof. Let $p||q$. We take $E \in \mathcal{E}$ such that $E = E^{-1}$ and $E[P] \in q$ for each $P \in p$. Since $(X, \mathcal{E})$ is almost finitary, there exist a bounded subset B of X and a natural number n such that $|E[x]| < n$ for each $x \in X \setminus B$. We put $Y = X \setminus E[B]$, note that $Y \in p$ and define a set-valued mapping $\mathcal{F} : X \longrightarrow [x]^{<\omega}$. $\mathcal{F}(x) = E[x]$ if $x \in Y$ and $\mathcal{F}(x) = \{x\}$ if $x = X \setminus Y$. By Theorem 1 from [10], there exists bijection $f_1, \dots, f_m$ of X such that $f_i(x) \in \mathcal{F}(x)$ and $f_1(x) \cup \dots \cup f_n(x) = \mathcal{F}(x)$. We take $i \in \{1, \dots, m\}$ such that $f_i(P) \in q$ for each $P \in p$ and put $g = f_i$.

The converse statement follows directly from the definition of the parallelity relation $||$. □

Corollary 2.4. *If $(X, \mathcal{E})$ is almost finitary, $p \in X^\sharp$ and p is an isolated point of $\overline{\overline{p}}$ then $\overline{\overline{p}}$ is discrete.*

Proof. We assume that some point $q \in \overline{\overline{p}}$ is not isolated in $\overline{\overline{p}}$, use Theorem 2.3 to choose a permutation g of X such that $gq = p$ and note that p is not isolated in $\overline{\overline{p}}$. □

For a subset A of $(X, \mathcal{E})$ and $p \in X^\sharp$, we denote $\Delta_p(A) = \overline{\overline{p}} \cap A^\sharp$.

Theorem 2.5. *An almost finitary coarse space $(X, \mathcal{E})$ is scattered if and only if, for every unbounded subset A of X, there exists $p \in A$ such that $\Delta_p(A)$ is finite.*

Proof. We suppose that X is scattered and choose $E \in \mathcal{E}$ such that A has an asymptotically isolated E-balls. For each $H \in \mathcal{E}$, we denote $P_H = \{x \in A : H_A[x] \setminus E_A[x] = \varnothing\}$ and take $p \in A^\sharp$ such that $P_H \in p$ for each $H \in \mathcal{E}$. If $q \in A^\sharp$ and $q||p$ then $E[P] \in q$ for each $P \in p$. We take the bijections $f_1, \dots, f_m$ from the proof of Theorem 2.3. Since $q = gp$ for some $g \in \{f_1, \dots, f_m\}$, we have $\Delta_p(A) \le m$.

Let $\Delta_p(A) = \{p_1, \dots, p_m\}$. For each $i \in \{1, \dots, m\}$, we pick $E_i \in \mathcal{E}$ such that $E_i[P] \in p_i$ for each $P \in p$. Then we take $E \in \mathcal{E}$ such that $E_i \subseteq E$ for each $i \in \{1, \dots, m\}$, and observe that A has an asymptotically isolated E-balls. □

Theorem 2.6. *Every orbitally discrete space is scattered.*

Proof. To apply Theorem 2.5, we take an arbitrary unbounded subset A of X and find $p \in A^\sharp$ such that $\Delta_p(A)$ is finite.

We use the Zorn lemma to choose a minimal (by inclusion) closed subset S of $A^\sharp$ such that $\Delta_q(A) \subseteq S$ for each $q \in S$. Let $p \in S$ but $\Delta_p(A)$ is infinite. We take the limit point q of $\Delta_p(A)$. By the minimality of S, we have $p \in cl\Delta_q(A)$. Applying Theorem 2.3, we conclude that p is not isolated in $\overline{\overline{p}}$. □

Question. *Let X be an almost finitary scattered space. Is X orbitally discrete?*

3. Comments

1. For a natural number n, a coarse space $(X, \mathcal{E})$ is called *n-thin* if, for every $E \in \mathcal{E}$, there exists a bounded subset B of X such that $|E[x]| \leq n$, for every $x \in X \setminus B$. A space $(X, \mathcal{E})$ is n-thin if and only if $|\overline{\overline{p}}| \leq n$ for each $p \in X^\sharp$.

   For finite partitions of an n-thin space into discrete subspaces, see [5], [14], [17], [1, Section 6].
2. A coarse space $(X, \mathcal{E})$ is called *sparse* if each orbit $\overline{\overline{p}}$, $p \in X^\sharp$ is finite. Sparse subsets of groups are studied in [4], [16]. For sparse metric spaces, see [9].
3. A coarse space $(X, \mathcal{E})$ is called *indiscrete* if each discrete subspace of X is bounded. By Theorem 3.15 from [11], a finitary indiscrete space has no unbounded orbitally discrete subspaces. We do not know whether this statement holds for any almost finitary indiscrete spaces.

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