APPLIED GENERAL TOPOLOGY

*Appl. Gen. Topol. 22, no. 1 (2021), 169-181*
*doi:10.4995/agt.2021.14446*


# Metric spaces related to Abelian groups

AMIR VEISI [a] AND ALI DELBAZNASAB [b]

[a] Faculty of Petroleum and Gas, Yasouj University, Gachsaran, Iran (aveisi@yu.ac.ir)
[b] Department of Mathematics, Shahid Bahonar University, Kerman, Iran (delbaznasab@gmail.com)

*Communicated by F. Lin*

ABSTRACT

*When working with a metric space, we are dealing with the additive group $(\mathbb{R}, +)$. Replacing $(\mathbb{R}, +)$ with an Abelian group $(G, *)$, offers a new structure of a metric space. We call it a G-metric space and the induced topology is called the G-metric topology. In this paper, we are studying G-metric spaces based on L-groups (i.e., partially ordered groups which are lattices). Some results in G-metric spaces are obtained. The G-metric topology is defined which is further studied for its topological properties. We prove that if G is a densely ordered group or an infinite cyclic group, then every G-metric space is Hausdorff. It is shown that if G is a Dedekind-complete densely ordered group, $(X, d)$ a G-metric space, $A \subseteq X$ and d is bounded, then $f : X \to G$ with $f(x) = d(x, A) := \inf\{d(x, a) : a \in A\}$ is continuous and further $x \in cl_X A$ if and only if $f(x) = e$ (the identity element in G). Moreover, we show that if G is a densely ordered group and further a closed subset of $\mathbb{R}$, $\mathcal{K}(X)$ is the family of nonempty compact subsets of X, $e < g \in G$ and d is bounded, then $d'(A, B) < g$ if and only if $A \subseteq N_d(B, g)$ and $B \subseteq N_d(A, g)$, where $N_d(A, g) = \{x \in X : d(x, A) < g\}$, $d_B(A) = \sup\{d(a, B) : a \in A\}$ and $d'(A, B) = \sup\{d_A(B), d_B(A)\}$.*

2010 MSC: *54C40; 06F20; 16H20.*

KEYWORDS: *G-metric space; L-group; Dedekind-complete group; densely ordered group; continuity.*

*Received 08 October 2020 – Accepted 29 January 2021*

## 1. Introduction

In this article, a group $(G, *)$ (briefly, $G$) is an Abelian group and for readability, we use $g_1g_2$ instead of $g_1 * g_2$. Let $X$ be a set and $\leq$ relation on $X$, we recall that the pair $(X, \leq)$ is a *partially ordered set* (in brief, a *poset*) if the following conditions hold: $x \leq x$, if $x \leq y$ and $y \leq x$, then $x = y$; if $x \leq y$ and $y \leq z$, then $x \leq z$. In a poset, the symbol $a \vee b$ denotes $\sup\{a, b\}$, i.e., the smallest element $c$, if one exists, such that $c \geq a$ and $c \geq b$. Likewise, $a \wedge b$ stands for $\inf\{a, b\}$. When both $a \vee b$ and $a \wedge b$ exist, for all $a, b \in A$, then $A$ is called a *lattice*. A subset $S$ is a *sublattice* of $A$ provided that, for all $x, y \in S$, the elements $x \vee y$ and $x \wedge y$ of $A$ belong to $S$. (Thus, it is not enough that $x$ and $y$ have a supremum and infimum in $S$.) For instance, $C(X)$, the ring of real-valued continuous functions on the topological space $X$ is a lattice. If $f, g \in C(X)$, then $f \vee g = \frac{f+g+|f-g|}{2} \in C(X)$ (note, $f \wedge g = -(-f \vee -g) = \frac{f+g-|f-g|}{2} \in C(X)$). In fact, $C(X)$ is a sublattice of $\mathbb{R}^X$, the ring of real-valued functions on the set $X$ (note, the partial ordering on $\mathbb{R}^X$ is: $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x$ in $X$). A poset in which every nonempty subset has both a supremum and an infimum is said to be a *lattice-complete*. For example, $P(X)$, the set of all subsets of $X$ with inclusion is lattice-complete. Union (resp. intersection) of sets is the supremum (resp. the infimum) of them. A *totally (*or *linearly) ordered set* is a poset in which every pair of elements is comparable, i.e., $x \leq y$ or $y \leq x$ for all $x$ and $y$ in $X$. We use "ordered sets" instead of "totally ordered sets". An ordered set is often referred to as a chain. A lattice need not be an ordered set, necessarily, but the converse is always true. We notice that $C(X)$ and $C_c(X)$, its subalgebra consisting of elements with countable image, are lattices, while they are not ordered sets, also, they are not lattice-complete necessarily. An ordered set is said to be *Dedekind-complete* provided that every nonempty subset with an upper bound has a supremum, or equivalently, every nonempty subset with a lower bound has an infimum. (For example, $\mathbb{R}$, the set of real numbers is Dedekind-complete, but not lattice-complete.) An ordered field $\mathbb{F}$ is said to be *archimedean* if $\mathbb{Z}$, the set of integers is *cofinal*, i.e., for every $x \in \mathbb{F}$, there exists $n \in \mathbb{Z}$ such that $n \geq x$. For instance, $\mathbb{Q}(\sqrt{n}) := \{a + b\sqrt{n} : a, b \in \mathbb{Q}, \sqrt{n} \notin \mathbb{Q}\}$ is an archimedean field.

**Theorem 1.1** ([3, Theorem 0.21])**.** *An ordered field is archimedean if and only if it is isomorphic to a subfield of the ordered field* $\mathbb{R}$.

A brief outline of this paper is as follows. In Section 2, we introduce the $G$-metric spaces related to $L$-groups (i.e., partially ordered groups which are lattices) and study them further. In Section 3, the basic topological properties based on the notion of $g$-disk are studied. We prove that if $G$ is a densely ordered group or an infinite cyclic group, then every $G$-metric space is Hausdorff. It is shown that if $G$ is a Dedekind-complete densely ordered group, $(X, d)$ a $G$-metric space, $d$ is bounded and $A \subseteq X$, then $f : X \to G$ given by $f(x) = d(x, A) := \inf\{d(x, a) : a \in A\}$ is continuous and further $x \in \mathrm{cl}_X A$ if and only if $f(x) = e$ (the identity element of $G$). Moreover, let

$\mathcal{F}(X)$ be the family of nonempty closed sets in $X$, $e < g \in G$, $A, B \in \mathcal{F}(X)$ and $d_A(B) = \sup\{d(b, A) : b \in B\}$. Then for the $G$-metric space $(\mathcal{F}(X), d')$ (note, $d'(A, B) = \sup\{d_A(B), d_B(A)\}$), we have $d'(A, B) \leq g$ if and only if $A \subseteq N_d(B, \bar{g})$ and $B \subseteq N_d(A, \bar{g})$, where $N_d(A, \bar{g}) = \{x \in X : d(x, A) \leq g\}$. Particularly, if $G$ is a densely ordered group and further a closed subset of $\mathbb{R}$, $X$ is a $G$-metric space and $\mathcal{K}(X)$ is the family of nonempty compact sets in $X$, then $d'(A, B) < g$ if and only if $A \subseteq N_d(B, g)$ and $B \subseteq N_d(A, g)$, where $N_d(A, g) = \{x \in X : d(x, A) < g\}$.

## 2. $G$-metric spaces

**Definition 2.1.** A group $G$ with a partial ordering relation $\leq$ is called a *partially ordered group* (in brief, a *poset group*) if the binary operation of $G$ preserves the order, i.e.,

$$g_1 \geq g_2 \text{ implies } g_1 g_3 \geq g_2 g_3 \text{ for all } g_1, g_2, g_3 \in G. \tag{$R_1$}$$

Moreover, if a poset group $G$ is a lattice then $G$ is called an *$L$-group*.

From the above definition, the following facts are evident: $g_1 \geq g_2$ if and only if $g_1 g_2^{-1} \geq e$; $g \geq e$ if and only if $g^{-1} \leq e$; if $g_1 \geq g_3$ and $g_2 \geq g_4$, then $g_1 g_2 \geq g_3 g_4$. For example, every archimedean field with the addition is an $L$-group. But $\mathbb{Z}_n$ with the addition of modulo $n$ is not a poset group yet, since this addition does not preserve the order. For an $L$-group $G$ and $g \in G$, we let $|g| = \sup\{g, g^{-1}\} = g \vee g^{-1} = |g^{-1}|$.

**Example 2.2.** Consider the group $G := \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}$ ($k$-times) with the usual addition and the identity element $e = (0, 0, \ldots, 0)$. Let $g_1 = (m_1, m_2, \ldots, m_k)$, $g_2 = (n_1, n_2, \ldots, n_k) \in G$. Define

$$g_1 \leq g_2 \text{ if and only if } m_i \leq n_i \text{ for all } i = 1, 2, \ldots, k.$$

We see that $\leq$ is a partial ordering relation on $G$. Also, the condition $(R_1)$ in the above definition is satisfied, i.e., $G$ is a poset group. Let $z_i = \max\{m_i, n_i\}$ and $z'_i = \min\{m_i, n_i\}$, where $i = 1, 2, \ldots, k$. Let $g_3 = (z_1, z_2, \ldots, z_k)$ and $g_4 = (z'_1, z'_2, \ldots, z'_k)$. Then we obtain $g_1 \vee g_2 = g_3$ and $g_1 \wedge g_2 = g_4$. Hence, $G$ is an $L$-group.

By an *ordered group*, we mean a poset group which is a totally ordered set by its partial ordering relation. It is clear that an ordered group is an $L$-group. Finally, by *Dedekind-complete group*, we mean an ordered group which is a Dedekind-complete set with its partial ordering relation, i.e., every nonempty subset with an upper bound has a supremum, or equivalently, every nonempty subset with a lower bound has an infimum. For example, every archimedean field with the addition is a Dedekind-complete group.

**Corollary 2.3.** *If $G$ is an $L$-group and $g_1, g_2 \in G$, then $|g_1 g_2| \leq |g_1||g_2|$.*

*Proof.* We note that $g_1, g_1^{-1} \leq |g_1|$ and $g_2, g_2^{-1} \leq |g_2|$. Definition 2.1 now gives $g_1 g_2 \leq |g_1||g_2|$ and also $g_1^{-1} g_2^{-1} \leq |g_1||g_2|$. So we have that $|g_1 g_2| = \sup\{g_1 g_2, (g_1 g_2)^{-1}\} \leq |g_1 g_2|$, and the result holds. □

**Definition 2.4.** Let $G$ be a poset group and $X$ a nonempty set. We say the function $d : X \times X \to G$ is a *$G$-metric* on $X$, whenever the following conditions hold, for every $x, y, z \in X$.

(i) $d(x, y) \geq e$ ($e$ is the identity element in $G$),
(ii) $d(x, y) = e$ if and only if $x = y$,
(iii) $d(x, y) = d(y, x)$,
(iv) $d(x, y) \leq d(x, z)d(z, y)$ (*triangle inequality*).

The pair $(X, d)$ (briefly, $X$) is called a *$G$-metric space*. Evidently, every metric space is a $G$-metric space, when $(G, *) = (\mathbb{R}, +)$. If all axioms but the second part of Definition 2.4 are satisfied, we call $d$ a *$G$-pseudometric* and then $X$ a *$G$-pseudometric space*. Defining $d(x, y) = e$ for all $x$ and $y$ in $X$, gives a $G$-pseudometric on $X$, called the trivial $G$-pseudometric, in this case, $d$ is a $G$-metric if and only if $X$ is the singleton set $\{x\}$. Although all the material of this section is developed for $G$-metric spaces, the basic results remain true for $G$-pseudometric spaces as well. If $(X, d)$ is a $G$-metric on $X$ and $A$ is a subset of $X$, then $A$ inherits a $G$-metric structure from $X$ in an obvious way, making $A$ a $G$-metric space.

In the following example, we will present some examples of $G$-metric spaces.

Before it, let $X = \mathbb{R}^n$, $G_1 = (\mathbb{R}, +), G_2 = ((0, +\infty), .), G_3 = (\mathbb{R} - \{-1\}, *)$ and $G_4 = (\mathbb{Z}_2, \oplus)$, where $+, .$ are usual addition and multiplication, the symbol $\oplus$ is the addition of modulo 2 and $*$ is defined as follows: $x * y = x + y + xy$. In $G_3$, the identity element is 0 and the inverse of $x$ is $x^{-1} = \frac{-x}{1+x}$. Checking of the associative property of $*$ is easy.
Moreover, let $\varphi_i : G_i \times G_i \to G_i$, where $i = 1, 2, 3, 4$, such that $\varphi_1(x, y) = x - y$, $\varphi_2(x, y) = \frac{x}{y}$, $\varphi_3(x, y) = \frac{-xy}{1+x}$ and $\varphi_4(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$ Then, since each $\varphi_i$ is continuous, each $G_i$ is a topological group as subspaces of $\mathbb{R}$ with the usual topology.

**Example 2.5.** Let $X = \mathbb{R}^n$, $G_i$, where $i = 1, 2, 3, 4$, be as defined in the previous discussion. For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in X$, $\|x - y\|$ is the usual norm, i.e., $\|x - y\| = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}}$. We claim that each $d_i$, the functions below, is a $G_i$-metric and therefore $(X, d_i)$ is a $G_i$-metric space. We only check that $d_2$ and $d_3$ satisfy (iv) of Definition 2.4. Other conditions are routine.

(1) Let $d_1 : X \times X \to G_1$ such that $d_1(x, y) = \|x - y\|$.
(2) Let $d_2 : X \times X \to G_2$ such that $d_2(x, y) = e^{\|x-y\|}$.
(3) Let $d_3 : X \times X \to G_3$ such that $d_3(x, y) = e^{\|x-y\|} - 1$.
(4) Let $d_4 : X \times X \to G_4$ such that $d(x, y) = 0$ if $x = y$; and 1 if $x \neq y$. $d_4$ is called a *discrete $G$-metric*.

We notice that the identity elements in $G_2$ and $G_3$ are 1 and 0 respectively. Moreover, $d_2(x, y) \geq 1$ and $d_3(x, y) \geq 0$. Now,

$$d_2(x, y) = e^{\|x-y\|} \leq e^{(\|x-z\|+\|z-y\|)} = e^{\|x-z\|}e^{\|z-y\|} = d_2(x, z)d_2(z, y).$$

Also, we have

$$\begin{aligned} d_3(x,y) &= e^{\|x-y\|} - 1 \le e^{\|x-z\|}e^{\|z-y\|} - 1 \\ &= (e^{\|x-z\|} - 1) + (e^{\|z-y\|} - 1) + (e^{\|x-z\|} - 1)(e^{\|z-y\|} - 1) \\ &= d_3(x,z) + d_3(z,y) + d_3(x,z)d_3(z,y) \\ &= d_3(x,z)d_3(z,y). \end{aligned}$$

So $d_2$ and $d_3$ satisfy the triangle inequality of Definition 2.4.

A $G$-metric $d$ on a set $X$ is called *bounded* if $d(x,y) \le g_0$, for all $x, y \in X$ and some $g_0 \in G$. Thus, the next result is now immediate.

**Corollary 2.6.** *Let $G$ be an $L$-group, $d$ a $G$-metric on $X$, $e < g_1$ a fixed element in $G$ and $d_1(x,y) = \inf\{d(x,y), g_1\}$. Then $d_1$ is a bounded $G$-metric.*

**Lemma 2.7.** *Let $G$ be an $L$-group; $A$ and $B$ are finite subsets of $G$ such that $A \ge e$ (i.e., $a \ge e$, for all $a \in A$) and $B \ge e$. Then*

- (i) *if $A \le B$ (i.e., for each $a \in A$ there is $b \in B$ such that $a \le b$) and $e \le g \in G$, then $\sup(gA) = g \sup A \le g \sup B = \sup(gB)$, where $gA = \{ga : a \in A\}$.*
- (ii) *if $A \ge B$ (i.e., for each $a \in A$ there is $b \in B$ such that $b \le a$) and $e \le g \in G$, then $\inf(gB) = g \inf B \le g \inf A = \inf(gA)$.*
- (iii) $\sup(AB) = \sup A \sup B$, *and also* $\inf(AB) = \inf A \inf B$, *where* $AB = \{ab : a \in A, b \in B\}$.

*Proof.* First, we note that by definition of an $L$-group, each of the finite sets $A, B$ and $AB$ has a supremum and an infimum in $G$. The proofs of (i) and (ii) are routine. (iii). Let $\sup A = \alpha$, $\sup B = \beta$ and $\sup(AB) = \gamma$. Since $G$ is an L-group, it is a poset group. So by Definition 2.1, we have $ab \le \alpha\beta$, for all $a \in A$ and $b \in B$. Evidently, $\gamma \le \alpha\beta$. Now, we are ready to show that $\gamma = \alpha\beta$. For the reverse inclusion, let $a \in A$ be fixed. Then $ab \le \gamma$ implies $b \le a^{-1}\gamma$. Therefore, $B$ is bounded by $a^{-1}\gamma$. So $\beta = \sup B \le a^{-1}\gamma$, in other words, $a \le \beta^{-1}\gamma$. Since $a \in A$ is arbitrary, we deduce that $A$ is bounded by $\beta^{-1}\gamma$. Thus, $\alpha \le \beta^{-1}\gamma$. This yields $\alpha\beta \le \gamma$, and we are through. The proof of another assertion (infimum) is done similarly. □

**Proposition 2.8.** *Let $G$ be an ordered group. Then defining*

$$d : G \times G \to G \text{ given by } d(g_1, g_2) = |g_1 g_2^{-1}|,$$

*turns $G$ into a $G$-metric space.*

*Proof.* We claim that $d$ is a $G$-metric on $G$. First, we note that since $G$ is an ordered group, it is an L-group and further $g \in G$ gives $g \ge e$ or $g^{-1} \ge e$. So $|g| = |g^{-1}| = \sup\{g, g^{-1}\} = g$ or $g^{-1}$. Hence, $|g| \ge e$ and therefore conditions (i)-(iii) of Definition 2.4 hold. Moreover, if $g_1, g_2, g_3 \in G$, then Corollary 2.3 implies

$$\begin{aligned} d(g_1,g_2) = |g_1g_2^{-1}| = |(g_1g_3^{-1})(g_3g_2^{-1})| &\le |g_1g_3^{-1}||g_3g_2^{-1}| \\ &= d(g_1,g_3)d(g_3,g_2) \end{aligned}$$

This gives $d$ satisfies the triangle inequality, i.e., it is a $G$-metric on $G$ and hence $G$ is a $G$-metric space. □

**Corollary 2.9.** *Let $G$ be an ordered group, $X$ a nonempty set and $f : X \to G$ a function. Then $d : X \times X \to G$ given by $d(x, y) = |f(x)f^{-1}(y)|$ is a $G$-pseudometric on $X$. Moreover, $d$ is a $G$-metric on $X$ if and only if $f$ is one-one.*

The next proposition generalizes Proposition 2.8.

**Proposition 2.10.** *Let $G$ be an ordered group. Then each of the following binary operations on $G^n$ (the $n$ product of $G$), turns it into a $G$-metric space, where $g = (g_1, g_2, \ldots, g_n)$ and $g' = (g'_1, g'_2, \ldots, g'_n)$ are arbitrary elements of $G^n$.*

(i) $d_1 : G^n \times G^n \to G$ *defined by* $d_1(g, g') = |g_1^{-1}g'_1||g_2^{-1}g'_2| \ldots |g_n^{-1}g'_n|$.

(ii) $d_2 : G^n \times G^n \to G$ *defined by*

$$d_2(g, g') = \sup\{|g_1^{-1}g'_1|, |g_2^{-1}g'_2|, \ldots, |g_n^{-1}g'_n|\}.$$

*Proof.* We only check that the triangle inequality for $d_1$ and $d_2$. Other conditions are routine. (i). Let $g'' = (g''_1, g''_2, \ldots, g''_n) \in G^n$. Then

$$\begin{aligned} d_1(g, g'') &= |g_1^{-1}g''_1||g_2^{-1}g''_2| \ldots |g_n^{-1}g''_n| \\ &= |(g_1^{-1}g'_1)(g_1'^{-1}g''_1)||(g_2^{-1}g'_2)(g_2'^{-1}g''_2)| \ldots |(g_n^{-1}g'_n)(g_n'^{-1}g''_n)| \\ &\leq |g_1^{-1}g'_1||g_1'^{-1}g''_1||g_2^{-1}g'_2||g_2'^{-1}g''_2| \ldots |g_n^{-1}g'_n||g_n'^{-1}g''_n| \\ &= \left(|g_1^{-1}g'_1|g_2^{-1}g'_2| \ldots |g_n^{-1}g'_n|\right)\left(|g_1'^{-1}g''_1||g_2'^{-1}g''_2| \ldots |g_n'^{-1}g''_n|\right) \\ &= d_1(g, g')d_1(g', g''). \end{aligned}$$

Notice that the above inequality is obtained by Corollary 2.3. (ii). Let $g'' = (g''_1, g''_2, \ldots, g''_n) \in G^n$ and let

$$A = \{|g_1^{-1}g''_1|, |g_2^{-1}g''_2|, \ldots, |g_n^{-1}g''_n|\},$$

$$B = \{|g_1^{-1}g'_1||g_1'^{-1}g''_1|, |g_2^{-1}g'_2||g_2'^{-1}g''_2|, \ldots, |g_n^{-1}g'_n||g_n'^{-1}g''_n|\},$$

$$\begin{aligned} B_1 &= \{|g_1^{-1}g'_1|, |g_2^{-1}g'_2|, \ldots, |g_n^{-1}g'_n|\}, \text{and} \\ B_2 &= \{|g_1'^{-1}g''_1|, |g_2'^{-1}g''_2|, \ldots, |g_n'^{-1}g''_n|\}. \end{aligned}$$

We notice that $G$ is an $L$-group. Now, according to Lemma 2.7, we have $A \leq B \leq B_1B_2$. Therefore,

$$d_2(g, g'') = \sup A \leq \sup B \leq \sup(B_1B_2) = \sup B_1 \sup B_2 = d_2(g, g')d_2(g', g''),$$

which completes the proof. □

 

## 3. Basic topological concepts in $G$-metric spaces and some related results

We begin with the following definition.

**Definition 3.1.** Let $G$ be a poset group, $(X, d)$ a $G$-metric space and $x$ a point of $X$. Given $e < g \in G$, we let

$$N_d(x, g) = \{y \in X : d(x, y) < g\},$$

and call it the $g$-disk centered at $x$. Also, we put $N_d(x, \overline{g}) = \{y \in X, d(x, y) \leq g\}$.

A subset $U$ of $X$ is said to be *open* in $X$ if either $U = \varnothing$ or for every $x \in U$ there is a $g \in G$ such that $N_d(x, g) \subseteq U$. Here, $x$ is called an *interior point* of $U$. The set of all interior points of $U$ is called the *interior* of $U$, denoted by $U^\circ$ (or $\text{int}_X U$). Also, a set $F$ is called *closed* if and only if its set-theoretic complement is an open set in $X$. Evidently, a set $F$ is closed if and if every $g$-disk centered at $x$ meets $F$, then $x \in F$.

**Corollary 3.2.** *Every $g$-disk $N_d(x, g)$ is an open set in $X$ (and hence $X \setminus N_d(x, g) = \{y \in X : d(x, y) \geq g\}$ is a closed set in $X$).*

*Proof.* Let $y \in N_d(x, g)$. Then $g_1 = d(x, y) < g$. We claim that $N_d(y, gg_1^{-1}) \subseteq N_d(x, g)$ (note, $gg_1^{-1} > e$). To see this, assume that $z \in N_d(y, gg_1^{-1})$. Hence, $d(z, x) \leq d(z, y)d(y, x) < gg_1^{-1}g_1 = g$. This yields $z \in N_d(x, g)$, i.e., $N_d(y, gg_1^{-1}) \subseteq N_d(x, g)$, and we are done. □

**Definition 3.3.** Let $X$ be a $G$-metric space and $A \subseteq X$. The *closure* of $A$ in $X$ is denoted by $\text{cl}_X A$ (or briefly clA) and defined by the set

$$\text{cl}A = \cap\{F \subseteq X : F \text{ is closed in } X \text{ and } A \subseteq F\}.$$

By the above definition, $A$ is closed if and only if $A = \text{cl}A$.

**Corollary 3.4.** *Let $G$ be a poset group, $(X, d)$ a $G$-metric space and $\bar{A} = \{x \in X : N_d(x, g) \cap A \neq \varnothing \text{ for all } e < g \in G\}$, where $A \subseteq X$. Then*

(1) $\overline{A} = clA$.

(2) *If $x \in X$ and $g \in G$, then $\overline{N_d(x, g)} \subseteq N_d(x, \overline{g})$.*

*Proof.* (1). Let $x \notin \text{cl}A$. Then $x \notin F$, for some closed set $F$ containing $A$. Now, since $X \setminus F$ is open, there exists $e < g \in G$, such that $x \in N_d(x, g) \subseteq X \setminus F$. So $x \notin \bar{A}$. Conversely, suppose that $x \notin \bar{A}$. So for some $e < g \in G$, $N_d(x, g) \cap A = \varnothing$. Therefore, the closed set $X \setminus N_d(x, g)$ contains $A$ but not $x$. This gives $x \notin \text{cl}A$, and we are done. (2). Suppose that $y \notin N_d(x, \overline{g})$. So $d(x, y) > g$ and hence $g_1 = d(x, y)g^{-1} > e$. We claim that $N_d(y, g_1) \cap N_d(x, g) = \varnothing$. Otherwise, for some $z \in N_d(y, g_1) \cap N_d(x, g)$, we have $d(x, y) \leq d(x, z)d(z, y) < gg_1 = d(x, y)$, a contradiction. Hence, $y \notin \overline{N_d(x, g)}$ and we are done. □

**Proposition 3.5.** *Let $G$ be an ordered group and $X$ a $G$-metric space. Then the open sets in $X$ have the following properties:*

(i) *$X$ and $\varnothing$ are both open.*

(ii) *Every union of open sets is open.*
(iii) *Every finite intersection of open sets is open.*

*Proof.* (i) and (ii) are clear. (iii). Let $x \in \bigcap_{i=1}^n U_i$, where $U_i$ is an open set in $X$. Take $g_i \in G$ such that $x \in N_d(x, g_i) \subseteq U_i$. Since $G$ is an ordered group, there exists $g \in G$ such that $g = \inf\{g_i\}_{i=1}^n$ (note, the elements $g_i$ form a chain and hence $g$ is one of them). Thus, $x \in N_d(x, g) \subseteq \bigcap_{i=1}^n N_d(x, g_i) \subseteq \bigcap_{i=1}^n U_i$, which completes the proof. □

By the above proposition, every $G$-metric $d$ on a set $X$ defines a topology $\tau_d$ on $X$; members of $\tau_d$, or, open subsets of $X$ are unions of $g$-disks. Clearly, the family of all $g$-disks is a base for $(X, \tau_d)$. We call $\tau_d$ *the topology induced by the G-metric d* (or *G-metric topology*).

*Remark* 3.6. Even if $G$ is a Dedekind-complete group, a countable intersection of open sets in a $G$-metric space need not be an open set necessarily. To see this, consider $\mathbb{R}$ as a $G$-metric space, where $G$ is $(\mathbb{R}, +)$ or $((0, \infty), .)$. Also, recall the fact that every point $a$ of $\mathbb{R}$ is a $G_\delta$-set, i.e., $\{a\} = \bigcap_{n=1}^\infty (a - \frac{1}{n}, a + \frac{1}{n})$.

In [2, I.3], an ordered set $X$ is called a *densely ordered set*, if no cut of $X$ is a jump, or equivalently, for every pair $x, y$ of elements of $X$ satisfying $x < y$, there exists a $z \in X$, such that $x < z < y$.

**Definition 3.7.** An ordered group $G$ is called a *densely ordered group* if it is a densely ordered set with its total ordering relation.

It is clear that densely ordered groups are infinite. For example, every archimedean field like as $\mathbb{R}$ and $\mathbb{Q}(\sqrt{n}) := \{a + b\sqrt{n} : a, b \in \mathbb{Q}, \sqrt{n} \notin \mathbb{Q}\}$ with the addition is a densely ordered group. $\mathbb{Z}$, the group of integers is an ordered group which is not a densely ordered group while $\mathbb{Z}_n$ with the addition of modulo $n$, is not a poset group yet.

From now on, the group $G$ is assumed to be a densely ordered group.

**Proposition 3.8.** *Let $G$ be a densely ordered group, $(X, d)$ a $G$-metric space, $x \in X$ and $g \in G$ be fixed, and $A = \{y \in X : d(x, y) > g\}$. Then $A$ is an open set in $X$ (and hence $N_d(x, \overline{g}) := \{y \in G : d(x, y) \leq g\}$ is closed).*

*Proof.* Let $y \in A$ be fixed. Then $d(x, y) > g$. We must show that $y$ is an interior point of $A$. Let $g_1 = d(x, y)g^{-1}$. Then $g_1 > e$ and $d(x, y) = gg_1$. Since $G$ is a densely ordered group, we take $e < g_2 < g_1$ and claim that $N_d(y, g_2)$ is contained in $A$ entirely. To see this, let $z \in N_d(y, g_2)$. Then we have $d(y, z) < g_2$ and so $g_2^{-1} < d^{-1}(y, z)$. Now, the inequality $d(x, y) \leq d(x, z)d(z, y)$ yields

$$g < gg_1g_2^{-1} < d(x, y)d^{-1}(y, z) \leq d(x, z).$$

(Notice that $g < gg_1g_2^{-1}$ if and only if $g_2 < g_1$.) Therefore, $g < d(x, z)$, i.e., $y$ is an interior point of $A$. So $A$ is an open set in $X$, and we are done. □

**Proposition 3.9.** *Let $G$ be an ordered group and $(X,d)$ a $G$-metric space. Then the following statements hold.*

(i) *If $G$ is a densely ordered group, then $X$ is Hausdorff.*
(ii) *If $G$ is an infinite cyclic group, then $X$ is discrete (and so it is first countable).*

*Proof.* (i). Let $x,y \in X$ and $d(x,y) = g > e$. By assumption, since $G$ is a densely ordered set, we can take $g_1, g_2 \in G$ such that $e < g_1 < g_2 < g$. Now, we claim that two disks $N_d(x,g_1)$ and $N_d(y,gg_2^{-1})$ are disjoint (note, $gg_2^{-1} > e$). Otherwise, for some $x' \in N_d(x,g_1) \cap N_d(y,gg_2^{-1})$, we have $d(x,x') < g_1$ and $d(x',y) < gg_2^{-1}$. Hence, $g = d(x,y) \leq d(x,x')d(x',y) < g_1gg_2^{-1}$. Therefore, $e < g_1g_2^{-1}$, or equivalently, $g_2 < g_1$, a contradiction. So we are done.

(ii). Let $e < g \in G$ be the generator of $G$. Then $G = \{g^n : n \in \mathbb{Z}\}$, in fact, we have $G \cong \mathbb{Z}$, the additive group of integers with the generator 1 (or $-1$). We note that the elements of $G$ form a chain. So we obtain

$$\ldots < g^{-3} < g^{-2} < g^{-1} < e < g < g^2 < g^3 < \ldots$$

Therefore, for each $x \in X$, $N_d(x,g) = \{y \in X : d(x,y) < g\} = \{y \in X : d(x,y) = e\} = \{x\}$. This yields $X$ is discrete (note, in this case $G$ is not a densely ordered group). □

*Remark* 3.10. By Proposition 3.9(ii), if $G$ is an infinite cyclic group then $X$ is first countable. But the converse of that result may be false, since every metric space is first countable, whereas the additive group $(\mathbb{R}, +)$ is not even countably generated.

In general, the converse of the above proposition does not need to be true. In the next example, we give examples of Hausdorff $G$-metric spaces such that the group $G$ is neither a densely ordered group nor an infinite cyclic group.

**Example 3.11.** (i) Let $d_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $d_1(m,n) = |m-n|$. Then, since $N_{d_1}(m,1) = \{m\}$, we obtain $\mathbb{Z}$ is a discrete $\mathbb{Z}$-metric space. So it is Hausdorff, whereas $\mathbb{Z}$ is not a densely ordered group. But if $\mathbb{Z}$ is considered as a $\mathbb{Q}$-metric space with the same definition, $d_1(m,n) = |m-n|$, it is a discrete $\mathbb{Q}$-metric space while $\mathbb{Q}$ is a densely ordered group.

(ii) Let $G := \mathbb{Z}\times\mathbb{Z}$ with the identity element $e = (0,0)$. Define $d_2 : \mathbb{Z}\times\mathbb{Z} \to G$ with $d_2(m,n) = (|m-n|, |m-n|)$. By example 2.2, $G$ is an $L$-group. It is easy to see that $d_2$ is a $G$-metric on $\mathbb{Z}$. Now, let $g = (1,1)$. Then

$$N_{d_2}(m,g) = \{n \in \mathbb{Z} : d_2(m,n) < g\} = \{m\}.$$

This yields $\mathbb{Z}$ is a discrete $G$-metric space, whereas $G = \mathbb{Z}\times\mathbb{Z}$ is not a cyclic group (note, it is a finitely generated group which is generated by the set $\{(0,1),(1,0)\}$).

**Definition 3.12.** If $(X,d_X)$ (resp. $(Y,d_Y)$) is a $G_1$- (resp. $G_2$-) metric space, a function $f : X \to Y$ is called *continuous* at $x_0 \in X$ if and only if for each $e_2 < g_2 \in G_2$ there is some $e_1 < g_1 \in G_1$ such that $d_Y(f(x_0), f(y)) < g_2$, whenever $d_X(x_0, y) < g_1$. $f$ is called *continuous* on $X$, if it is continuous at every $x \in X$.

A simple translation of the above definition is:

**Corollary 3.13.** *A function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if for each $g_2$-disk $N_{d_Y}(f(x_0), g_2)$ centered at $f(x_0)$, there is some $g_1$-disk $N_{d_X}(x_0, g_1)$ centered at $x_0$, such that $f(N_{d_X}(x_0, g_1)) \subseteq N_{d_Y}(f(x_0), g_2)$.*

**Theorem 3.14.** *If $(X, d_X)$ and $(Y, d_Y)$ are $G_1$- and $G_2$-metric spaces respectively, a function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if for each open set $V$ of $Y$ containing $f(x_0)$, there exists an open set $U$ of $X$ containing $x_0$ such that $f(U) \subseteq V$.*

*Proof.* $(\Rightarrow)$ : Suppose that $f$ is continuous at $x_0$ and $V$ is an open set in $Y$ containing $f(x_0)$. By definition of open sets, there is $g_2 \in G_2$ such that $f(x_0) \in N_{d_Y}(f(x_0), g_2) \subseteq V$. By Corollary 3.13, there exists a $g_1$-disk $N_{d_X}(x_0, g_1)$ centered at $x_0$ such that $f(N_{d_X}(x_0, g_1)) \subseteq N_{d_Y}(f(x_0), g_2) \subseteq V$, where $g_1 \in G_1$. It now suffices to choose $U = N_{d_X}(x_0, g_1)$.

$(\Leftarrow)$ : Consider $e_2 < g_2 \in G_2$ and $N_{d_Y}(f(x_0), g_2)$ as an open set in $Y$ containing $f(x_0)$. By hypothesis, there exists an open set $U$ in $X$ containing $x_0$ such that $f(U) \subseteq N_{d_Y}(f(x_0), g_2)$. Also, we can take $e_1 < g_1 \in G_1$ such that $N_{d_X}(x_0, g_1) \subseteq U$. So $f(N_{d_X}(x_0, g_1)) \subseteq f(U) \subseteq N_{d_Y}(f(x_0), g_2)$, and we are done. □

The following lemma is the counterpart of Lemma 2.7 for a Dedekind-complete group $G$. The only difference is that there $A$ and $B$ were finite subsets of $G$ but here these sets must be bounded.

**Lemma 3.15.** *Let $G$ be a Dedekind-complete group and; $A$ and $B$ are bounded subsets of $G$ such that $A \geq e$ (i.e., $a \geq e$, for all $a \in A$) and $B \geq e$. Then*

(i) *if $A \leq B$ (i.e., for each $a \in A$ there is $b \in B$ such that $a \leq b$) and $e \leq g \in G$, then $\sup(gA) = g \sup A \leq g \sup B = \sup(gB)$, where $gA = \{ga : a \in A\}$.*

(ii) *if $A \geq B$ (i.e., for each $a \in A$ there is $b \in B$ such that $b \leq a$) and $e \leq g \in G$, then $\inf(gB) = g \inf B \leq g \inf A = \inf(gA)$.*

(iii) *$\sup(AB) = \sup A \sup B$, and also $\inf(AB) = \inf A \inf B$, where $AB = \{ab : a \in A, b \in B\}$.*

In the remainder of this article, $G$ is assumed to be a Dedekind-complete densely ordered group (i.e., a densely ordered group in which every bounded nonempty subset has a supremum and an infimum in $G$), $(X, d)$ a $G$-metric space, and $d$ is bounded. The *distance* of a point $x$ to a set $A\,(\subseteq X)$ is defined by $d(x, A) = \inf\{d(x, a) : a \in A\}$, if $A \neq \varnothing$, and $d(x, \varnothing) = e$.

**Theorem 3.16.**

(i) *The mapping $f : X \to G$ defined by $f(x) = d(x, A)$ is continuous.*

(ii) *$x \in cl_X A$ if and only if $f(x) = d(x, A) = e$, in fact, $cl_X A = f^{-1}(e)$.*

*Proof.* (i). First, by Proposition 2.8, we have $(G, d')$ is a $G$-metric space, where $d'(g_1, g_2) = |g_1 g_2^{-1}|$. Let $x_0 \in X, g_0 \in G$ and $N_{d'}(f(x_0), g_0)$ be an open set containing $f(x_0)$. Then

$$d(x_0, a) \leq d(x_0, x)d(x, a), \text{ and } d(x, a) \leq d(x, x_0)d(x_0, a). \qquad (R_2)$$

Now, if we let $G_1 = \{d(x_0, a) : a \in A\}$ and $G_2 = \{d(x_0, x)d(x, a) : a \in A\}$, then $G_1$ and $G_2$ are two subsets of $G$ with the same cardinality and $G_2 \geq G_1$. By Lemma 3.15 (ii), we have $\inf_{a\in A} G_1 \leq \inf_{a\in A} G_2$. In other words, taking infimum on both sides of each of the inequalities in $(R_2)$ with respect to $a \in A$, we obtain

$$\inf_{a\in A} d(x_0, a) \leq d(x_0, x) \inf_{a\in A} d(x, a), \text{and} \inf_{a\in A} d(x, a) \leq d(x, x_0) \inf_{a\in A} d(x_0, a).$$

Thus, $f(x_0) \leq d(x, x_0)f(x)$ and $f(x) \leq d(x, x_0)f(x_0)$. Hence, $f(x_0)f^{-1}(x) \leq d(x, x_0)$ and also $f(x)f^{-1}(x_0) \leq d(x, x_0)$, i.e., $d(x, x_0)$ is a common upper bound for $f(x_0)f^{-1}(x)$ and $f^{-1}(x_0)f(x)$. Therefore,

$$d'(f(x_0), f(x)) = |f(x)f^{-1}(x_0)| = \sup\{f(x_0)f^{-1}(x), f^{-1}(x_0)f(x)\} \leq d(x, x_0).$$

Now, for the $g_0$-disk $N_d(x_0, g_0)$ we have $f(N_d(x_0, g_0)) \subseteq N_{d'}(f(x_0), g_0)$, and we are through.

(ii). *Necessity:* First, we note that by Corollary 3.4, $\text{cl}A = \bar{A} = \{x \in X : N_d(x, g) \cap A \neq \varnothing, \text{for all } g > e\}$. If $d(x, A) = g > e$ then $d(x, a) \geq g > e$, for all $a \in A$. By assumption, since $G$ is a densely ordered group, we can take $g_1 \in G$ such that $g > g_1 > e$. Now, we observe that $N_d(x, g_1) \cap A = \varnothing$. Hence, $x \notin \overline{A}$.

*Sufficiency:* Let $x \notin \overline{A}$. Then $N_d(x, g) \cap A = \varnothing$, for some $e < g \in G$. Hence, $d(x, a) \geq g$, for all $a \in A$. Therefore, $d(x, A) \geq g > e$. So $d(x, A) \neq e$, and we are done. □

**Theorem 3.17.** *Let $G$ be a Dedekind-complete densely ordered group, $(X, d)$ a $G$-metric space, $d$ is bounded, $g \in G$, and let $\mathcal{F}(X)$ be the family of all nonempty closed subsets of $X$. For $A, B \in \mathcal{F}(X)$ define*

$$d_B(A) = \sup\{d(a, B) : a \in A\}, \textit{and } d'(A, B) = \sup\{d_A(B), d_B(A)\}.$$

*Then the following statements hold.*

(1) *$d'$ is a $G$-metric on $\mathcal{F}(X)$. We call it the Hausdorff $G$-metric on $\mathcal{F}(X)$.*
(2) *$d'(A, B) \leq g$ if and only if $A \subseteq N_d(B, \bar{g})$ and $B \subseteq N_d(A, \bar{g})$, where $N_d(A, \bar{g}) = \{x \in X : d(x, A) \leq g\}$.*

*Proof.* (1). (i) and (iii) of Definition 2.4 are evident. Let $d'(A, B) = e$. Then $d_B(A) = e = d_A(B)$. So $d(a, B) = e$ for all $a \in A$. By Theorem 3.16 (ii), $a \in \text{cl}B = B$, i,e., $A \subseteq B$. Similarly, $B \subseteq A$. This proves (ii) of Definition 2.4. For the proof of triangle inequality, let $A, B, C \in \mathcal{F}(X)$ and $a \in A, b \in B, c \in C$. We notice that $d(a, B) \leq d(a, b)$ and $d(b, C) \leq d_C(B)$. Thus,

$$d(a, B) \leq d(a, b) \leq d(a, c)d(c, b).$$

Taking infimum on both sides of the above inequality with respect to $c \in C$ plus Lemma 3.15 yield

$$d(a,B) \leq \inf_{c\in C}\{d(a,c)d(c,b)\} = \inf_{c\in C} d(a,c) \inf_{c\in C} d(c,b).$$

Therefore, $d(a,B) \leq d(a,C)d(b,C)$. Since $d(b,C) \leq d_C(B)$, we have $d(a,B) \leq d(a,C)d_C(B)$. Taking supremum on both sides of the latter inequality with respect to $a \in A$, we obtain

$$d_B(A) \leq d_C(A)d_C(B). \qquad (R_3)$$

On the other hand, taking infimum over $c \in C$ on both sides of the inequalities $d(b,A) \leq d(a,b) \leq d(a,c)d(c,b)$ we obtain $d(b,A) \leq d(a,C)d(b,C)$ (Lemma 3.15). Furthermore, $d(a,C) \leq d_C(A)$ gives $d(b,A) \leq d_C(A)d(b,C)$. Now, take supremum on both sides of the latter inequality respect to $b \in B$. Thus,

$$d_A(B) \leq d_C(A)d_C(B). \qquad (R_4)$$

Combining $(R_3)$ and $(R_4)$ we get

$$d'(A,B) \leq d_C(A)d_C(B) \leq d'(A,C)d'(C,B).$$

Hence, $d'$ satisfies (iv) of Definition 2.4, and we are done.

(2). $(\Rightarrow)$ : Let $d'(A,B) \leq g$. Then $d_B(A) \leq g$ and $d_A(B) \leq g$. Hence, $d(a,B) \leq g$, for all $a$ in $A$. So $A \subseteq N_d(B,\bar{g})$. Similarly, $B \subseteq N_d(A,\bar{g})$.

$(\Leftarrow)$ : Since $A \subseteq N_d(B,\bar{g})$, it gives $d(a,B) \leq g$, for all $a$ in $A$, and therefore $d_B(A) = \sup_{a\in A} d(a,B) \leq g$. The assertion $d_A(B) \leq g$ is deduced similarly. So $d'(A,B) \leq g$, and we are through. □

**Corollary 3.18.** *Let $G$ be a densely ordered group and further a closed subset of $\mathbb{R}$, $\mathcal{K}(X)$ the family of nonempty compact subsets of $X$ and $A,B \in \mathcal{K}(X)$ such that $X$, $d$, $g$, $d_A$ and $d'$ be as defined in Theorem 3.17. Then $d'(A,B) < g$ if and only if $A \subseteq N_d(B,g)$ and $B \subseteq N_d(A,g)$, where $N_d(A,g) = \{x \in X : d(x,A) < g\}$.*

*Proof.* We first recall the fact that a nonempty subset of $\mathbb{R}$ has the least-upper-bound property (equivalently, the greatest-lower-bound property) if and only if it is closed in $\mathbb{R}$. So $G$ has the least-upper-bound property and hence it is a Dedekind-complete densely ordered group. Moreover, by Proposition 3.9, $X$ is Hausdorff and therefore every compact set in $X$ is closed. Thus, the conditions of Theorem 3.17 are satisfied. The necessary condition is obvious. To prove the sufficiency, let us define

$$f_1, f_2 : X \to G \text{ with } f_1(x) = d(x,A) \text{ and } f_2(x) = d(x,B).$$

Now, since $A$ and $B$ are compact subsets of $X$ and further; $f_1$ and $f_2$ are continuous functions on $X$ (Theorem 3.16), $f_1(B)$ and $f_2(A)$ are compact sets in $G$ (note, since $G$ is closed, $f_1$ and $f_2$ are well defined). Therefore, $\sup f_1(B) \in f_1(B)$ and also $\sup f_2(A) \in f_2(A)$. So we have

$$d_A(B) = \sup f_1(B) = f_1(b_1) = d(b_1,A), \text{ for some } b_1 \in B,$$

and also

$$d_B(A) = \sup f_2(A) = f_2(a_2) = d(a_2,B), \text{ for some } a_2 \in A.$$

By assumption, we now get $d_A(B) < g$ and $d_B(A) < g$. Hence, $d'(A, B) < g$, and we are through. □

Acknowledgements. *The authors are grateful to the referee for providing helpful comments and recommendations to improve the quality of the paper.*

## References

[1] A. Arhangel'skii and M. Tkachenko, Topological groups and related structures, Atlantis press/ World Scientific, Amsterdam-Paris, 2008.
[2] R. Engelking, General topology, Sigma Ser. Pure Math., Vol. 6, Heldermann Verlag, Berlin, 1989.
[3] L. Gillman and M. Jerison, Rings of Continuous Functions, Springer-Verlag, Berlin/Heidelberg/New York, 1976.
[4] I. Kaplansky, Infinite Abelian Groups, University of Michigan Press, 1954.
[5] O. A. S. Karamzadeh, M. Namdari and S. Soltanpour, On the locally functionally countable subalgebra of $C(X)$, Appl. Gen. Topol. 16, no. 2 (2015), 183–207.
[6] M. Namdari and A. Veisi, Rings of quotients of the subalgebra of $C(X)$ consisting of functions with countable image, Inter. Math. Forum 7 (2012), 561–571.
[7] D. J. S. Robinson, A course in the theory of groups, second edition, Springer-Verlag New York, Inc. 1996.
[8] J. Rotman, An Introduction to the Theory of Groups, Vol. 148, 4th edition Springer, New York, 1995.
[9] A. Veisi, The subalgebras of the functionally countable subalgebra of $C(X)$, Far East J. Math. Sci. (FJMS) 101, no. 10 (2017), 2285–2297.
[10] A. Veisi, Invariant norms on the subalgebras of $C(X)$ consisting of bounded functions with countable image, JP Journal of Geometry and Topology 21, no. 3 (2018), 167–179.
[11] A. Veisi, $e_c$-Filters and $e_c$-ideals in the functionally countable subalgebra of $C^*(X)$, Appl. Gen. Topol. 20, no. 2 (2019), 395–405.
[12] A. Veisi and A. Delbaznasab, New structure of norms on $\mathbb{R}^n$ and their relations with the curvature of the plane curves, Ratio Mathematica 39 (2020), 55–67.
[13] S. Willard, General Topology, Addison-Wesley, 1970.