

Topologically mixing extensions of endomorphisms on Polish groups

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ABSTRACT

In this paper we study the dynamics of continuous endomorphisms on Polish groups. We offer necessary and sufficient conditions for a continuous endomorphism on a Polish group to be weakly mixing. We prove that any continuous endomorphism of an abelian Polish group can be extended in a natural way to a topologically mixing endomorphism on the countable infinite product of said group.

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1. INTRODUCTION

The theory of discrete dynamical systems is concerned with the behavior of the iterates of a continuous map on a (usually compact) metric space. The most interesting and studied examples include maps that, in some sense, ‘mix’ the space. For a nice survey on the subject see the article by Kolyada and Snoha [7].

Formally, let X be a topological space and $f : X \rightarrow X$ a continuous map. Let

$$f^n(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n\text{-fold}}$$

denote the n – th iteration of the map f .

We say f is topologically transitive if given any two non-empty open subsets U and V of X there exists a natural number $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$.

A continuous map $f : X \rightarrow X$ is said to be topologically mixing if given any two non-empty open subsets U and V of X there exists a natural number N such that $f^n(U) \cap V \neq \emptyset$, whenever $n > N$. A map f such that $f \times f$ is transitive in $X \times X$ is called a weakly mixing map. Topological mixing is a stronger condition than weak mixing which is generally stronger than topological transitivity. For example, the irrational rotation of the circle is a topologically transitive map that is not weakly mixing.

In the setting of linear operators acting on a Banach space, a very celebrated result is the set of sufficient conditions for topological mixing known as the Hypercyclicity Criterion. The result first appeared in Kitai [6] and was later independently rediscovered by Gethner and Shapiro [5]. In [1], Bes and Peris showed that satisfying slightly laxer hypothesis than those in the Hypercyclicity Criterion is equivalent to the weak mixing of the operator in question. In [8] Reed and De la Rosa constructed a topologically transitive operator on a Banach Space that is not weak mixing and hence does not satisfy the Hypercyclicity Criterion, settling the question of whether the conditions were necessary.

Notice that the general theory of discrete dynamical systems is usually not concerned with any underlying algebraic structure of the space X . Operator theorists studying linear dynamics, on the other hand, only consider maps preserving the linear structure of the underlying space. In this note we will revisit this theme in a more general setting; we will study the dynamics of continuous endomorphisms on Polish groups. In particular, we conclude that every topologically mixing and weakly mixing continuous endomorphism on a Polish group satisfies conditions analogous to those in the Hypercyclicity Criterion.

2. MAIN RESULTS

In what follows G will denote a metric, complete, separable topological group: a Polish group for short.

If G has no isolated points, then a result by G. D. Birkhoff [2] tells us that a continuous map $\varphi : G \rightarrow G$ is topologically transitive if and only if there exists an element x_0 such that the orbit $orb(x_0, \varphi) = \{x_0, \varphi(x_0), \varphi^2(x_0), \varphi^3(x_0), \dots\}$ is dense in X . In other words, any element of G can be arbitrarily approximated by a sequence of elements in the orbit of x_0 . In this setting, φ is weakly mixing if there exists (h, f) in $G \times G$ whose orbit under $\varphi \times \varphi$ is dense.

Recall that for a group G , with the group operation written as multiplication, an endomorphism is a map $\varphi : G \rightarrow G$ such that $\varphi(gh) = \varphi(g)\varphi(h)$ for all g and h in G .

Before we continue with our discussion, we introduce the following definition:

Definition 2.1 (Weak Mixing Criterion). Let G be a Polish semigroup with identity e . We say that a continuous endomorphism $\varphi : G \rightarrow G$ satisfies the Weak Mixing Criterion if there exists an increasing sequence $\{n_k\}$ of natural

numbers, dense sets $F, H \subset G$, and maps $\psi_{n_k} : F \rightarrow G$ such that for any $f \in F$ and $h \in H$:

- (i) $\varphi^{n_k}(h) \rightarrow e$ as $k \rightarrow \infty$.
- (ii) $\psi_{n_k}(f) \rightarrow e$ as $k \rightarrow \infty$.
- (iii) $\varphi^{n_k}\psi_{n_k}(f) \rightarrow f$ as $k \rightarrow \infty$.

Chan [3] and Moothatu [9] independently remarked that if φ is a continuous endomorphism on a Polish group satisfying the Weak Mixing Criterion in the particular case $n_k = k, k \geq 0$, then φ is topologically mixing. In the same paper Moothatu showed that if G is a compact Hausdorff group and φ is a topologically transitive endomorphism, then φ is weakly mixing.

In view of Chan and Moothatu’s observation we ask if every weak mixing continuous endomorphism on a Polish group must satisfy the Weak Mixing Criterion. We answer the question in the affirmative for compact Polish groups in contrast to de la Rosa and Reed result for Banach spaces presented in [8]. The connection between the two settings comes from the fact that continuous linear operators are endomorphisms on the additive abelian group of a topological vector space. Indeed, we have:

Theorem 2.2. *Let G be a Polish group with identity e and $f : G \rightarrow G$ a continuous endomorphism. Then φ satisfies the Weak Mixing Criterion if and only if φ is weakly mixing.*

Proof. Assume φ satisfies the the Weak Mixing Criterion for a sequence $\{n_k\}$ and dense sets F and H . Let U_1, U_2, V_1 and V_2 be nonempty open subsets of G . In order to show that φ is weakly mixing, it suffices to exhibit $n \in N$ such that $(\varphi^n \times \varphi^n)(U_1 \times U_2) \cap (V_1 \times V_2) \neq \emptyset$. Since F and H are dense in G , we can choose $f_1 \in V_1 \cap F, f_2 \in V_2 \cap F, h_1 \in U_1 \cap H$, and $h_2 \in U_2 \cap H$. Then

$$h_1\psi_{n_k}(f_1) \rightarrow h_1e = h_1 \in U_1 \text{ as } k \rightarrow \infty$$

Similarly,

$$h_2\psi_{n_k}(f_2) \rightarrow h_2e = h_2 \in U_2 \text{ as } k \rightarrow \infty,$$

so for sufficiently large k

$$(h_1\psi_{n_k}(f_1), h_2\psi_{n_k}(f_2)) \in U_1 \times U_2.$$

Now, notice that

$$\varphi^{n_k}(h_1\psi_{n_k}(f_1)) = \varphi^{n_k}(h_1)\varphi^{n_k}(\psi_{n_k}(f_1)) \rightarrow ef_1 = f_1 \in V_1 \text{ as } k \rightarrow \infty$$

and also

$$\varphi^{n_k}(h_2\psi_{n_k}(f_2)) = \varphi^{n_k}(h_2)\varphi^{n_k}(\psi_{n_k}(f_2)) \rightarrow ef_2 = f_2 \in V_2 \text{ as } k \rightarrow \infty.$$

Again, for k large enough,

$$(\varphi^{n_k}(U_1) \times \varphi^{n_k}(U_2)) \cap (V_1 \times V_2) \neq \emptyset,$$

and we can conclude φ is weakly mixing.

Now, we assume that is φ weakly mixing with (h, f) an element whose orbit is dense in $G \times G$. Let id be the identity map on G . Since $id \times \varphi^n$ commutes

with $\varphi \times \varphi$ and its image is dense in $G \times G$, we have that for any $n \in \mathbb{N}$, the orbit of $(h, \varphi^n(f))$ is also dense. This implies that for all $U \subset G$ open, there is $u \in U$ such that (h, u) has dense orbit under $\varphi \times \varphi$. We will now exhibit the sets H and F and construct the sequence $\{n_k\}$ in the statement of the Weak Mixing Criterion.

For each natural number $k > 1$, consider B_k be the open ball in G centered at e of radius $1/k$. Since B_k is open in G , and left multiplication is continuous, the set $B_k \times hB_k$ is open in $G \times G$. By the observation above, we can pick $w_k \in B_k$ such that, the orbit of (h, w_k) under $\varphi \times \varphi$ is dense in $G \times G$. It follows that there exists an integer $n_k > k$ such that

$$(\varphi^{n_k} \times \varphi^{n_k})(h, w_k) \in (B_k \times hB_k).$$

That is, we can inductively construct an increasing sequence of positive integer $\{n_k\}$ such that:

- (i) $\varphi^{n_k}(h) \in B_k$ and
- (ii) $\varphi^{n_k}(w_k) \in hB_k$.

Letting $k \rightarrow \infty$, we have that:

$w_k \rightarrow e$ because $w_k \in B_k$, $\varphi^{n_k}(h) \rightarrow e$ by (i), and $\varphi^{n_k}(w_k) \rightarrow h$ by (ii). Now, let $F = H = \{h, \varphi(h), \varphi^2(h), \dots\}$ and define $\psi_{n_k} : F \rightarrow G$ by

$$\psi_{n_k}(\varphi^j(h)) = \varphi^j(w_k).$$

We have:

- (i) $\varphi^{n_k}(\varphi^j(h)) = \varphi^j(\varphi^{n_k}(h)) \rightarrow \varphi^j(e) = e$,
- (ii) $\psi_{n_k}(\varphi^j(h)) = \varphi^j(w_k) \rightarrow \varphi^j(e) = e$, and
- (iii) $\varphi^{n_k}(\psi_{n_k}(\varphi^j(h))) = \varphi^{n_k}(\varphi^j(w_k)) = \varphi^j(\varphi^{n_k}(w_k)) \rightarrow \varphi^j(h)$,

which is what we needed to show. □

It is worth noting again, that if the subsequence $\{n_k\}$ is given by $n_k = k$ then we can conclude that φ is topologically mixing. We can now state:

Corollary 2.3. *Let G be a compact Polish group and let φ be a continuous homomorphism on G . Then φ is topologically transitive if and only if it satisfies the Weak Mixing Criterion.*

Proof. If φ is topologically transitive, then by a result of Moothatu [9] φ must be weak mixing and by Theorem 2.2 it satisfies the Weak Mixing Criterion. Conversely, if φ satisfies that Weak Mixing Criterion then its transitivity follows immediately from Theorem 2.2. □

A natural question is whether any Polish group supports a topologically mixing endomorphism. We show that if the group in question is an abelian group that can be written as a countable infinite product of isomorphic copies of one of its closed subgroups, then the answer is affirmative. More precisely, we show that any continuous endomorphism on an abelian Polish group can be extended in a natural way to a topologically mixing continuous endomorphism of the infinite direct product of said group. The result is analogous to that of Chan [4].

Theorem 2.4. *Let G be an abelian Polish group, and $\varphi : G \rightarrow G$ a continuous endomorphism. There exists a continuous endomorphism $\Phi : \prod_{i=1}^{\infty} G \rightarrow \prod_{i=1}^{\infty} G$ such that:*

- (i) Φ is topologically mixing, and
- (ii) $j \circ \varphi = \Phi \circ j$ where $j : G \rightarrow \prod_{i=1}^{\infty} G$ is defined by $j(g) = (g, e, e, \dots)$

Proof. We will construct the endomorphism Φ and check it is topologically mixing by verifying it satisfies the Weak Mixing Criterion.

Consider a bounded metric d on G , we can define a metric ρ on $\prod_{i=1}^{\infty} G$ by

$$\rho(g, h) = \sum_{i=1}^{\infty} \frac{d(g_i, h_i)}{2^i}, \text{ for } f, g \in G$$

which induces the product topology on $\prod_{i=1}^{\infty} G$.

For each $g \in \prod_{i=1}^{\infty} G$, write

$$g = (g_1, g_2, g_3, \dots), \quad g_i \in G,$$

and define the maps $\Phi, \Psi : \prod_{i=1}^{\infty} G \rightarrow \prod_{i=1}^{\infty} G$ by

$$\Phi(g) = (\varphi(g_1)g_2, g_3, g_4, \dots)$$

and

$$\Psi(g) = (e, g_1, g_2, g_3, \dots)$$

where e is the identity in G .

First we verify that Φ is an endomorphism in $\prod_{i=1}^{\infty} G$. Indeed, let $f, g \in \prod_{i=1}^{\infty} G$ and we have

$$\begin{aligned} \Phi(fg) &= \Phi((f_1, f_2, f_3, \dots)(g_1, g_2, g_3, \dots)) \\ &= \Phi(f_1g_1, f_2g_2, f_3g_3, \dots) \\ &= (\varphi(f_1g_1)f_2g_2, f_3g_3, \dots) \\ &= (\varphi(f_1)f_2\varphi(g_1)g_2, f_3g_3, \dots) \\ &= (\varphi(f_1)f_2, f_3, \dots)(\varphi(g_1)g_2, g_3, \dots) \\ &= \Phi(f)\Phi(g). \end{aligned}$$

Notice that we have used the fact that G is abelian on the third equality from the bottom.

The continuity of Φ follows from the fact that the group operation is continuous. Now, we will check that $j \circ \varphi = \Phi \circ j$. Indeed, for any $g \in G$,

$$\begin{aligned} (j \circ \varphi)(g) &= j(\varphi(g)) \\ &= (\varphi(g), e, e, \dots) \\ &= (\varphi(g)e, e, \dots) \\ &= \Phi((\varphi(g), e, e, \dots)) \\ &= \Phi(j(g)) \\ &= (\Phi \circ j)(g) \end{aligned}$$

Now, note that for all $g \in \prod_{i=1}^{\infty} G$, $\Phi(\Psi(g)) = g$ and $\Psi^n g \rightarrow \tilde{e}$ as $n \rightarrow \infty$ where $\tilde{e} = (e, e, e, \dots)$ is the identity element of $\prod_{i=1}^{\infty} G$. We will now verify that Φ is mixing. Let H be a dense set in G and consider the subgroup \tilde{D} in $\prod_{i=1}^{\infty} G$ whose elements are of the form $(h_1, h_2, h_3, \dots, h_k, e, e, e, \dots)$ for some natural k and $h_i \in H$. This is clearly dense in $\prod_{i=1}^{\infty} G$ and for any element $h = (h_1, h_2, h_3, \dots, h_k, e, e, e, \dots) \in \tilde{D}$, we have

$$\Phi^k(h) = (\varphi^k(h_1)\varphi^{k-1}(h_2) \cdots \varphi(h_{k-1})h_k, e, e, \dots)$$

and

$$\Psi^k(\Phi^k(h)) = (\underbrace{e, e, \dots, e}_k, \varphi^k(h_1)\varphi^{k-1}(h_2) \cdots \varphi(h_{k-1})h_k, e, e, \dots).$$

Notice that:

$$\begin{aligned} \rho(\Psi^k(\Phi^k(h)), \tilde{e}) &= \frac{d(\varphi^k(h_1)\varphi^{k-1}(h_2) \cdots \varphi(h_{k-1})h_k, e)}{2^k} \\ &\leq \frac{1}{2^{k+1}} \end{aligned}$$

So we conclude that as $n \rightarrow \infty$, $\rho(\Psi^k(\Phi^k(h)), \tilde{e}) \rightarrow 0$.

Now, consider the set:

$$\tilde{F} = \{g(\Psi^n(\Phi^n(g)))^{-1} : g \in \tilde{D}, n \in \mathbb{N}\},$$

and note that

$$\begin{aligned} \lim_{n \rightarrow \infty} g(\Psi^n(\Phi^n(g)))^{-1} &= g \lim_{n \rightarrow \infty} (\Psi^n(\Phi^n(g)))^{-1} \\ &= g(\lim_{n \rightarrow \infty} (\Psi^n(\Phi^n(g^{-1}))) \\ &= g\tilde{e} \\ &= g \end{aligned}$$

which shows that \tilde{F} is dense in $\prod_{i=1}^{\infty} G$.

Also, let $f \in \tilde{F}$, so $f = g(\Psi^n(\Phi^n(g)))^{-1}$ for some $g \in \tilde{D}$ and we have

$$\begin{aligned} \Phi^n(f) &= \Phi^n(g(\Psi^n(\Phi^n(g)))^{-1}) \\ &= \Phi^n(g)\Phi^n(\Psi^n(\Phi^n(g)))^{-1} \\ &= \Phi^n(g)\Phi^n(\Psi^n(\Phi^n(g^{-1}))) \\ &= \Phi^n(g)(\Phi^n(g))^{-1} \\ &= \tilde{e}. \end{aligned}$$

The third equality follows from the fact that Ψ is a left inverse for Φ . We can then conclude that $\Phi^n \rightarrow \tilde{e}$ on \tilde{F} and by the results of Chan [3] and Moothathu [9], we conclude Φ is mixing. \square

Now, we define a weakly mixing continuous endomorphism on the infinite product in an analogous fashion to Theorem 2.4.

Theorem 2.5. *Let G be an abelian Polish group and let $\varphi : G \rightarrow G$ be a continuous endomorphism. Then the map $\Phi : \prod_{i=1}^{\infty} G \rightarrow \prod_{i=1}^{\infty} G$ defined by:*

$$\Phi(a_1, a_2, a_3, \dots) = (\varphi(a_1^{-1})a_2^{-1}, a_3^{-1}, a_4^{-1}, \dots)$$

is weakly mixing.

Proof. This proof will be a slight modification of the proof of Theorem 2.4. We follow the notation in the proof of Theorem 2.4 and let Ψ and \tilde{D} be defined in the same manner. It is not hard to verify that Φ is a continuous endomorphism.

As in the proof of Theorem 2.4, we have $\Psi^k(g) \rightarrow \tilde{e}$ as $k \rightarrow \infty$ and thus $\Psi^{2k}(g) \rightarrow \tilde{e}$ as $k \rightarrow \infty$. Also observe that $\Phi^{2k}(\Psi^{2k}(g)) = g$. Next note that since G is an abelian group we have,

$$\Phi^2(a_1, a_2, a_3, \dots) = (\varphi^2(a_1)\varphi(a_2)a_3, a_4, a_5, \dots).$$

One can thus verify that

$$\Psi^{2k}(\Phi^{2k}(h)) = (\underbrace{e, e, \dots, e}_{2k \text{ positions}}, \varphi^{2k}(h_0)\varphi^{2k-1}(h_1) \cdots \varphi(h_{2k-1})h_{2k}, e, e, \dots).$$

and

$$\begin{aligned} \rho(\Psi^{2k}(\Phi^{2k}(h)), \tilde{e}) &= \frac{\hat{d}(\varphi^{2k}(h_0)\varphi^{2k-1}(h_1) \cdots \varphi(h_{2k-1})h_{2k}, e)}{2^{2k}} \\ &\leq \frac{1}{2^{2k+1}} \end{aligned}$$

We conclude that as $n \rightarrow \infty$, $\rho(\Psi^{2k}(\Phi^{2k}(h)), \tilde{e}) \rightarrow 0$.

Now, let D be a dense set in G and define \tilde{F} as follows,

$$\tilde{F} = \{g(\Psi^{2n}(\Phi^{2n}(g)))^{-1} : g \in D, n \in \mathbb{N}\}.$$

Notice that

$$\begin{aligned} \lim_{n \rightarrow \infty} g(\Psi^{2n}(\Phi^{2n}(g)))^{-1} &= g(\lim_{n \rightarrow \infty} (\Psi^{2n}(\Phi^{2n}(g)))^{-1}) \\ &= g(\lim_{n \rightarrow \infty} (\Psi^{2n}(\Phi^{2n}(g)))^{-1}) \\ &= g\tilde{e} \\ &= g \end{aligned}$$

which shows that \tilde{F} is dense in $\prod_{i=1}^{\infty} G$.

Also, let $f \in \tilde{F}$, so $f = g(\Psi^{2n}(\Phi^{2n}(g)))^{-1}$ for some $g \in \prod_{i=1}^{\infty} G$ and we have

$$\begin{aligned} \Phi^{2n}(f) &= \Phi^{2n}(g(\Psi^{2n}(\Phi^{2n}(g)))^{-1}) \\ &= \Phi^{2n}(g)\Phi^{2n}(\Psi^{2n}(\Phi^{2n}(g)))^{-1} \\ &= \Phi^{2n}(g)(\Phi^{2n}(g))^{-1} \\ &= \tilde{e}. \end{aligned}$$

The third equality follows from the fact that Ψ is a left inverse for Φ . Hence, $\Phi^{2n}(f) \rightarrow \tilde{e}$ on \tilde{F} and by Theorem 2.2, we can conclude Φ is weakly mixing. \square

3. SOME GENERAL RESULTS

The first result in this section is concerned with the size of the orbit of an endomorphism; the second result gives an idea of how common are group elements with a dense orbit.

Before we do so, we will fix some notation. For a topological space X and A a subset of X , we denote the closure of A by \bar{A} and its interior by A° . We say that a subset of a topological space is somewhere dense when $\bar{A}^\circ \neq \emptyset$.

Proposition 3.1. *Let T be a continuous endomorphism on a Polish group. If T has an element g with a somewhere dense orbit then the subgroup generated by g is clopen in G .*

Proof. It is a well-known fact that if a subgroup of a topological group has non-empty interior then the subgroup is open. Since

$$\overline{\text{orb}(T, g)}^\circ \subset \overline{\langle \text{orb}(T, x) \rangle} \neq \emptyset$$

we have that $\overline{\langle \text{orb}(T, x) \rangle}$ is open but being the closure of the orbit, it is closed. \square

We immediately get the following.

Corollary 3.2. *If G is a connected Polish group, then the subgroup generated by an element with a somewhere dense orbit is dense.*

Proof. The result follows from the basic fact that the only clopen subsets of a connected space are the empty set and the space itself. \square

Now, we turn our attention to the algebraic structure of the set of group elements whose orbit is dense. We begin with the observation made by Kitai in [6] that a continuous self-map f on a complete metrizable space X without isolated points is topologically transitive if and only if the set of transitive elements $tr(f) = \{x \in X; \text{orb}(x, f) \text{ is dense}\}$ is a dense G_δ set.

We then have :

Proposition 3.3. *Let G be a Polish group that admits a topologically transitive endomorphism, then every element $x \in G$ is the product of two elements whose orbits are dense under T .*

Proof. Let T be a topologically transitive endomorphism on G and let $x \in G$. We have that both $tr(T)$ and $tr(T)x^{-1}$ are dense G_δ sets, by Baire's Theorem their intersection is not empty. Let h be an element in the intersection, so $h = gx^{-1}$ with g and h with dense orbit. \square

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