

## Some generalizations for mixed multivalued mappings

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### ABSTRACT

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*In this paper, we first introduce a new concept of KW-type  $m$ -contraction mapping. Then, we obtain some fixed point results for these mappings on  $M$ -metric spaces. Thus, we extend many well-known results for both single valued mappings and multivalued mappings such as the main results of Klim and Wardowski [13] and Altun et al. [4]. Also, we provide an interesting example to show the effectiveness of our result.*

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### 1. INTRODUCTION

In 1922, Banach [7] proved an important theorem which is known as Banach contraction principle. This principle is an important tool in the fixed point theory and has been accepted as starting of the fixed point theory in metric spaces. Due to its applicability, many authors have studied to generalize this principle by considering different kinds of contractions or abstract spaces [2, 10, 11, 12, 19]. Taking into account multivalued mappings, Nadler [17] proved

one of the interesting and famous generalizations of this result in metric spaces as follows:

**Theorem 1.1.** *Let  $T : X \rightarrow CB(X)$  be a multivalued mapping on a complete metric space  $(X, d)$  where  $CB(X)$  is the family of all nonempty bounded and closed subsets of  $X$ . Suppose that there exists  $k$  in  $[0, 1)$  satisfying*

$$H_d(Tx, Ty) \leq kd(x, y)$$

for all  $x, y \in X$  where  $H_d : CB(X) \times CB(X) \rightarrow \mathbb{R}$  is a Pompei-Hausdorff metric defined as

$$H_d(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right\}$$

for all  $A, B \in CB(X)$ . Then,  $T$  has a fixed point in  $X$ .

Then, a lot of fixed point theorems for multivalued mappings have been obtained [15, 20]. In this sense, Nadler's result has been extended by Feng and Liu [9] by taking into account  $C(X)$ , which is the family of all nonempty closed subsets of a metric space  $(X, d)$  valued mappings instead of  $CB(X)$  as follows:

**Theorem 1.2.** *Let  $T : X \rightarrow C(X)$  be a multivalued mapping on a complete metric space  $(X, d)$ . Suppose that for all  $x \in X$  there exists  $y \in I_\lambda^x = \{z \in Tx : \lambda d(x, z) \leq d(x, Tx)\}$  such that*

$$d(y, Ty) \leq \gamma d(x, y).$$

If the function  $g(x) = d(x, Tx)$  is lower semicontinuous (briefly l.s.c.) on  $X$  and  $0 < \gamma < \lambda < 1$ , then  $T$  has a fixed point in  $X$ .

Later, Klim and Wardowski [13] generalized Theorem 1.2 by taking into account a nonlinear contraction:

**Theorem 1.3.** *Let  $T : X \rightarrow C(X)$  be a multivalued mapping on a complete metric space  $(X, d)$ . If there exist  $\lambda$  in  $(0, 1)$  and  $\varphi : [0, \infty) \rightarrow [0, \lambda)$  such that*

$$(1.1) \quad \lim_{s \rightarrow u^+} \sup \varphi(s) < \lambda$$

for all  $u \in [0, \infty)$  and there is  $y \in I_\lambda^x$  for all  $x \in X$  satisfying

$$d(y, Ty) \leq \varphi(d(x, y)) d(x, y),$$

then  $T$  has a fixed point provided that  $g(x) = d(x, Tx)$  is lower-semicontinuous function on  $X$ .

On the other hand, introducing the concept of partial metric, Matthews [14] obtained another generalization of the Banach contraction principle. Now, we give the definition of the partial metric space.

**Definition 1.4** ([14]). Let  $X$  be a nonempty set and  $p : X \times X \rightarrow [0, \infty)$  be a function satisfying following conditions for all  $x, y, z \in X$ .

- p1)  $p(x, x) = p(x, y) = p(y, y)$  if and only if  $x = y$
- p2)  $p(x, x) \leq p(x, y)$

- p3)  $p(x, y) = p(y, x)$
- p4)  $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$

Then,  $p$  is said to be a partial metric. Also, the pair  $(X, p)$  is called partial metric space.

It is clear that every metric space is a partial metric space, but the converse may not be true. For some examples of partial metric space, we refer to [1, 5, 8, 21]. Let  $(X, p)$  be a partial metric space. Recently, Asadi et al. [6] introduced a nice concept of  $M$ -metric which includes the notion of the partial metric. Then, they proved a version of the Banach contraction principle on these spaces. After that, many authors have proved many fixed point results for multivalued and single valued mappings [3, 16, 18] on  $M$ -metric spaces. Now, we recall some notations and properties of an  $M$ -metric space.

**Definition 1.5.** Let  $X$  be a nonempty set, and  $m : X \times X \rightarrow [0, \infty)$  be a function. Then,  $m$  is said to be an  $M$ -metric if the following conditions hold for all  $x, y, z \in X$ :

- m1)  $m(x, y) = m(x, x) = m(y, y)$  if and only if  $x = y$ ,
- m2)  $m_{xy} = \min \{m(x, x), m(y, y)\} \leq m(x, y)$ ,
- m3)  $m(x, y) = m(y, x)$ ,
- m4)  $m(x, y) - m_{xy} \leq (m(x, z) - m_{xz}) + (m(z, y) - m_{zy})$ .

Also,  $(X, m)$  is called an  $M$ -metric space. It is obvious that every standard metric and partial metric space is an  $M$ -metric space but the converse may not be true. Indeed, let  $X = [0, \infty)$  and  $m : X \times X \rightarrow [0, \infty)$  be a function defined by  $m(x, y) = \frac{x+y}{2}$ . Hence,  $(X, m)$  is an  $M$ -metric space, but neither a partial metric space nor a standard metric. Let  $(X, m)$  be an  $M$ -metric space. Then, the  $M$ -metric  $m$  generates a  $T_0$  topology  $\tau_m$  on  $X$  which has as a base the family open balls

$$\{B_m(x, r) : x \in X, r > 0\}$$

where

$$B_m(x, r) = \{y \in X : m(x, y) < m_{xy} + r\}$$

for all  $x \in X$  and  $r > 0$ . Let  $\{x_n\}$  be a sequence in  $X$  and  $x \in X$ . It can be seen that the sequence  $\{x_n\}$   $M$ -converges to  $x$  with respect to  $\tau_m$  if and only if

$$\lim_{n \rightarrow \infty} (m(x_n, x) - m_{x_n x}) = 0.$$

If  $\lim_{n, m \rightarrow \infty} m(x_n, x_m)$  exists and is finite, then  $\{x_n\}$  is said to be an  $M$ -Cauchy sequence. If every  $M$ -Cauchy sequence  $\{x_n\}$  converges to a point  $x$  in  $M$  such that

$$\lim_{n, k \rightarrow \infty} m(x_n, x_k) = m(x, x),$$

then  $(X, m)$  is said to be  $M$ -complete.

The following proposition is important for our main results.

**Proposition 1.6.** *Let  $(X, m)$  be an  $M$ -metric space,  $A \subseteq X$  and  $x \in X$ .*

$$m(x, A) = 0 \implies x \in \overline{A}^m$$

where  $\overline{A}^m$  is the closure of  $A$  with respect to  $\tau_m$ .

The converse of Proposition 1.6 may not be true. Indeed, let  $X = [2, \infty)$  and  $m : X \times X \rightarrow \mathbb{R}$  be a function defined by  $m(x, y) = \min\{x, y\}$ . Then,  $(X, m)$  is an  $M$ -metric space. Let  $A = (3, 5)$  and  $x = 2$ . It can be seen that  $x \in \overline{A}^m$ , but  $m(x, A) = 2 > 0$ .

Note that, since every metric space is a  $T_1$ -space, every singleton is a closed set. Therefore, Theorem 1.2 is an extension of some fixed point result for single-valued mappings. However,  $\tau_m$  may not be a  $T_1$ -space, and thus each singleton does not have to be closed. Therefore, the fixed point results obtained for multivalued mappings on an  $M$ -metric space may not be valid for single-valued mappings unlike in the settings of metric spaces. To overcome this problem, we will use the notion of mixed multivalued mapping introduced by Romaguera [22].

In the current paper, we first introduce a new concept of  $KW$ -type  $m$ -contraction for the mixed multivalued mapping. Then, we obtain some fixed point results on  $M$ -metric spaces for these mappings. Hence, we extend some well known results in the literature such as Theorem 1.3. Also, we provide a noteworthy example to show the effectiveness of our results.

## 2. MAIN RESULTS

We start this section with the definition of  $KW$ -type  $m$ -contraction for mixed multivalued mapping.

**Definition 2.1.** Let  $T : X \rightarrow X \cup C_m(X)$  be a mixed multivalued mapping on an  $M$ -metric space  $(X, m)$  where  $C_m(X)$  is the family of all closed subsets of  $X$  w.r.t.  $\tau_m$ . Then,  $T$  is called  $KW$ -type  $m$ -contraction mapping if there exists  $\lambda, \alpha \in (0, 1)$  and  $\varphi : [0, \infty) \rightarrow [0, \lambda)$  satisfying  $\lim_{s \rightarrow u^+} \sup \varphi(s) < \lambda$  for all  $u \in [0, \infty)$  and for all  $x \in X$  with  $m(x, Tx) > 0$  there is  $y \in T_x^\lambda(m) = \{z \in Tx : \lambda m(x, z) \leq m(x, Tx)\}$  such that

$$m(y, Ty) \leq \varphi(m(x, y))m(x, y)$$

and

$$\alpha m(y, y) \leq m(x, y).$$

**Theorem 2.2.** *Let  $T : X \rightarrow X \cup C_m(X)$  be a  $KW$ -type  $m$ -contraction on an  $M$ -complete  $M$ -metric space  $(X, m)$ . If the function  $g : X \rightarrow \mathbb{R}$  defined by  $g(x) = m(x, Tx)$  is l.s.c., then  $T$  has a fixed point in  $X$ .*

*Proof.* Let  $x_0 \in X$  be an arbitrary point. If there exists  $n_0 \in \mathbb{N}$  such that  $m(x_{n_0}, Tx_{n_0}) = 0$ , then  $x_{n_0} \in \overline{Tx_{n_0}} = Tx_{n_0}$  (or  $x_{n_0} = Tx_{n_0}$ ), and so  $x_{n_0}$  is a fixed point of  $T$ . Assume that  $m(x_n, Tx_n) > 0$  for all  $n \geq 1$ . Now, we consider the following cases:

Case 1. Let  $|Tx_0| = 1$ . Since  $T$  is a  $KW$ -type  $m$ -contraction mapping, there exists  $x_1 = Tx_0$  such that

$$m(x_1, Tx_1) \leq \varphi(m(x_0, x_1))m(x_0, x_1)$$

and

$$\alpha m(x_1, x_1) \leq m(x_0, x_1).$$

Now, if  $|Tx_1| = 1$ , since  $T$  is a  $KW$ -type  $m$ -contraction mapping, there exists  $x_2 = Tx_1$  such that

$$m(x_2, Tx_2) \leq \varphi(m(x_1, x_2))m(x_1, x_2)$$

and

$$\alpha m(x_2, x_2) \leq m(x_1, x_2),$$

If  $|Tx_1| > 1$ , since  $T$  is a  $KW$ -type  $m$ -contraction mapping, there exists  $x_2 \in T_\lambda^{x_1}(m)$  such that

$$m(x_2, Tx_2) \leq \varphi(m(x_1, x_2))m(x_1, x_2)$$

and

$$\alpha m(x_2, x_2) \leq m(x_1, x_2).$$

Case 2. Let  $|Tx_0| > 1$ . Since  $T$  is a  $KW$ -type  $m$ -contraction mapping, there exists  $x_1 \in T_\lambda^{x_0}(m)$  such that

$$m(x_1, Tx_1) \leq \varphi(m(x_0, x_1))m(x_0, x_1)$$

and

$$\alpha m(x_1, x_1) \leq m(x_0, x_1).$$

If  $|Tx_1| = 1$ . Since  $T$  is a  $KW$ -type  $m$ -contraction mapping, there exists  $x_2 = Tx_1$  such that

$$m(x_2, Tx_2) \leq \varphi(m(x_1, x_2))m(x_1, x_2)$$

and

$$\alpha m(x_2, x_2) \leq m(x_1, x_2).$$

Now, if  $|Tx_1| > 1$ . Since  $T$  is a  $KW$ -type  $m$ -contraction mapping, there exists  $x_2 \in T_\lambda^{x_1}(m)$  such that

$$m(x_2, Tx_2) \leq \varphi(m(x_1, x_2))m(x_1, x_2)$$

and

$$\alpha m(x_2, x_2) \leq m(x_1, x_2).$$

Repeating this process, we can construct a sequence  $\{x_n\}$  such that for  $x_{n+1} \in T_\lambda^{x_n}(m)$ ,

$$m(x_{n+1}, Tx_{n+1}) \leq \varphi(m(x_n, x_{n+1}))m(x_n, x_{n+1})$$

and

$$(2.1) \quad \alpha m(x_{n+1}, x_{n+1}) \leq m(x_n, x_{n+1})$$

for all  $n \geq 1$ . Since  $x_{n+1} \in T_\lambda^{x_n}(m)$  for all  $n \geq 1$ , we have

$$(2.2) \quad \lambda m(x_n, x_{n+1}) \leq m(x_n, Tx_n)$$

for all  $n \geq 1$ . Hence, we get

$$\begin{aligned}
 m(x_n, x_{n+1}) &\leq \frac{m(x_n, Tx_n)}{\lambda} \\
 &\leq \frac{\varphi(m(x_{n-1}, x_n))m(x_{n-1}, x_n)}{\lambda} \\
 (2.3) \qquad &< m(x_{n-1}, x_n)
 \end{aligned}$$

and

$$\begin{aligned}
 m(x_n, Tx_n) - m(x_{n+1}, Tx_{n+1}) &\geq \lambda m(x_n, x_{n+1}) \\
 &\quad - \varphi(m(x_n, x_{n+1}))m(x_n, x_{n+1}) \\
 &= (\lambda - \varphi(m(x_n, x_{n+1})))m(x_n, x_{n+1}) \\
 (2.4) \qquad &> 0
 \end{aligned}$$

for all  $n \geq 1$ . From inequalities (2.3) and (2.4),  $(m(x_n, x_{n+1}))$  and  $(m(x_n, Tx_n))$  are decreasing sequences in  $\mathbb{R}$ , and so they are convergent. Because of the fact that

$$\lim_{n \rightarrow \infty} m(x_n, x_{n+1}) = r \geq 0$$

and

$$\lim_{s \rightarrow u^+} \sup \varphi(s) < \lambda,$$

we can find  $q \in [0, \lambda)$  satisfying

$$\lim_{n \rightarrow \infty} \sup \varphi(m(x_n, x_{n+1})) = q.$$

Therefore, for any  $\lambda_0 \in (q, \lambda)$ , there exists  $n_0 \in \mathbb{N}$  such that

$$\varphi(m(x_n, x_{n+1})) < \lambda_0$$

for all  $n \geq n_0$ . From (2.4), we have

$$\begin{aligned}
 m(x_n, Tx_n) - m(x_{n+1}, Tx_{n+1}) &\geq (\lambda - \varphi(m(x_n, x_{n+1})))m(x_n, x_{n+1}) \\
 (2.5) \qquad &\geq (\lambda - \lambda_0)m(x_n, x_{n+1})
 \end{aligned}$$

for all  $n \geq n_0$ . Hence, for all  $n \geq n_0$ , we have

$$\begin{aligned}
 m(x_n, Tx_n) &\leq \varphi(m(x_{n-1}, x_n))m(x_{n-1}, x_n) \\
 &\leq \frac{\varphi(m(x_{n-1}, x_n))}{\lambda}m(x_{n-1}, Tx_{n-1}) \\
 &\leq \frac{\varphi(m(x_{n-1}, x_n))\varphi(m(x_{n-2}, x_{n-1}))}{\lambda}m(x_{n-2}, x_{n-1}) \\
 &\leq \frac{\varphi(m(x_{n-1}, x_n))\varphi(m(x_{n-2}, x_{n-1}))}{\lambda^2}m(x_{n-2}, Tx_{n-2}) \\
 &\vdots \\
 &\leq \frac{\varphi(m(x_{n-1}, x_n)) \cdots \varphi(m(x_0, x_1))}{\lambda^n}m(x_0, Tx_0) \\
 &= \frac{\varphi(m(x_{n-1}, x_n)) \cdots \varphi(m(x_{n_0}, x_{n_0+1}))}{\lambda^{n-n_0}} \\
 &\quad \times \frac{\varphi(m(x_{n_0-1}, x_{n_0})) \cdots \varphi(m(x_0, x_1))}{\lambda^{n_0}}m(x_0, Tx_0) \\
 &< \left(\frac{\lambda_0}{\lambda}\right)^{n-n_0} \frac{\varphi(m(x_{n_0-1}, x_{n_0})) \cdots \varphi(m(x_0, x_1))}{\lambda^{n_0}}m(x_0, Tx_0).
 \end{aligned}$$

Then, since  $\lim_{n \rightarrow \infty} \left(\frac{\lambda_0}{\lambda}\right)^{n-n_0} = 0$ , we have

$$(2.6) \quad \lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0.$$

Hence, for all  $k > n \geq n_0$ , from (2.5), we get

$$\begin{aligned}
 m(x_n, x_k) - m_{x_n x_k} &\leq (m(x_n, x_{n+1}) - m_{x_n x_{n+1}}) + (m(x_{n+1}, x_{n+2}) - m_{x_{n+1} x_{n+2}}) \\
 &\quad + \cdots + (m(x_{k-1}, x_k) - m_{x_{k-1} x_k}) \\
 &\leq m(x_n, x_{n+1}) + \cdots + m(x_{k-1}, x_k) \\
 &= \sum_{j=n}^{k-1} m(x_j, x_{j+1}) \\
 &\leq \frac{1}{\lambda - \lambda_0} \sum_{j=n}^{k-1} (m(x_j, Tx_j) - m(x_{j+1}, Tx_{j+1})) \\
 &= \frac{1}{\lambda - \lambda_0} (m(x_n, Tx_n) - m(x_k, Tx_k))
 \end{aligned}$$

From (2.6), we have

$$\lim_{n, k \rightarrow \infty} (m(x_n, x_k) - m_{x_n x_k}) = 0$$

Also, from (2.1), (2.2) and (2.6) we have  $\lim_{n \rightarrow \infty} m(x_n, x_n) = 0$ , and so

$$\lim_{n, k \rightarrow \infty} m(x_n, x_k) = 0.$$

Then,  $\{x_n\}$  is an  $M$ -Cauchy sequence in  $(X, m)$ . Since  $(X, m)$  is an  $M$ -complete  $M$ -metric space, there exists  $x^* \in M$  such that

$$\lim_{n \rightarrow \infty} (m(x_n, x^*) - m_{x_n x^*}) = 0$$

and

$$\lim_{n, k \rightarrow \infty} m(x_n, x_k) = m(x^*, x^*)$$

Now, we shall show that  $x^*$  is a fixed point of  $T$ . Since  $g(x) = m(x, Tx)$  is a l.w.s.c. function and  $\lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0$ , we have

$$0 \leq m(x^*, Tx^*) = g(x^*) \leq \liminf_{n \rightarrow \infty} g(x_n) = \liminf_{n \rightarrow \infty} m(x_n, Tx_n) = 0$$

Hence,  $m(x^*, Tx^*) = 0$ , and so we have  $x^* \in \overline{Tx^*}^m = Tx^*$ . Therefore,  $x^*$  is a fixed point of  $T$ .  $\square$

The following example is important to show the effectiveness of our result.

**Example 2.3.** Let  $X = [0, 4]$  and  $m : X \times X \rightarrow [0, \infty)$  be a function defined by

$$m(x, y) = \frac{x + y}{2}$$

Then,  $(X, m)$  is an  $M$ -complete  $M$ -metric space. Define mappings  $T : X \rightarrow X \cup C_m(X)$  and  $\varphi : [0, \infty) \rightarrow [0, \frac{3}{4})$  by

$$Tx = \left[ 0, \frac{x^2}{16} \right],$$

and

$$\varphi(u) = \begin{cases} \frac{3}{4}u & , \quad u < 1 \\ \frac{1}{2} & , \quad u \geq 1 \end{cases} ,$$

respectively. Then, we have

$$g(x) = m(x, Tx) = \frac{x}{2}$$

for all  $x \in X$ . It can be seen that  $g$  is l.s.c. with respect to  $\tau_m$ . Now, we shall show that  $KW$ -type  $m$ -contraction mapping. Let  $x$  be an arbitrary point in  $X$  with  $m(x, Tx) > 0$ . Also, we have

$$\begin{aligned} T_{\frac{3}{4}}^x &= \left\{ y \in Tx : \frac{3}{4}m(x, y) \leq m(x, Tx) \right\} \\ &= \left\{ y \in Tx : y \leq \frac{x}{3} \right\} \end{aligned}$$

Choose  $\alpha = \frac{1}{2}$ . Then, for  $y \in T_{\frac{3}{4}}^x$ , we have

$$\begin{aligned} m(y, Ty) &= \frac{y}{2} \\ &\leq \frac{x}{6} \\ &\leq \frac{3}{8}(x + y)^2 \\ &= \varphi(m(x, y))m(x, y), \end{aligned}$$



and

$$\frac{1}{2}m(y, y) \leq m(x, y).$$

Hence, all hypotheses of Theorem 2.2 hold, and so  $T$  has a fixed point in  $X$ .

If we take  $\varphi(u) = \gamma \in (0, \lambda)$  for all  $u \in [0, \infty)$  in Definition 2.1, we obtain the following fixed point result which is a generalization result of [4].

**Corollary 2.4.** *Let  $T : X \rightarrow X \cup C_m(X)$  be a multivalued mapping on a  $M$ -complete  $M$ -metric space  $(X, m)$ . Assume that the following conditions hold:*

- (i) *the function  $g : X \rightarrow \mathbb{R}$  defined by  $g(x) = m(x, Tx)$  is l.s.c.*
- (ii) *there exist  $\lambda, \gamma, \alpha \in (0, 1)$  with  $\gamma < \lambda$  such that for any  $x \in X$  with  $m(x, Tx) > 0$ ,*

$$m(y, Ty) \leq \gamma m(x, y)$$

and

$$\alpha m(y, y) \leq m(x, y)$$

for some  $y \in T_\lambda^x(m)$ .

Then,  $T$  has a fixed point in  $X$ .

### 3. CONCLUSION

In this paper, we extend the result given by Klim and Wardowski [13] to  $M$ -metric spaces. Also, we generalize the main result of Altun et al. [4]. Since an  $M$ -metric space may not be a  $T_1$ -space, we first introduce a new concept of  $KW$ -type  $m$ -contraction mapping to obtain a real generalization of the results obtained for the single valued mappings. Then we obtain some fixed point results for these mappings in  $M$ -metric spaces. Moreover, we provide an interesting example to show the effectiveness of our results.

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