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Common new fixed point results on b-cone Banach spaces over Banach algebras

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Abstract

Recently Zhu and Zhai studied the concepts of cone b-norm and cone b-Banach space as generalizations of cone b-metric spaces and they gave a definition of ϕ-operator and obtained some new fixed point theorems in cone b-Banach spaces over Banach algebras by using ϕ-operator. In this paper we propose a notion of quasi-cone over Banach algebras, then by utilizing some new conditions and following their work with introducing two mappings $\mathcal{T}$ and $\mathcal{S}$ we improve the fixed point theorems to the common fixed point theorems. An example is given to illustrate the usability of the obtained results.

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1. Introduction

The notion of b-metric was proposed by Czerwik [12, 13] to generalize the concept of distance. The analog of the famous Banach fixed point theorem was proved by Czerwik in the frame of complete b-metric spaces, see also [9, 10, 11]. In [19] E. Karapinar generalized some conclusions on the cone Banach space,

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in the literature [2] and obtained the existence results of fixed points for self-mappings. Also, the cone metric space over Banach algebra, proposed by Liu and Xu (see [23]) and they considered some fixed point results on such new space.

In 2001 Hussain and Shah [17] introduced the notation of cone b-metric space. Many researchers continued the work of Hussain and Shah, and proved some fixed point theorems and common fixed point theorems for multiple operators on these new spaces, and also used them to investigate the existence of the solutions of fractional integral equations (see [3, 5, 4, 6, 8, 14, 15, 16, 18, 24, 25, 27, 28, 26, 20, 21]).

Recently Zhu and Zhai [30] studied the concepts of cone b-norm and cone b-Banach space as generalizations of cone b-metric spaces. Also they introduced the operator ϕ and obtained some new fixed point theorems in cone b-Banach spaces over Banach space utilizing the ϕ-operator.
In this paper by introducing a notion of quasi-cone over Banach space and also with applying different conditions we examine the existence of some common fixed points of two self-mappings $\mathcal{S}$ and $\mathcal{T}$ that has led to the development of similar results in the literature.

2. Preliminaries

Let $(\mathrm{E}, \|\cdot\|)$ be a real Banach space, $\mathrm{P} \subset \mathrm{E}$ a cone and θ be the zero of E, also there is a partial ordering $\leq$ such that $\xi \leq \zeta$ iff $\zeta - \xi \in \mathrm{P}$. Write $\xi \ll \zeta$ for $\zeta - \xi \in int\mathrm{P}$, where $int\mathrm{P}$ is the interior set of P. We say that P is normal if there exists $N > 0$ such that $\theta \leq \xi \leq \zeta$ implies $\| \xi \| \leq N \| \zeta \|$, for $\xi, \zeta \in \mathrm{E}$. P is called to be solid if $int\mathrm{P} \neq \varnothing$.

Definition 2.1 (see [17]). Let $\mathrm{X} \neq \varnothing$ and $s \geq 1$, a mapping $\varrho : \mathrm{X} \times \mathrm{X} \to \mathrm{E}$ is a cone b-metric if;
(i) $\theta < \varrho(\xi, \zeta)$ with $\xi \neq \zeta$ and $\varrho(\xi, \zeta) = \theta$ iff $\xi = \zeta$;
(ii) $\varrho(\xi, \zeta) = \varrho(\zeta, \xi)$;
(iii) $\varrho(\xi, \zeta) \leq s[\varrho(\xi, \eta) + \varrho(\eta, \zeta)]$,
for all $\xi, \zeta, \eta \in \mathrm{X}$. The pair (X, ϱ) is said a cone b-metric space, in short, CBMS.

Lemma 2.2 (see [17]). *If* (X, ϱ) *is a CBMS. Then;*
$(p1)$ *If* $\xi \ll \zeta$ *and* $\zeta \ll \eta$, *then* $\xi \ll \eta$.
$(p2)$ *If* $\xi \ll \zeta$ *and* $\zeta \ll \eta$, *then* $\xi \ll \eta$.
$(p3)$ *If* $\theta \leq \xi \ll c$ *for* $c \in int\mathrm{P}$, *then* $\xi = \theta$.
$(p4)$ *If* $c \in int\mathrm{P}$, $\theta \leq \xi_n$ *and* $\xi_n \to \theta$, *then there exists* n_0 *such that* $\xi_n \ll c$ *for* $n > n_0$.
$(p5)$ *Suppose* $\theta \ll c$, *if* $\theta \leq \varrho(\xi_n, \xi) \leq \zeta_n$ *and* $\zeta_n \to \theta$, *then eventually* $\varrho(\xi_n, \xi) \ll c$, *where* $\xi \in \mathrm{X}$ *and* $\{\xi_n\}_{n \geq 1}$ *is a sequence in* X.
$(p6)$ *If* $\theta \leq \xi_n \leq \zeta_n$ *and* $\xi_n \to \xi$, $\zeta_n \to \zeta$, *then* $\xi \leq \zeta$, *for each cone* P.
$(p7)$ *If* $\xi \leq \lambda \xi$ *where* $\xi \in \mathrm{P}$ *and* $0 \leq \lambda < 1$, *then* $\xi = \theta$.

Definition 2.3 (see [19]). Let X be a vector space over R. For a cone $\mathrm{P} \subset \mathrm{E}$ and a mapping $\| \cdot \|_{\mathrm{E}}$: $\mathrm{X} \to \mathrm{E}$ if we have;

(i) $\| \xi \|_{\mathrm{E}} \geq \theta$ for $\xi \in$ X and $\| \xi \|_{\mathrm{E}} = \theta$ iff $\xi = \theta$;
(ii) $\| \xi + \zeta \|_{\mathrm{E}} \leq \| \xi \|_{\mathrm{E}} + \| \zeta \|_{\mathrm{E}}$ for $\xi, \zeta \in X$;
(iii) $\| k\xi \|_{\mathrm{E}} = | k | \| \xi \|_{\mathrm{E}}$ for $k \in$ R.
Then $\| \cdot \|_{\mathrm{E}}$ is said a cone norm on X, and $(\mathrm{X}, \| \cdot \|_{\mathrm{E}})$ is said a cone normed space (CNS). If we set $\varrho(\xi, \zeta) = \| \xi - \zeta \|_{\mathrm{E}}$, then every CNS is a CMS.

Definition 2.4 (see [19])**.** Let $1 \leq s \leq 2$, X be a vector space over R, cone $\mathrm{P} \subset \mathrm{E}$. If $\| \cdot \|_{\mathrm{P}}$: $\mathrm{X} \to \mathrm{E}$ satisfies;
(i) $\| \xi \|_{\mathrm{P}} \geq \theta$ for $\xi \in$ X and $\| \xi \|_{\mathrm{P}} = \theta$ iff $\xi = \theta$;
(ii) $\| \xi - \zeta \|_{\mathrm{P}} = \| \zeta - \xi \|_{\mathrm{P}}$ for $\xi, \zeta \in$ X;
(iii) $\| \xi + \zeta \|_{\mathrm{P}} \leq s[\| \xi \|_{\mathrm{P}} + \| \zeta \|_{\mathrm{P}}]$ for $\xi, \zeta \in$ X;
(iv) $\| k\xi \|_{\mathrm{P}} = | k |^s \| \xi \|_{\mathrm{P}}$ for $k \in$ R.
Then we call $\| \cdot \|_{\mathrm{P}}$ a cone-norm on X, and $(\mathrm{X}, \| \cdot \|_{\mathrm{P}})$, we call it a cone-normed space (CNS). Obviously, each CNS is a CMS. In fact, we only need to set $\varrho(\xi, \zeta) = \| \xi - \zeta \|_{\mathrm{P}}$.

Definition 2.5 (see [30])**.** Suppose that $(X, \| \cdot \|_{\mathrm{P}})$ is a cone b-normed space, $\mathrm{P} \subset \mathrm{E}$ is a solid cone, $\xi \in$ X and $\{\xi_n\}_{n \geq 1}$ is a sequence in X. Then;
(i) we say that $\{\xi_n\}_{n \geq 1}$ converges to ξ if for $c \in$ E with $\theta \ll c$, there is a natural number N satisfying $\| \xi_n - \xi \|_{\mathrm{P}} \ll c$ for $n \geq N$. We denote $\lim_{n \to \infty} \xi_n = \xi$ or $\xi_n \to \xi$;
(ii) we say that $\{\xi_n\}_{n \geq 1}$ is a Cauchy if for $c \in$ E with $\theta \ll c$, there exists a natural number N satisfying $\| \xi_n - \xi_m \|_{\mathrm{P}} \ll c$ for all $n, m \geq N$;
(iii) we say that $(\mathrm{X}, \| \cdot \|_{\mathrm{P}})$ is complete if every Cauchy is convergent.

Lemma 2.6 (see [30])**.** *Suppose* $(\mathrm{X}, \| \cdot \|_{\mathrm{P}})$ *is a cone b-normed space,* P *is a solid cone,* $\xi \in$ X *and* $\{\xi_n\}_{n \geq 1}$ *is a sequence in* X*. Then the following conclusions hold:*
(i) $\| \xi_n - \xi \|_{\mathrm{P}} \to \theta (n \to \infty)$ *iff* $\{\xi_n\}$ *converges to* ξ*.*
(ii) $\| \xi_n - \xi_m \|_{\mathrm{P}} \to \theta (n, m \to \infty)$ *iff* $\{\xi_n\}$ *is a Cauchy.*

Lemma 2.7 ([29])**.** *Suppose* $(\mathrm{E}, \| \cdot \|)$ *is a real Banach space and* P *is a normal cone in* E*, then there is an equivalent norm* $\| \cdot \|_1$*, which satisfies* $\theta \leq \xi \leq \zeta \Longrightarrow \| \xi \|_1 \leq \| \zeta \|_1$*, for* $\xi, \zeta \in$ E*, that is, norm* $\| \cdot \|_1$ *is monotonous.*

Remark 2.8. Suppose E is a linear space, $\| \cdot \|_1$ and $\| \cdot \|_2$ are two given norms in E, we say that $\| \cdot \|_2$ is stronger than $\| \cdot \|_1$ if $\| \xi_n \|_2 \to 0 \Longrightarrow \| \xi_n \|_1 \to 0$ $(n \to \infty)$. If $\| \cdot \|_2$ is stronger than $\| \cdot \|_1$, and $\| \cdot \|_1$ is stronger than $\| \cdot \|_2$, then $\| \cdot \|_1$ is equivalent $\| \cdot \|_2$.

Definition 2.9 ([29, 7])**.** Let E be a real Banach algebra, that is, for $\xi, \zeta, \eta \in$ E, $a \in$ R,
(i) $\xi(\zeta\eta) = (\xi\zeta)\eta$;
(ii) $\xi(\zeta + \eta) = \xi\zeta + \xi\eta$, $(\xi + \zeta)\eta = \xi\eta + \zeta\eta$;
(iii) $a(\xi\zeta) = (a\xi)\zeta = (a\zeta)\xi$;
(iv) $\| \xi\zeta \| \leq \| \xi \| \| \zeta \|$.
If Banach algebra E with unit element e, such that $\xi e = e\xi = \xi$ for all $\xi \in$ E, then $\| e \| = 1$. If every non-zero element of E has an inverse element in E, then E is called a divisible Banach algebra.

Definition 2.10 ([7]). Let E be a Banach algebra with unit element e and $\mathrm{P} \subseteq \mathrm{E}$ be a cone. P is called algebra cone if $e \in \mathrm{P}$ and for each $\xi, \zeta \in \mathrm{P}$, $\xi\zeta \in \mathrm{P}$.

In our following discussions, $\xi = (\mathrm{X}, \|\cdot\|_{\mathrm{P}})$ is a cone b-Banach space, P is a solid cone and $\mathcal{S}$ is a operator defined on D of X. Let $\mathrm{E} := (\mathrm{E}, \|\cdot\|)$ be a divisible Banach algebra with unit element e. Let $\mathrm{P_E}$ be a normal algebra cone in E with a normal constant N.

Definition 2.11. Let $(\mathrm{E}, \|\cdot\|)$ be a divisible Banach algebra. $\mathrm{P_E}$ is a normal algebra cone in E. We call the mapping $\phi : \mathrm{P_E} \to \mathrm{P_E}$ is a ϕ-operator if it satisfies

(i) ϕ is an increasing operator;

(ii) ϕ is a continuous bijection and has an inverse mapping ϕ^{-1} which is also continuous and increasing;

(iii) $\phi(\xi + \zeta) \leq \phi(\xi) + \phi(\zeta)$ for all $\xi, \zeta \in \mathrm{P_E}$;

(iv) $\phi(\xi\zeta) = \phi(\xi)\phi(\zeta)$ for all $\xi, \zeta \in \mathrm{P_E}$.

Remark 2.12. By Definition 2.11, the part of (iii), we can get $\phi^{-1}(\xi)+\phi^{-1}(\zeta) \leq \phi^{-1}(\xi + \zeta)$ for all $\xi, \zeta \in \mathrm{P_E}$. In fact, note that $\phi(\xi + \zeta) \leq \phi(\xi) + \phi(\zeta)$ for all $\xi, \zeta \in \mathrm{P_E}$ and ϕ^{-1} is also a continuous and increasing operator, then

$$\phi^{-1}(\phi(\xi + \zeta)) \leq \phi^{-1}(\phi(\xi) + \phi(\zeta)),$$

which yields that

$$\xi + \zeta \leq \phi^{-1}(\phi(\xi) + \phi(\zeta)).$$

Hence,

$$\phi^{-1}(\phi(\xi)) + \phi^{-1}(\phi(\zeta)) \leq \phi^{-1}(\phi(\xi) + \phi(\zeta)).$$

Since $\phi : \mathrm{P_E} \to \mathrm{P_E}$ is a continuous bijection, thus $\phi^{-1}(\xi)+\phi^{-1}(\zeta) \leq \phi^{-1}(\xi+\zeta)$, for all $\xi, \zeta \in \mathrm{P_E}$.

Remark 2.13. By Definition 2.11, the part of (iv), we can obtain $\phi^{-1}(\xi\zeta) = \phi^{-1}(\xi)\phi^{-1}(\zeta)$, for all $\xi, \zeta \in \mathrm{P_E}$.

Indeed, from $\phi(\xi\zeta) = \phi(\xi)\phi(\zeta)$ for all $\xi, \zeta \in \mathrm{P_E}$ and $\phi^{-1} : \mathrm{P_E} \to \mathrm{P_E}$ is also a continuous, we get

$$\phi^{-1}(\phi(\xi\zeta)) = \phi^{-1}(\phi(\xi)\phi(\zeta)),$$

which yields that

$$\xi\zeta = \phi^{-1}(\phi(\xi)\phi(\zeta)).$$

Then

$$\phi^{-1}(\phi(\xi))\phi^{-1}(\phi(\zeta)) = \phi^{-1}(\phi(\xi)\phi(\zeta)).$$

Thanks to that $\phi : \mathrm{P_E} \to \mathrm{P_E}$ is a continuous bijection, $\phi^{-1}(\xi\zeta) = \phi^{-1}(\xi)\phi^{-1}(\zeta)$, for all $\xi, \zeta \in \mathrm{P_E}$.

Remark 2.14. For example, let $\mathrm{E} = \mathrm{R}$, a divisible Banach algebra, $\mathrm{P_E} = \{\xi \in \mathrm{E} \mid \xi \geq 0\}$ be a normal cone in E, suppose $\phi : \mathrm{P_E} \to \mathrm{P_E}$, defined by $\phi(\xi) = \xi^{\frac{1}{5}}$ and then $\phi^{-1}(\xi) = \xi^5$, for $\xi \in \mathrm{P_E}$. We can prove it also satisfies the above conditions.

3. Main results

Theorem 3.1. *Let* X *be a cone-b-Banach space with the coefficient* $1 \leq s \leq 2$, E_1 *and* E_2 *be divisible Banach space with identity elements* e_1 *and* e_2*, also* $\mathrm{P}_{\mathrm{E}_1}$ *and* $\mathrm{P}_{\mathrm{E}_2}$ *be normal algebra cones in* E_1 *and* E_2 *(respectively). If* D *and* $\mathrm{D}' \subset \mathrm{X}$ *with* $\mathrm{D} \cap \mathrm{D}' \neq \varnothing$ *are closed and convex, also* $\phi : \mathrm{P}_{\mathrm{E}_1} \cup \mathrm{P}_{\mathrm{E}_2} \to \mathrm{P}_{\mathrm{E}_1} \cup \mathrm{P}_{\mathrm{E}_2}$ *is* ϕ*-operator and* $\mathcal{T} : \mathrm{D} \to \mathrm{D}'$, $\mathcal{S} : \mathrm{D}' \to \mathrm{D}$ *satisfying the followings*

$$\begin{aligned}
&\phi(\varrho(\eta, \mathcal{S}\xi')) + \phi(\varrho(\eta', \mathcal{T}\xi)) \leq k_1 \phi(\varrho(\eta, \xi)), \\
&\phi(\varrho(\xi', \mathcal{T}\eta)) + \phi(\varrho(\xi, \mathcal{S}\eta')) \leq k_2 \phi(\varrho(\xi', \eta')),
\end{aligned} \tag{3.1}$$

for all $\xi, \eta \in \mathrm{D}$, $\xi', \eta' \in \mathrm{D}'$*, where* $\phi(2^s e_1) \leq k_1 < \phi(2^{s+1} e_1)$, $\phi(2^s e_2) \leq k_2 < \phi(2^{s+1} e_2)$ *in* $\mathrm{P}_{\mathrm{E}_1}$ *and* $\mathrm{P}_{\mathrm{E}_2}$ *(respectively). Then* $\mathcal{S}$ *and* $\mathcal{T}$ *have a common fixed point in* $\mathrm{D} \cap \mathrm{D}'$.

Proof. Let $\xi_1 \in \mathrm{D}$, $\eta_1 \in \mathrm{D}'$ be arbitrary. We introduce two sequences $\{\xi_n\}, \{\eta_n\} \in \mathrm{D} \cup \mathrm{D}'$, defined by

$$\begin{aligned}
\xi_2 &= \tfrac{\eta_1 + \mathcal{T}\xi_1}{2} \in \mathrm{D}', \\
\eta_2 &= \tfrac{\xi_1 + \mathcal{S}\eta_1}{2} \in \mathrm{D}, \\
\xi_3 &= \tfrac{\eta_2 + \mathcal{S}\xi_2}{2} \in \mathrm{D}', \\
\eta_3 &= \tfrac{\xi_2 + \mathcal{T}\eta_2}{2} \in \mathrm{D}, \\
&\vdots \\
\xi_{2n} &= \tfrac{\eta_{2n-1} + \mathcal{T}\xi_{2n-1}}{2}, \ n = 1, 2, \cdots, \\
\eta_{2n} &= \tfrac{\xi_{2n-1} + \mathcal{S}\eta_{2n-1}}{2}, \ n = 1, 2, \cdots, \\
\xi_{2n+1} &= \tfrac{\eta_{2n} + \mathcal{S}\xi_{2n}}{2}, \ n = 1, 2, \cdots, \\
\eta_{2n+1} &= \tfrac{\xi_{2n} + \mathcal{T}\eta_{2n}}{2}, \ n = 1, 2, \cdots.
\end{aligned}$$

We get

$$\eta_{2n} - \mathcal{S}\xi_{2n} = 2(\eta_{2n} - (\frac{\eta_{2n} + \mathcal{S}\xi_{2n}}{2})) = 2(\eta_{2n} - \xi_{2n+1}),$$

$$\xi_{2n} - \mathcal{T}\eta_{2n} = 2(\xi_{2n} - (\frac{\xi_{2n} + \mathcal{T}\eta_{2n}}{2})) = 2(\xi_{2n} - \eta_{2n+1}),$$

$$\eta_{2n+1} - \mathcal{T}\xi_{2n+1} = 2(\eta_{2n+1} - (\frac{\eta_{2n+1} + \mathcal{T}\xi_{2n+1}}{2})) = 2(\eta_{2n+1} - \xi_{2n+2}),$$

$$\xi_{2n+1} - \mathcal{S}\eta_{2n+1} = 2(\xi_{2n+1} - (\frac{\xi_{2n+1} + \mathcal{S}\eta_{2n+1}}{2})) = 2(\xi_{2n+1} - \eta_{2n+2}),$$

which is equivalent to

$$\begin{aligned}
\varrho(\eta_{2n}, \mathcal{S}\xi_{2n}) &= \| \eta_{2n} - \mathcal{S}\xi_{2n} \|_{\mathrm{P}_{\mathrm{E}_1}} \\
&= \| 2(\eta_{2n} - \xi_{2n+1}) \|_{\mathrm{P}_{\mathrm{E}_1}} \\
&= 2^s \| \eta_{2n} - \xi_{2n+1} \|_{\mathrm{P}_{\mathrm{E}_1}} \\
&= 2^s \varrho(\eta_{2n}, \xi_{2n+1}),
\end{aligned}$$

$$\begin{aligned}\varrho(\xi_{2n}, \mathcal{T}\eta_{2n}) &= \| \xi_{2n} - \mathcal{T}\eta_{2n} \|_{\mathrm{P}_{\mathrm{E}_2}} \\ &= \| 2(\xi_{2n} - \eta_{2n+1}) \|_{\mathrm{P}_{\mathrm{E}_2}} \\ &= 2^s \| \xi_{2n} - \eta_{2n+1} \|_{\mathrm{P}_{\mathrm{E}_2}} \\ &= 2^s \varrho(\xi_{2n}, \eta_{2n+1})\end{aligned}$$

and

$$\begin{aligned}\varrho(\eta_{2n+1}, \mathcal{T}\xi_{2n+1}) &= \| \eta_{2n+1} - \mathcal{T}\xi_{2n+1} \|_{\mathrm{P}_{\mathrm{E}_2}} \\ &= \| 2(\eta_{2n+1} - \xi_{2n+2}) \|_{\mathrm{P}_{\mathrm{E}_2}} \\ &= 2^s \| \eta_{2n+1} - \xi_{2n+2} \|_{\mathrm{P}_{\mathrm{E}_2}} \\ &= 2^s \varrho(\eta_{2n+1}, \xi_{2n+2}),\end{aligned} \tag{3.2}$$

$$\begin{aligned}\varrho(\xi_{2n+1}, \mathcal{S}\eta_{2n+1}) &= \| \xi_{2n+1} - \mathcal{S}\eta_{2n+1} \|_{\mathrm{P}_{\mathrm{E}_1}} \\ &= \| 2(\xi_{2n+1} - \eta_{2n+2}) \|_{\mathrm{P}_{\mathrm{E}_1}} \\ &= 2^s \| \xi_{2n+1} - \eta_{2n+2} \|_{\mathrm{P}_{\mathrm{E}_1}} \\ &= 2^s \varrho(\xi_{2n+1}, \eta_{2n+2}).\end{aligned} \tag{3.3}$$

Substituting $\xi' = \xi_{2n}, \xi = \xi_{2n+1}$ and $\eta = \eta_{2n}, \eta' = \eta_{2n+1}$ in (3.1), we can obtain

$$\phi(\varrho(\eta_{2n}, \mathcal{S}\xi_{2n})) + \phi(\varrho(\eta_{2n+1}, \mathcal{T}\xi_{2n+1})) \leq k_1 \phi(\varrho(\eta_{2n}, \xi_{2n+1})).$$

We get

$$\phi(2^s \varrho(\eta_{2n}, \xi_{2n+1})) + \phi(2^s \varrho(\eta_{2n+1}, \xi_{2n+2})) \leq k_1 \phi(\varrho(\eta_{2n}, \xi_{2n+1})).$$

According to the condition (*iii*) of ϕ-operator,

$$\phi(2^s(\varrho(\eta_{2n}, \xi_{2n+1}) + \varrho(\eta_{2n+1}, \xi_{2n+2}))) \leq k_1 \phi(\varrho(\eta_{2n}, \xi_{2n+1})).$$

Remark 2.13 and the property of ϕ^{-1} operator, we can get

$$2^s(\varrho(\eta_{2n}, \xi_{2n+1}) + \varrho(\eta_{2n+1}, \xi_{2n+2})) \leq \phi^{-1}(k_1) \varrho(\eta_{2n}, \xi_{2n+1}),$$

by simplifying, we get

$$\varrho(\eta_{2n+1}, \xi_{2n+2}) \leq (\frac{\phi^{-1}(k_1)}{2^s} - e_1) \varrho(\eta_{2n}, \xi_{2n+1}).$$

Substituting $\xi' = \xi_{2n}, \xi = \xi_{2n+1}$ and $\eta = \eta_{2n}, \eta' = \eta_{2n+1}$ in (3.1). Then one can obtain

$$\phi(\varrho(\xi_{2n}, \mathcal{T}\eta_{2n})) + \phi(\varrho(\xi_{2n+1}, \mathcal{S}\eta_{2n+1})) \leq k_2 \phi(\varrho(\xi_{2n}, \eta_{2n+1})).$$

By (3.2) and (3.3) we get

$$\phi(2^s \varrho(\xi_{2n}, \eta_{2n+1})) + \phi(2^s \varrho(\xi_{2n+1}, \eta_{2n+2})) \leq k_2 \phi(\varrho(\xi_{2n}, \eta_{2n+1})).$$

According to the condition (*iii*) of ϕ-operator,

$$\phi(2^s(\varrho(\xi_{2n}, \eta_{2n+1}) + \varrho(\xi_{2n+1}, \eta_{2n+2}))) \leq k_2 \phi(\varrho(\xi_{2n}, \eta_{2n+1})).$$

By Remark 2.13 and the property of ϕ^{-1} operator, we can get

$$2^s(\varrho(\xi_{2n}, \eta_{2n+1}) + \varrho(\xi_{2n+1}, \eta_{2n+2})) \leq \phi^{-1}(k_2) \varrho(\xi_{2n}, \eta_{2n+1}),$$

by simplifying, we get

$$\varrho(\xi_{2n+1},\eta_{2n+2})\leq(\frac{\phi^{-1}(k_2)}{2^s}-e_2)\varrho(\xi_{2n},\eta_{2n+1}).$$

Thus, $\varrho(\eta_{2n+1},\xi_{2n+2})\leq k_1'\varrho(\eta_{2n},\xi_{2n+1})$ and $\varrho(\xi_{2n+1},\eta_{2n+2})\leq k_2'\varrho(\xi_{2n},\eta_{2n+1})$, where $k_1'=\frac{\phi^{-1}(k_1)}{2^s}-e_1$ and $k_2'=\frac{\phi^{-1}(k_2)}{2^s}-e_2$. Repeating this relations, we get

$$\begin{aligned}\varrho(\eta_{2n+1},\xi_{2n+2})&\leq {k_2'}^n{k_1'}^n\varrho(\eta_1,\xi_2),\\ \varrho(\xi_{2n+1},\eta_{2n+2})&\leq {k_2'}^n{k_1'}^n\varrho(\xi_1,\eta_2).\end{aligned}\tag{3.4}$$

For any $m\geq 1,p\geq 1$, we have one of the following two cases:
(i) $m+p=2r-1,\ r\geq 1,r\in\mathrm{N}$, then we get

$$\begin{aligned}\varrho(\eta_{m+p},\xi_m)&\leq s[\varrho(\eta_{m+p},\xi_{m+p-1})+\varrho(\xi_{m+p-1},\xi_m)]\\
&\leq s\varrho(\eta_{m+p},\xi_{m+p-1})+s^2[\varrho(\xi_{m+p-1},\eta_{m+p-2})+\varrho(\eta_{m+p-2},\xi_m)]\\
&\leq s\varrho(\eta_{m+p},\xi_{m+p-1})+s^2\varrho(\xi_{m+p-1},\eta_{m+p-2})+s^3\varrho(\eta_{m+p-2},\xi_{m+p-3})+\cdots\\
&\quad+s^{p-1}\varrho(\xi_{m+2},\eta_{m+1})+s^{p-1}\varrho(\eta_{m+1},\xi_m)\\
&\leq s{k_2'}^{r-2}{k_1'}^{r-1}\varrho(\eta_2,\xi_1)+s^2{k_2'}^{r-3}{k_1'}^{r-2}\varrho(\xi_2,\eta_1)+s^3{k_2'}^{r-4}{k_1'}^{r-3}\varrho(\eta_2,\xi_1)+\cdots\\
&\quad+s^{p-1}{k_2'}^{2r-p-1}{k_1'}^{2r-p}\varrho(\xi_2,\eta_1)+s^{p-1}{k_2'}^{2r-p-2}{k_1'}^{2r-p-1}\varrho(\eta_2,\xi_1)\\
&=(s{k_2'}^{r-2}{k_1'}^{r-1}+s^3{k_2'}^{r-4}{k_1'}^{r-3}+\cdots+s^{p-1}{k_2'}^{2r-p-2}{k_1'}^{2r-p-1})\varrho(\eta_2,\xi_1)\\
&\quad+(s^2{k_2'}^{r-3}{k_1'}^{r-2}+\cdots+s^{p-1}{k_2'}^{2r-p-1}{k_1'}^{2r-p})\varrho(\xi_2,\eta_1)\\
&=\frac{s{k_2'}^{r-2}{k_1'}^{r-1}(e_1e_2-\frac{s^2}{(k_2'k_1')^2})^{\frac{p+1}{2}}}{e_1e_2-\frac{s^2}{(k_2'k_1')^2}}\varrho(\eta_2,\xi_1)\\
&\quad+\frac{s^2{k_2'}^{r-3}{k_1'}^{r-2}(e_1e_2-\frac{s^2}{(k_2'k_1')^2})^{\frac{p-1}{2}}}{e_1e_2-\frac{s^2}{(k_2'k_1')^2}}\varrho(\xi_2,\eta_1).\end{aligned}$$

(ii) $m+p=2r,\ r\geq 1,r\in\mathrm{N}$, then we get

$$\begin{aligned}\varrho(\xi_{m+p},\eta_m)&\leq s[\varrho(\xi_{m+p},\eta_{m+p-1})+\varrho(\eta_{m+p-1},\eta_m)]\\
&\leq s\varrho(\xi_{m+p},\eta_{m+p-1})+s^2[\varrho(\eta_{m+p-1},\xi_{m+p-2})+\varrho(\xi_{m+p-2},\eta_m)]\\
&\leq s\varrho(\xi_{m+p},\eta_{m+p-1})+s^2\varrho(\eta_{m+p-1},\xi_{m+p-2})+s^3\varrho(\xi_{m+p-2},\eta_{m+p-3})+\cdots\\
&\quad+s^{p-1}\varrho(\eta_{m+2},\xi_{m+1})+s^{p-1}\varrho(\xi_{m+1},\eta_m)\\
&\leq s{k_2'}^{r-1}{k_1'}^{r-1}\varrho(\xi_2,\eta_1)+s^2{k_2'}^{r-2}{k_1'}^{r-2}\varrho(\eta_2,\xi_1)+s^3{k_2'}^{r-3}{k_1'}^{r-3}\varrho(\xi_2,\eta_1)+\cdots\\
&\quad+s^{p-1}{k_2'}^{2r-p}{k_1'}^{2r-p}\varrho(\eta_2,\xi_1)+s^{p-1}{k_2'}^{2r-p-1}{k_1'}^{2r-p-1}\varrho(\xi_2,\eta_1)\end{aligned}$$

$$\begin{aligned}
&= (s{k'_2}^{r-1}{k'_1}^{r-1} + s^3{k'_2}^{r-3}{k'_1}^{r-3} + \cdots + s^{p-1}{k'_2}^{2r-p-1}{k'_1}^{2r-p-1})\varrho(\xi_2,\eta_1) \\
&\quad +(s^2{k'_2}^{r-2}{k'_1}^{r-2} + \cdots + s^{p-1}{k'_2}^{2r-p}{k'_1}^{2r-p})\varrho(\eta_2,\xi_1) \\
&= \frac{s{k'_2}^{r-1}{k'_1}^{r-1}(e_1e_2 - \frac{s^2}{(k'_2k'_1)^2})^{\frac{p+1}{2}}}{e_1e_2 - \frac{s^2}{(k'_2k'_1)^2}}\varrho(\xi_2,\eta_1) \\
&\quad + \frac{s^2{k'_2}^{r-2}{k'_1}^{r-2}(e_1e_2 - \frac{s^2}{(k'_2k'_1)^2})^{\frac{p-1}{2}}}{e_1e_1 - \frac{s^2}{(k'_2k'_1)^2}}\varrho(\eta_2,\xi_1).
\end{aligned}$$

Since $\phi(2^se_1) \leq k'_2 < \phi(2^{s+1}e_1)$ in $\mathrm{P}_{\mathrm{E}_1}$ and $\phi(2^se_2) \leq k'_1 < \phi(2^{s+1}e_2)$ in $\mathrm{P}_{\mathrm{E}_2}$ with $1 \leq s \leq 2$, we know $\theta_2 \leq k'_2 < e_1$, $\theta_1 \leq k'_1 < e_2$, thus $\theta_2 < se_2 - k'_2 \leq se_2$, $\theta_1 < se_1 - k'_1 \leq se_1$. Further,

$$\| {k'_2}^{r-p} - \theta_2 \| = \| {k'_2}^{r-p} \| \leq \| k'_2 \|^{r-p},$$

$$\| {k'_1}^{r-p} - \theta_1 \| = \| {k'_1}^{r-p} \| \leq \| k'_1 \|^{r-p}, \tag{3.5}$$

since $\theta_2 \leq k'_2 < e_2$, $\theta_1 \leq k'_1 < e_1$ and P_E is a normal cone in E, by Lemma 2.7 we know there is an equivalent norm $\| \cdot \|_1$ and thus

$$\begin{aligned} 0 &\leq \| k'_2 \|_1 < \| e_2 \|_1 = 1, \\ 0 &\leq \| k'_1 \|_1 < \| e_1 \|_1 = 1. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we get

$$\| {k'_2}^{r-p} - \theta_2 \|_1 \leq \| k'_2 \|_1^{r-p} \to 0((r-p) \to \infty),$$

$$\| {k'_1}^{r-p} - \theta_1 \|_1 \leq \| k'_1 \|_1^{r-p} \to 0((r-p) \to \infty). \tag{3.7}$$

From Remark (2.8) and (3.7),

$$\| {k'_2}^{r-p} - \theta_2 \| \leq \| k'_2 \|^{r-p} \to 0((r-p) \to \infty),$$

$$\| {k'_1}^{r-p} - \theta_1 \| \leq \| k'_1 \|^{r-p} \to 0((r-p) \to \infty).$$

Thus,

$$\lim_{(r-p)\to\infty} {k'_2}^{(r-p)} \to \theta_2,$$

$$\lim_{(r-p)\to\infty} {k'_1}^{(r-p)} \to \theta_1. \tag{3.8}$$

Let $\theta_1, \theta_2 \ll c$ be given. By (3.8),

$$\frac{s{k'_2}^{r-1}{k'_1}^{r}(e_1e_2 - \frac{s^2}{(k'_2k'_1)^2})^{\frac{p+1}{2}}}{e_1e_2 - \frac{s^2}{(k'_2k'_1)^2}}\varrho(\eta_2,\xi_1) + \frac{s^2{k'_2}^{r-2}{k'_1}^{r-1}(e_1e_2 - \frac{s^2}{(k'_2k'_1)^2})^{\frac{p-1}{2}}}{e_1e_2 - \frac{s^2}{(k'_2k'_1)^2}}\varrho(\xi_2,\eta_1) \to \theta_1,$$

$$\frac{s{k'_2}^{r}{k'_1}^{r}(e_1e_2 - \frac{s^2}{(k'_2k'_1)^2})^{\frac{p+1}{2}}}{e_1e_2 - \frac{s^2}{(k'_2k'_1)^2}}\varrho(\xi_2,\eta_1) + \frac{s^2{k'_2}^{r-1}{k'_1}^{r-1}(e_1e_2 - \frac{s^2}{(k'_2k'_1)^2})^{\frac{p-1}{2}}}{e_1e_1 - \frac{s^2}{(k'_2k'_1)^2}}\varrho(\eta_2,\xi_1) \to \theta_2,$$

as $(r-p) \to \infty$. Making full use of Lemma 2.2 ($p4$), we find $m_0 \in N$, such that

$$\frac{sk_2'^{\,r-1}k_1'^{\,r}(e_1e_2 - \frac{s^2}{(k_2'k_1')^2})^{\frac{p+1}{2}}}{e_1e_2 - \frac{s^2}{(k_2'k_1')^2}}\varrho(\eta_2,\xi_1) + \frac{s^2k_2'^{\,r-2}k_1'^{\,r-1}(e_1e_2 - \frac{s^2}{(k_2'k_1')^2})^{\frac{p-1}{2}}}{e_1e_2 - \frac{s^2}{(k_2'k_1')^2}}\varrho(\xi_2,\eta_1) \ll c,$$

$$\frac{sk_2'^{\,r}k_1'^{\,r}(e_1e_2 - \frac{s^2}{(k_2'k_1')^2})^{\frac{p+1}{2}}}{e_1e_2 - \frac{s^2}{(k_2'k_1')^2}}\varrho(\xi_2,\eta_1) + \frac{s^2k_2'^{\,r-1}k_1'^{\,r-1}(e_1e_2 - \frac{s^2}{(k_2'k_1')^2})^{\frac{p-1}{2}}}{e_1e_1 - \frac{s^2}{(k_2'k_1')^2}}\varrho(\eta_2,\xi_1) \ll c,$$

for each $m > m_0$. Thus

$$\frac{sk_2'^{\,r-1}k_1'^{\,r}(e_1e_2 - \frac{s^2}{(k_2'k_1')^2})^{\frac{p+1}{2}}}{e_1e_2 - \frac{s^2}{(k_2'k_1')^2}}\varrho(\eta_2,\xi_1) + \frac{s^2k_2'^{\,r-2}k_1'^{\,r-1}(e_1e_2 - \frac{s^2}{(k_2'k_1')^2})^{\frac{p-1}{2}}}{e_1e_2 - \frac{s^2}{(k_2'k_1')^2}}\varrho(\xi_2,\eta_1) \ll c,$$

$$\frac{sk_2'^{\,r}k_1'^{\,r}(e_1e_2 - \frac{s^2}{(k_2'k_1')^2})^{\frac{p+1}{2}}}{e_1e_2 - \frac{s^2}{(k_2'k_1')^2}}\varrho(\xi_2,\eta_1) + \frac{s^2k_2'^{\,r-1}k_1'^{\,r-1}(e_1e_2 - \frac{s^2}{(k_2'k_1')^2})^{\frac{p-1}{2}}}{e_1e_1 - \frac{s^2}{(k_2'k_1')^2}}\varrho(\eta_2,\xi_1) \ll c,$$

for $m > m_0$ and each p. Considering the upper relations we can get;

$$\varrho(\xi_{m+p},\xi_m) \le s(\varrho(\xi_{m+p},\eta_m) + \varrho(\eta_m + \xi_m)),$$
$$\varrho(\eta_{m+p},\eta_m) \le s(\varrho(\eta_{m+p},\xi_m) + \varrho(\xi_m + \eta_m)).$$

Now by Lemma **??** part ($p1$), we can claim that $\{\xi_n\}$ and $\{\eta_n\}$ are Cauchy sequences in D. Note that D and D$'$ are closed and convex and $\{\xi_{2n}\}$, $\{\eta_{2n}\}$ converges to some ζ,ζ', that is, $\xi_{2n},\eta_{2n} \to \zeta,\zeta' \in \mathrm{D}\cup\mathrm{D}'$. Regarding the inequality

$$\varrho(\zeta,\mathcal{S}\eta_{2n+1}) \le s[\varrho(\zeta,\xi_{2n+1}) + \varrho(\xi_{2n+1},\mathcal{S}\eta_{2n+1})],$$
$$\varrho(\zeta',\mathcal{T}\xi_{2n+1}) \le s[\varrho(\zeta',\eta_{2n+1}) + \varrho(\eta_{2n+1},\mathcal{T}\xi_{2n+1})],$$

and from (3.3), we obtain

$$\varrho(\zeta,\mathcal{S}\eta_{2n+1}) \le s[\varrho(\zeta,\xi_{2n+1}) + 2^s\varrho(\xi_{2n+1},\eta_{2n+2})], \tag{3.8}$$
$$\varrho(\zeta',\mathcal{T}\xi_{2n+1}) \le s[\varrho(\zeta',\eta_{2n+1}) + 2^s\varrho(\eta_{2n+1},\xi_{2n+2})],$$

let $n \to \infty$, then $\mathcal{S}\eta_{2n+1} \to \zeta$, $\mathcal{T}\xi_{2n+1} \to \zeta$. Finally, replacing $\eta_{2n+1} = \zeta$ in (3.8). Then one can obtain

$$\phi(\varrho(\zeta,\mathcal{S}\zeta')) \le s[\varrho(\zeta,\xi_{2n+1}) + 2^s\varrho(\xi_{2n+1},\eta_{2n+2})],$$

and if $\zeta = \xi_{2n+1}$

$$\phi(\varrho(\zeta',\mathcal{T}\zeta)) \le s[\varrho(\zeta',\eta_{2n+1}) + 2^s\varrho(\eta_{2n+1},\xi_{2n+2})],$$

and by making use of the property *iv* of ϕ-operator, we obtain, $\phi(e_1) = e_1, \phi(e_2) = e_2$. So we get

$$\phi(\varrho(\zeta,\mathcal{S}\zeta') = e_1,$$
$$\phi(\varrho(\zeta',\mathcal{T}\zeta) = e_2.$$

Therefore as $n \to \infty$, we can obtain $\mathcal{S}\zeta' = \zeta, \mathcal{T}\zeta = \zeta'$. Hence considering $\zeta = \zeta'$, we conclude $\zeta = T\zeta = S\zeta$. □

Corollary 3.2. *Let* X *be a cone-b-Banach space with the coefficient* $1 \leq s \leq 2$, E *be a divisible Banach algebra with identity element* e*, and also* $\mathrm{P_E}$ *be a normal algebra cone in* E*. If* $\mathrm{D} \subset \mathrm{X}$ *is closed and convex,* $\phi : \mathrm{P_E} \to \mathrm{P_E}$ *is an* ϕ*-operator and* $\mathcal{S}, \mathcal{T} : \mathrm{D} \to \mathrm{D}$ *are mappings satisfying the conditions*

$$\phi(\varrho(\eta, \mathcal{S}\xi)) + \phi(\varrho(\eta, \mathcal{T}\xi)) \leq k\phi(\varrho(\eta, \xi)),$$
$$\phi(\varrho(\xi, \mathcal{T}\eta)) + \phi(\varrho(\xi, \mathcal{S}\eta)) \leq k\phi(\varrho(\xi, \eta)), \tag{3.9}$$

for all $\xi, \eta \in \mathrm{D}$*, where* $\phi(2^s e) \leq k < \phi(2^{s+1}e)$ *in* P_E*. Then* $\mathcal{S}$ *and* $\mathcal{T}$ *have a common fixed point in* D*.*

Proof. If in Theorem 3.1 we set, $\mathcal{T} = \mathcal{S}$ and $\mathrm{D} = \mathrm{D}'$, considering the condition of (3.10) and by the proof similar to the proof of Theorem 3.1 we deduce the result. □

Example 3.3. Let $\mathrm{X} = \mathrm{R}^2$ and $\mathrm{E} = \mathrm{R}^2$ endowed with partial ordered $\xi = (\xi_1, \xi_2) \leq \zeta = (\zeta_1, \zeta_2)$ iff $\xi_1 \leq \zeta_1, \xi_2 \leq \zeta_2$. If $\mathrm{P} = \{(\xi_1, \xi_2) \in \mathrm{E} : \xi_1 \geq 0, \xi_2 \geq 0\}$, we define $\| (\xi_1, \xi_2) \|_{\mathrm{P}} = (| \xi_1 |^2, | \xi_2 |^2)$. Then $(\mathrm{X}, \| \cdot \|_{\mathrm{P}})$ is a cone b-Banach space with $s = 2$.
For $\xi = (\xi_1, \xi_2)$ and $\zeta = (\zeta_1, \zeta_2)$ we define; $\xi.\zeta = (\xi_1\xi_2, \zeta_1\zeta_2)$. By the the mentioned definition P is a Banach algebra and $\mathrm{E} := (\mathrm{E}, \| \cdot \|)$ is a divisible Banach algebra with unit element $e = (1, 1)$, because $\xi e = e\xi = \xi, \|e\| = 1$ and hence e is a multiplicative identity. If we put $\phi : P \to P$ with $\phi(\xi = (\xi_1, \xi_2)) = (\sqrt{\xi_1}, \sqrt{\xi_2})$, then ϕ satisfies the conditions (i)-$(i\mathrm{v})$ of Definition 3.4. Also we set; $\varrho(\xi, \zeta) = \| \xi - \zeta \|_{\mathrm{P}} = (| (\xi_1 - \zeta_1 |^2, | \xi_2 - \zeta_2 |^2), \varrho(\xi, A) = \inf\{\varrho(\xi, \zeta) : \zeta \in A\}$ and $\mathcal{S}\xi = \frac{\xi}{2}, \mathcal{T}\xi = \frac{\xi^2}{4}$. Now we define the region D as the following;

$$\mathrm{D} = \{(\xi_n, \eta_n) : |\eta_n - \frac{\xi_n}{2}| + |\eta_n - \frac{{\xi_n}^2}{4}| \leq 2.8|\eta_n - \xi_n|, |\xi_n - \frac{\eta_n}{2}| + |\xi_n - \frac{{\eta_n}^2}{4}| \leq 2.8|\xi_n - \eta_n|, n = 1, 2\}.$$

Obviously D is closed and convex.

$$\begin{aligned}
&\phi(\varrho(\eta, \mathcal{S}\xi)) + \phi(\varrho(\eta, \mathcal{T}\xi)) \leq \phi(\varrho(\eta, \tfrac{\xi}{2})) + \phi(\varrho(\eta, \tfrac{\xi^2}{4}))\\
&\leq \phi(|\eta_1 - \tfrac{\xi_1}{2}|^2, |\eta_2 - \tfrac{\xi_2}{2}|^2) + \phi(|\eta_1 - \tfrac{{\xi_1}^2}{4}|^2, |\eta_2 - \tfrac{{\xi_2}^2}{4}|^2)\\
&= (|\eta_1 - \tfrac{\xi_1}{2}|, |\eta_2 - \tfrac{\xi_2}{2}|) + (|\eta_1 - \tfrac{{\xi_1}^2}{4}|, |\eta_2 - \tfrac{{\xi_2}^2}{4}|)\\
&= (|\eta_1 - \tfrac{\xi_1}{2}| + |\eta_1 - \tfrac{{\xi_1}^2}{4}|, |\eta_2 - \tfrac{\xi_2}{2}| + |\eta_2 - \tfrac{{\xi_2}^2}{4}|)\\
&\leq (|\eta_1 - \tfrac{\xi_1}{2}| + |\eta_1 - \tfrac{{\xi_1}^2}{4}|, |\eta_2 - \tfrac{\xi_2}{2}| + |\eta_2 - \tfrac{{\xi_2}^2}{4}|),
\end{aligned}$$

also

$$\begin{aligned}
&\phi(\varrho(\xi,\mathcal{T}\eta))+\phi(\varrho(\xi,\mathcal{S}\eta))\leq\phi(\varrho(\xi,\tfrac{\eta^2}{4}))+\phi(\varrho(\xi,\tfrac{\eta}{2}))\\
&\leq\phi(|\xi_1-\tfrac{\eta_1{}^2}{4}|^2,|\xi_2-\tfrac{\eta_2{}^2}{4}|^2)+\phi(|\xi_1-\tfrac{\eta_1}{2}|^2,|\xi_2-\tfrac{\eta_2}{2}|^2)\\
&=(|\xi_1-\tfrac{\eta_1{}^2}{4}|,|\xi_2-\tfrac{\eta_2{}^2}{4}|)+(|\xi_1-\tfrac{\eta_1}{2}|,|\xi_2-\tfrac{\eta_2}{2}|)\\
&=(|\xi_1-\tfrac{\eta_1}{2}|+|\xi_1-\tfrac{\eta_1{}^2}{4}|,|\xi_2-\tfrac{\eta_2}{2}|+|\xi_2-\tfrac{\eta_2{}^2}{4}|)\\
&\leq(|\xi_1-\tfrac{\eta_1}{2}|+|\xi_1-\tfrac{\eta_1{}^2}{4}|,|\xi_2-\tfrac{\eta_2}{2}|+|\xi_2-\tfrac{\eta_2{}^2}{4}|).
\end{aligned}$$

Considering

$$\phi(2^s e)=\phi(4,4)=(2,2)\leq k<\phi(2^{s+1}e)=\phi(8,8)=(2\sqrt{2},2\sqrt{2}),$$

we should have

$$\begin{aligned}
&(|\eta_1-\tfrac{\xi_1}{2}|+|\eta_1-\tfrac{\xi_1{}^2}{4}|,|\eta_2-\tfrac{\xi_2}{2}|+|\eta_2-\tfrac{\xi_2{}^2}{4}|)\\
&\leq(2.8,2.8)(|\eta_1-\xi_1|,|\eta_2-\xi_2|)=(2.8|\eta_1-\xi_1|,2.8|\eta_2-\xi_2|),\\
&(|\xi_1-\tfrac{\eta_1}{2}|+|\xi_1-\tfrac{\eta_1{}^2}{4}|,|\xi_2-\tfrac{\eta_2}{2}|+|\xi_2-\tfrac{\eta_2{}^2}{4}|)\\
&\leq(2.8,2.8)(|\xi_1-\eta_1|,|\xi_2-\eta_2|)=(2.8|\xi_1-\eta_1|,2.8|\xi_2-\eta_2|).
\end{aligned}$$

So according to the definition of region D the conditions of Corollary 3.2 are satisfied. Hence $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point.

Corollary 3.4. *Let* X *be a cone-b-Banach space with the coefficient* $1\leq s\leq 2$, E *be a divisible Banach algebra with identity element* e, *and also* $\mathrm{P_E}$ *be a normal algebra cone in* E *. If* $\mathrm{D}\subset\mathrm{X}$ *is closed and convex,* $\phi:\mathrm{P_E}\to\mathrm{P_E}$ *is an* ϕ*-operator and* $\mathcal{T}:\mathrm{D}\to\mathrm{D}$ *is a mapping satisfying the condition*

$$\phi(\varrho(\eta,\mathcal{T}\eta))+\phi(\varrho(\xi,\mathcal{T}\xi))\leq k\phi(\varrho(\eta,\xi)),$$

for all $\xi,\eta\in\mathrm{D}$, *where* $\phi(2^s e)\leq k<\phi(2^{s+1}e)$ *in* $\mathrm{P_E}$. *Then* $\mathcal{T}$ *has a fixed point in* D.

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