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On the group of homeomorphisms on $\mathbb{R}$: A revisit

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Abstract

In this article, we prove that the group of all increasing homeomorphisms on $\mathbb{R}$ has exactly five normal subgroups, and the group of all homeomorphisms on $\mathbb{R}$ has exactly four normal subgroups. There are several results known about the group of homeomorphisms on $\mathbb{R}$ and about the group of increasing homeomorphisms on $\mathbb{R}$ ([2], [6], [7] and [8]), but beyond this there is virtually nothing in the literature concerning the topological structure in the aspects of topological dynamics. In this paper, we analyze this structure in some detail.

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1. Introduction

There have been several papers discussing about the normal subgroups of the group of homeomorphisms on various metric spaces ([2], [6], [7] and [8]). It is natural to ask: Which subsets will arise as normal subgroups of the group of homeomorphisms on a metric space? We provide a proof in the case of the group of (increasing) homeomorphisms on $\mathbb{R}$. This paper contains a detailed

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proof that highlights the differences and similarities between our results and those given in the references.

A dynamical system is simply a pair (X, f) where X is a metric space and $f : X \to X$ is a continuous function. A point $x \in X$ is said to be periodic with period n if $f^n(x) = x$ for some $n \in \mathbb{N}$, and $f^m(x) \neq x$ for $1 \leq m < n$ where $f^n = f \circ f \circ ... \circ f$ is the composition of f with itself n times. If $f(x) = x$ then we say that x is a fixed point of f. We denote the set of all fixed points of f by $Fix(f)$, and the complement of $Fix(f)$ by $Fix(f)^c$. Two dynamical systems (X, f) and (Y, g) are said to be conjugate (we simply say f is conjugate to g), if there exists a homeomorphism h from X to Y such that $h \circ f = g \circ h$. Being conjugate is an equivalence relation in the class of dynamical systems. A homeomorphism h from X to X such that $h \circ f = f \circ h$ is called a conjugacy of (X, f) or simply a conjugacy. Let A, B be two subgroups of a group G. Then A is invariant in B if $A \subset B$ and if $bAb^{-1} \subset A$ for every $b \in B$. A subgroup N of a group G is normal if and only if it is invariant under conjugation, if and only if it is a union of conjugacy classes of G. Since our group in this paper is a group of homeomorphisms, the algebraic notion of conjugacy here coincides with topological conjugacy in the sense of dynamical systems theory. For preliminaries from topological dynamics and group theory, the reader may refer [3], [4] and [5].

In this paper, we study the normal subgroups of the group of (increasing) homeomorphisms of $\mathbb{R}$ and analyze the topological structure in the aspects of topological dynamics in some detail. The proofs are different from those given in the above references. The classification here is only based on the set of fixed points of members of the normal subgroups. In articles [2], [6] and [8], the set of fixed points or the support of the homeomorphisms are used to classify normal subgroups.

First we discuss the ideas of proof involved in the references [2], [6] and [8] to convince the reader that our proof is different from the known ones. For a set X, let $\pi(X)$ be the group of all permutations (bijections) on X and G be a subgroup of $\pi(X)$. For a topological space X, let $H(X)$ be the group of homeomorphisms on X. Suppose $\mathcal{F}$ is a non-empty family of subsets of X. We define $S(\mathcal{F}, G) = \{g \in G : Fix(g) \supset F \text{ for some } F \in \mathcal{F}\}$. For $S(\mathcal{F}, H(X))$, we shall write $S(\mathcal{F})$. We say that the family $\mathcal{F}$ is ecliptic relative to G whenever it satisfies the following two conditions.

(1) If $F_1, F_2 \in \mathcal{F}$, then there exists an $F_3 \in \mathcal{F}$ such that $F_3 \subset F_1 \cap F_2$,
(2) If $F_1 \in \mathcal{F}$ and $g \in G$, then there exists an $F_2 \in \mathcal{F}$ such that $F_2 \subset g(F_1)$.

An ecliptic family which satisfies the following additional condition will be called strictly ecliptic.

(3) If $F \in \mathcal{F}$ and $U \subset X$ is open $(U \neq \varnothing)$, then there exists an $h \in H(X)$ such that $h(F^c) \subset U$, where F^c is the complement of F in X.

The objective of the reference [8] is to investigate the normal subgroups for a class of spaces which includes the n-cell B_n and the author proved that some of these normal subgroups can be defined in terms of the family of fixed point sets of their elements. For a family $\mathcal{F}$ of subsets of X, define $\mathcal{S}(\mathcal{F}, G) = \{g \in G : Fix(g) \supset F$ for some $F \in \mathcal{F}\}$. It is proved that $\mathcal{S}(\mathcal{F}, G)$ is a subgroup of G and if $\mathcal{F}$ is ecliptic relative to G, then $\mathcal{S}(\mathcal{F}, G)$ is a normal subgroup of G. If X is a topological space such that for any non-empty, open set U, there is an open set $V \subset U$ which is homeomorphic to an open ball in a Euclidean space of positive dimension and supposing there is a strictly ecliptic family $\mathcal{F}$ on X relative to $H(X)$ and if N is a normal subgroup of $H(X)$, then the author proved that either $N \supset S(\mathcal{F})$ or N consists of the identity 1. They also proved that if N is a normal subgroup of $H(B_n)$ which contains an element not in $H_0(B_n) = \{h \in H(B_n) : Fix(h) \supset S_{n-1}$(the boundary of B_n)$\}$, then $N \supset H_0(B_n)$.

The objective of the reference [2] is to analyze the algebraic structure of $H(I)$ in some detail for some interval I. For a subgroup H of $\pi(X)$, $x \in X$, let $H_x :=$ isotropy subgroup of H at x. The authors proved that every translation is a product of two involutions and every element of $H(I)$ is a product of at most four involutions. They considered a signature theorem, which provides a useful criteria for the conjugacy in $H(I)$. Using this idea, they enumerate completely the normal subgroups of $H := H(I) \leq \pi(I)$. Let F be the isotropy subgroup H_0 of $H(\mathbb{R})$. The idea of proof is as follows. For an interval I, we denote I^0 for the interior of I. For a map $f : I \to I$, $S(f, x) :=$sign$(f(x) - x)$. An element $t \in F$ is a translation if it does not have interior points as fixed points and let T denote the set of all translations. Also $T^+ := \{t \in T : S(t, x) > 0\}$ and $T^- := \{t \in T : S(t, x) < 0\}$. Now, let S be a semi-group which is invariant in F, $Q_a = \{f \in H : f(x) = x$ for all x in some neighborhood $N_f(a)\}$ for $a \in I$ and $Q = Q_0 \cap Q_1$. The authors proved that if $S \not\subset Q_0$, then S contains an element with at most one interior fixed point. Also T^+ and T^- are complete conjugacy classes in F, T is a complete conjugacy in H and $F = TT$. Using these ideas, the authors also proved that if N is an invariant subgroup of F, then $N \subset Q_0$ or $N \subset Q_1$ or $N = F$. If N is an invariant subgroup of H then either $N = H$, $N = F$ or $N \subset Q$. Now the only subgroups of H are H, F, Q and $\{1\}$ since Q is simple. If N is normal in Q_0 (respectively in Q_1), then either $N = \{1\}$, $N = Q$ or $N = Q_0$ (respectively in Q_1). Hence the only normal subgroups of F are F, Q_0, Q_1, Q and $\{1\}$.

The reference [6] is an expository paper, the author provides a relatively complete but concise account of the classification of $H := H(I)$, in terms of a suitable topological signature concept. For $\phi \in H$, the author first associated the function $s(\phi) : \mathbb{R} \to S = \{-1, 0, 1\}$ defined by $s(\phi)(x) = sign(\phi(x) - x)$. For $s \in \Sigma := \{h : \mathbb{R} \to S : h$ is continuous$\}$, let $Spt(s) = \mathbb{R} \setminus int(s^{-1}(0))$, $H_a = \{\phi \in H : Spt(\phi)$ is bounded above$\}$, $H_b = \{\phi \in H : Spt(\phi)$ is bounded below$\}$ and $H_c = \{\phi \in H : Spt(\phi)$ is bounded$\}$.

The author first observed the following facts:

(1) $s(\phi^{-1}) = -s(\phi)$
(2) For $\phi_1, \phi_2 \in H^+$ (the set of all increasing homeomorphisms) with $s(\phi_1) \geq 0$ and $s(\phi_2) \geq 0$, it holds $s(\phi_1 \circ \phi_2) \geq \max\{s(\phi_1), s(\phi_2)\}$.

These facts provide a one-to-one correspondence between the collection of normal subgroups N of H (resp. H^+) and $\Sigma(N)$, the family of s-functions closed under the operation $(s(\phi), s(\psi)) \to s(\phi^{-1} \circ \psi)$ and closed under topological equivalence.

Consider the group $H(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : \ f \textit{ is a homeomorphism}\}$ under composition of functions, and its subgroups $IH(\mathbb{R}) = \{f \in H(\mathbb{R}) : \ f \textit{ is increasing}\}$, $H_l = \{f \in IH(\mathbb{R}) : Fix(f)^c \textit{ is bounded above}\}$ and $H_r = \{f \in IH(\mathbb{R}) : Fix(f)^c \textit{ is bounded below}\}$.

Our main results prove that:

(1) The group $IH(\mathbb{R})$ has exactly five normal subgroups. They are:
   - (a) The whole group $IH(\mathbb{R})$
   - (b) The trivial group $\{1\}$
   - (c) H_l
   - (d) H_r
   - (e) $H = H_l \cap H_r$.

(2) For $H(\mathbb{R})$ there are exactly four normal subgroups. They are:
   - (a) The whole group $H(\mathbb{R})$
   - (b) The trivial group $\{1\}$
   - (c) $H = \{f \in IH(\mathbb{R}) : Fix(f)^c \text{ is bounded}\}$
   - (d) $IH(\mathbb{R})$.

2. Main results

Let $IH([a, b])$ denote the group (under composition of functions) of all increasing homeomorphisms on the closed interval $[a, b]$ and let $H([a, b])$ denote the group (under composition of functions) of all homeomorphisms on the closed interval $[a, b]$. In fact $H(\mathbb{R})$ and $H([a, b])$ are topological groups with respect to compact-open topology. This happens since the homeomorphism group on a locally connected and locally compact second countable space is a topological group (see [1]). Consider $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$ with order topology. Any closed interval $[a, b]$ in $\mathbb{R}$ is homeomorphic to $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$, and the groups $IH([a, b])$ and $IH(\mathbb{R}^*)$ are isomorphic.

We write:

(1) $C_A([a, b]) = \{f : [a, b] \to [a, b] : \ f \textit{ is a homeomorphism such that } f(t) > t \ \forall \ t \ \in \ (a, b)\}$
(2) $C_B([a, b]) = \{f : [a, b] \to [a, b] : \ f \textit{ is a homeomorphism such that } f(t) < t \ \forall \ t \ \in \ (a, b)\}$.
(3) $H = H_l \cap H_r = \{f \in IH(\mathbb{R}) : Fix(f)^c \textit{ is bounded}\}$

For $f \in C_A([a, b]) \cup C_B([a, b])$, we have $f(a) = a$ and $f(b) = b$. Hence $C_A([a, b])$ and $C_B([a, b])$ are subsets of $IH([a, b])$. We define $C_A([a, b]) \circ C_B([a, b]) := \{f \circ g : f \in C_A([a, b]), g \in C_B([a, b])\}$. Two continuous maps $f, g : \mathbb{R} \to \mathbb{R}$ are said to be *order conjugate* if there exists an increasing homeomorphism h on $\mathbb{R}$ such that $h \circ f = g \circ h$. The maps $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$ and $g(x) = x - 1$ are conjugate to each other through $h(x) = -x + \frac{1}{2} \in H(\mathbb{R})$. But f and g are not order conjugate. Contrary to assume there is an $h \in IH(\mathbb{R})$ such that $h\circ f = g\circ h$. Then $h(x+1) = h(x)-1$. That is, $h(x+1)-h(x) = -1 < 0$. Which is a contradiction to the assumption that $h \in IH(\mathbb{R})$ and hence the maps f and g are not order conjugate.

Lemma 2.1.

(1) *Assume that $f, g \in IH(\mathbb{R})$ are such that $Fix(f) = Fix(g)$*
  (a) *f is conjugate to g;*
  (b) *If for every $t \in \mathbb{R}$ it holds $(f(t)-t)(g(t)-t) \geq 0$ then f and g are order conjugate.*

(2) *Assume that $f, g \in IH([a,b])$ are such that $Fix(f) = Fix(g)$*
  (a) *f is conjugate to g;*
  (b) *If for every $t \in [a,b]$ it holds $(f(t)-t)(g(t)-t) \geq 0$ then f and g are order conjugate.*

Proof. (1) Assume that $f, g \in IH(\mathbb{R})$ are such that $Fix(f) = Fix(g)$

(a) Case 1: $Fix(f) = Fix(g) = \varnothing$
Assume that $f(0) > 0$. For $n \in \mathbb{N}$, inductively define $f^{-n} = f^{-n+1} \circ f^{-1}$. Since f is increasing $(f^n(0))$ increases and thus diverges to ∞, and $(f^{-n}(0))$ decreases and diverges to $-\infty$. Moreover, for $t \in \mathbb{R}$ there exists unique $n \in \mathbb{Z}$ such that, $f^n(0) \leq t < f^{n+1}(0)$. Consider this unique n, and define $k : (-\infty, f(0)) \to (-\infty, 1)$ by $k(t) = \frac{t}{f(0)}$ and $h : \mathbb{R} \to \mathbb{R}$ by $h(t) = k(f^{-n}(t)) + n$ for $t \in \mathbb{R}$. Note that $h(f(0)) = 1$ and h is a homeomorphism of $\mathbb{R}$. Then $h \circ f(t) = h(t) + 1 \ \forall \ t \in \mathbb{R}$. This h gives a conjugacy from f to $x+1$. Similarly we can prove that, f is conjugate to $x-1$ if $f(0) < 0$. The maps $x+1$ and $x-1$ are conjugate to each other. Hence the proof.
Case 2: $Fix(f) = Fix(g) \neq \varnothing$
In this case, define $\tilde{f} : \mathbb{R}^* \to \mathbb{R}^*$ by $\tilde{f}|_{\mathbb{R}}$ (the restriction map $\tilde{f}$ to $\mathbb{R}$)$= f$, $\tilde{f}(-\infty) = -\infty$ and $\tilde{f}(\infty) = \infty$. Similarly define $\tilde{g}$ also. Let $Fix(\tilde{f})^c = Fix(\tilde{g})^c = \cup_\alpha (a_\alpha, b_\alpha)$ (disjoint union of open intervals). The restriction maps $\tilde{f}|_{[a_\alpha,b_\alpha]} : [a_\alpha, b_\alpha] \to [a_\alpha, b_\alpha]$ and $\tilde{g}|_{[a_\alpha,b_\alpha]} : [a_\alpha, b_\alpha] \to [a_\alpha, b_\alpha]$ are increasing with $a_\alpha = \tilde{f}(a_\alpha) = \tilde{g}(a_\alpha)$ and $b_\alpha = \tilde{f}(b_\alpha) = \tilde{g}(b_\alpha)$ and $Fix(\tilde{f}|_{(a,b)}) = Fix(\tilde{g}|_{(a,b)}) = \varnothing$. Let h_α be a conjugacy from $\tilde{f}|_{[a_\alpha,b_\alpha]}$ to $\tilde{g}|_{[a_\alpha,b_\alpha]}$ for every α. By Case 1, this conjugacy h_α exists for every α. Define $h : \mathbb{R}^* \to \mathbb{R}^*$ as $h(x) = \begin{cases} h_\alpha(x) & \text{if } x \in (a_\alpha, b_\alpha) \\ x & \text{otherwise} \end{cases}$. Then h is a conjugacy from $\tilde{f}$ to $\tilde{g}$. Hence $h|_{\mathbb{R}}$ is a conjugacy from f to g.

(b) Suppose $(f(t)-t)(g(t)-t) \geq 0$ for all $t \in \mathbb{R}$. If $Fix(f) = Fix(g) = \varnothing$ then $f(0)g(0) > 0$. This implies either both $f(0)$ and $g(0)$ are positive or both $f(0)$ and $g(0)$ are negative. Hence both f and g are either order conjugate to $x+1$ or to $x-1$. If $Fix(f) = Fix(g) \neq \varnothing$ then consider the maps $\tilde{f}, \tilde{g}$ as in Case 2 of (1) (a) in the proof of Lemma 2.1. If $Fix(\tilde{f})^c = Fix(\tilde{g})^c = \cup_\alpha (a_\alpha, b_\alpha)$ (disjoint union of open intervals) then the restriction maps $\tilde{f}|_{[a_\alpha,b_\alpha]} : [a_\alpha, b_\alpha] \to [a_\alpha, b_\alpha]$ and $\tilde{g}|_{[a_\alpha,b_\alpha]} : [a_\alpha, b_\alpha] \to [a_\alpha, b_\alpha]$ are increasing with $a_\alpha = \tilde{f}(a_\alpha) = \tilde{g}(a_\alpha)$ and $b_\alpha = \tilde{f}(b_\alpha) = \tilde{g}(b_\alpha)$, and $(\tilde{f}(t)-t)(\tilde{g}(t)-t) > 0$ for all $t \in (a_\alpha, b_\alpha)$. If h_α is an order conjugacy from $\tilde{f}|_{[a_\alpha,b_\alpha]}$ to $\tilde{g}|_{[a_\alpha,b_\alpha]}$ for every α then the map $h : \mathbb{R}^* \to \mathbb{R}^*$ defined by $h(x) = \begin{cases} h_\alpha(x) & \text{if } x \in (a_\alpha, b_\alpha) \\ x & \text{otherwise} \end{cases}$ is an order conjugacy from f to g. Hence the proof follows.

(2) Assume that $f, g \in IH([a,b])$ are such that $Fix(f) = Fix(g)$. Without loss of generality, we can assume that $f, g \in IH(\mathbb{R}^*)$. Then $f(-\infty) = -\infty$ and $f(\infty) = \infty$. Hence $Fix(f|_\mathbb{R}) = Fix(g|_\mathbb{R})$.
   (a) By (1) (a) of Lemma 2.1, $f|_\mathbb{R}$ is conjugate to $g|_\mathbb{R}$ and hence f is conjugate to g;
   (b) By (2) (a) of Lemma 2.1, if for every $t \in \mathbb{R}$ it holds $(f(t)-t)(g(t)-t) \geq 0$ then $f|_\mathbb{R}$ and $g|_\mathbb{R}$ are order conjugate and hence f is order conjugate to g.

□

For a map $f : \mathbb{R} \to \mathbb{R}$, if $t_1, t_2 \in Fix(f)$ and $s \notin Fix(f)$ for all $s \in (t_1, t_2)$ then we say that t_1 and t_2 are adjacent.

Lemma 2.2. *Let $f \in IH(\mathbb{R})$ and let $\{\{a_\alpha, b_\alpha\}\}_\alpha$ be the pairs of adjacent fixed points. Define $g : \mathbb{R} \to \mathbb{R}$ by*
$$g(x) = \begin{cases} \frac{x+f(x)}{2} & \text{if } a_\alpha < x < b_\alpha \text{ for some } \alpha \\ f(x) & \text{otherwise} \end{cases}$$
Then f is order conjugate to g.

Proof. The proof follows from Lemma 2.1. □

Proposition 2.3. $C_A([a,b]) \circ C_B([a,b]) = IH([a,b])$.

Proof. Let $h \in IH([a,b])$. Define $h_A(x) := \begin{cases} h(x) & \text{if } h(x) > x \\ x & \textit{otherwise} \end{cases}$ and $h_B(x) := \begin{cases} h(x) & \text{if } h(x) < x \\ x & \textit{otherwise} \end{cases}$. Then $h = h_A \circ h_B$. But $h_A \notin C_A([a,b])$ and $h_B \notin C_B([a,b])$. Now consider $g(x) = x + 1$. Then $g(x) > x$ for all $x \in \mathbb{R}$. Let $h'_A = h_A \circ g$ and $h'_B = g^{-1} \circ h_B$. Then $h \circ g(x) > h(x)$ for all x. Therefore $h \circ g(x) > x$ whenever $h(x) > x$. Hence $h'_A(x) > x$ if $h(x) > x$. If $h(x) \leq x$ then $h'_A(x) = g(x) > x$. Hence $h'_A \in C_A([a,b])$. Similarly we can prove that $h'_B(x) \in C_B([a,b])$. Hence the proof follows since $h = h'_A \circ h'_B$. □

Corollary 2.4. *If N is a normal subgroup of $IH([a,b])$ that contains an element of either $C_A([a,b])$ or $C_B([a,b])$ then $N = IH([a,b])$.*

Proof. By Lemma 2.1, the sets $C_A([a,b])$ and $C_B([a,b])$ are exactly the conjugacy classes of $IH([a,b])$, and $C_A([a,b]) = \{f^{-1} : f \in C_B([a,b])\}$. So if subgroup N is normal and intersect either $C_A([a,b])$ or $C_B([a,b])$, then it automatically contains these sets. Hence $N = IH([a,b])$ by Proposition 2.3. □

We introduce the following notation:
For $f \in IH(\mathbb{R})$ and $t_0 \in Fix(f)$, we denote
$f_{t_0}(x) := \begin{cases} \frac{x+f(x)}{2} & \text{if } x \geq t_0 \\ f(x) & \text{if } x < t_0 \end{cases}$, $f^*_{t_0}(x) := \begin{cases} f(x) & \text{if } x \geq t_0 \\ x & \text{if } x < t_0 \end{cases}$, and $f^{**}_{t_0}(x) := \begin{cases} x & \text{if } x \geq t_0 \\ f(x) & \text{if } x < t_0 \end{cases}$.

Lemma 2.5. *Let $f \in IH(\mathbb{R})$ and let $t_0 \in Fix(f)$. Then $f^*_{t_0}$ is order conjugate to $f^{-1}_{t_0} \circ f$.*

Proof. For $f \in IH(\mathbb{R})$, first observe that $f_{t_0}|_{[t_0,\infty)}$ is $\frac{x+f(x)}{2}$ and $f^*_{t_0}|_{[t_0,\infty)}$ is $f(x)$. Hence by Lemma 2.1, $f_{t_0}|_{[t_0,\infty)}$ is order conjugate to $f^*_{t_0}|_{[t_0,\infty)}$. For $t \in [t_0, \infty)$, first suppose that $f(t) - t \geq 0$. Then $f(t) \geq \frac{t+f(t)}{2}$. Which implies $f^{-1}_{t_0}(f(t)) \geq f^{-1}_{t_0}(\frac{t+f(t)}{2}) = t$. If $f(t) - t \leq 0$ then we can prove that $f^{-1}_{t_0}(f(t)) \leq t$. Hence $(f(t) - t)((f^{-1}_{t_0} \circ f)(t) - t) \geq 0$ for all $t \in [t_0, \infty)$. By Lemma 2.1, $f^*_{t_0}$ is order conjugate to $f^{-1}_{t_0} \circ f$ on $[t_0, \infty)$. Also $f^{-1}_{t_0} \circ f|_{(-\infty,t_0)}$ is the identity function. Hence the proof follows. □

Corollary 2.6. *Let N be a normal subgroup of $IH(\mathbb{R})$. Let $f \in N$ and let $t_0 \in Fix(f)$. Then $f^*_{t_0} \in N$.*

Proof. By Lemma 2.1, f_{t_0} is order conjugate to f and by Lemma 2.5, $f^*_{t_0}$ is order conjugate to $f^{-1}_{t_0} \circ f$. Hence the proof follows. □

Proposition 2.7. *Let N be a normal subgroup of $IH(\mathbb{R})$. If there exists an $f \in N$ with $Fix(f) \neq \varnothing$ is bounded above then $H_r \subset N$.*

Proof. Let N be a normal subgroup of $IH(\mathbb{R})$ and let $f \in N$ such that $Fix(f)$ is bounded above and let $t_0 = \operatorname{Sup} Fix(f)$, the supremum of $Fix(f)$. If $g \in N$ with $g(t_0) = t_0$ and $h \in IH(\mathbb{R})$ with $h(t_0) = t_0$ then $h|_{[t_0,\infty)} \circ g|_{[t_0,\infty)} \circ h^{-1}|_{[t_0,\infty)}$ is the same as $h \circ g \circ h^{-1}|_{[t_0,\infty)}$. Hence $N|_{[t_0,\infty)} = \{g|_{[t_0,\infty)} : g \in N, \ g(t_0) = t_0\}$ is a normal subgroup of $IH([t_0, \infty))$. Since $t_0 = Sup\ Fix(f)$, either $f(t) > t$ or $f(t) < t$ on (t_0, ∞). That is, $f|_{[t_0,\infty]} \in C_A([t_0, \infty])$ or $C_B([t_0, \infty])$. Then by Corollary 2.1, it follows that $N|_{[t_0,\infty)} = IH([t_0, \infty))$. Now let $\phi \in H_r$. Choose a fixed point s_0 of ϕ such that every number less than s_0 is also a fixed point of ϕ. Consider $\chi(t) = t_0 - s_0 + \phi(t - t_0 + s_0)$. If $\tau(t) = t - t_0 + s_0$ then $\phi \circ \tau = \tau \circ \chi$. Hence ϕ is order conjugate to χ. Observe that $\chi = \chi^*_{t_0}$. Hence the order conjugate χ of ϕ is the identity outside $[t_0, \infty)$. Now $\chi|_{[t_0,\infty)} \in IH([t_0, \infty))$. Hence there exists $\tilde{\chi} \in N$ such that $\tilde{\chi} = \chi$ on $[t_0, \infty)$. Thus $\tilde{\chi}^*_{t_0} = \chi$, and hence $\chi \in N$. Then $\phi \in N$ since N is normal. Hence $H_r \subset N$. □

Remark 2.8. Let N be a normal subgroup of $IH(\mathbb{R})$. If there exists $f \in N$ such that $Fix(f) \neq \varnothing$ is bounded below then by considering analogues arguments involved in the proof of Proposition 2.7, we have $H_l \subset N$.

Remark 2.9. Let N be a subgroup of $IH(\mathbb{R}^*)$. If $Fix(f) = \{-\infty, \infty\}$ for all $f \in N$ then either $f \in C_A([-\infty, \infty])$ or $f \in C_B([-\infty, \infty])$. Hence by

Proposition 2.3, $N = IH(\mathbb{R}^*)$. From this it follows that, if N is a subgroup of $IH(\mathbb{R})$ with $Fix(f) = \varnothing$ then $N = IH(\mathbb{R})$.

Remark 2.10. If N be a normal subgroup of $IH(\mathbb{R})$ with $f \in N$ and $t_0 \in Fix(f)$, then analogues to Corollary 2.6, we can prove that $f^{**}_{t_0} \in N$.

Corollary 2.11. *Let $t_1 < t_2$ be adjacent fixed points of some $f \in N$ where N is a normal subgroup of $IH(\mathbb{R})$. If* $g(x) = \begin{cases} f(x) & \text{if } t_1 < x < t_2 \\ x & \text{otherwise} \end{cases}$ *then $g \in N$.*

Proof. Observe that $g = f^*_{t_1} \circ {f^*_{t_2}}^{-1}$. Hence $g \in N$ by Corollary 2.6. □

The following two lemmas are important to prove our main theorem. We consider these lemmas before considering our main theorem. We first make a back ground to complete the proof of following lemma. Let $f : \mathbb{R} \to \mathbb{R} \in IH(\mathbb{R})$ with unique fixed point a. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = f(a + t) - a$ for $t \in \mathbb{R}$. Then $g(0) = 0$ if and only if $f(a) = a$, and g is order conjugate to f by the order conjugacy $h(t) = a + t$ for $t \in \mathbb{R}$. By Lemma 2.1, there are only 3 elements in $IH(\mathbb{R})$ with a unique fixed point upto order conjugacy. Let $f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ \frac{x}{2} & \text{if } x < 0 \end{cases}$ for $x \in \mathbb{R}$ and $g(x) := \begin{cases} 2x & \text{if } x \geq 1 \\ \frac{1}{2}(3x+1) & \text{if } -1 \leq x \leq 1 \\ \frac{x-1}{2} & \text{if } x \leq -1 \end{cases}$.
Observe that f has a unique fixed point at 0 and g has unique fixed point at -1. Then the map $g \circ f$ has no fixed points. By Corollary 2.6 and Remark 2.10, it follows that, if N is a non-trivial normal subgroup of $IH(\mathbb{R})$ and contains an element with a unique fixed point then it contains an element without fixed points. Next, let $f, g : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = \begin{cases} x^2 + 1 & \text{if } 0 \leq x \leq 1 \\ x + 1 & \text{otherwise} \end{cases}$ and $g(x) = x - 1$. Then f and g has no fixed points, and $g \circ f$ has only two adjacent fixed points 0 and 1.

Lemma 2.12. *The group H is the smallest non-trivial proper normal subgroup of $IH(\mathbb{R})$.*

Proof. Let N be a non-trivial normal subgroup of $IH(\mathbb{R})$. Suppose there exists $f \in N$ with adjacent fixed points $t_1 < t_2$. If ϕ is an element of $IH([t_1, t_2])$ such that $\phi = f|_{[t_1,t_2]}$ then by Corollary 2.11, the extension $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined by $\tilde{\phi} = \begin{cases} \phi(x) & \text{if } t_1 < x < t_2 \\ x & \text{otherwise} \end{cases}$ is also in N. Let $h \in H = \{f \in IH(\mathbb{R}) : Fix(f)^c \textit{ is bounded}\}$. Without loss of generality assume that t_1 be the infimum of $Fix(h)^c$ and t_2 be the supremum of $Fix(h)^c$. Then $t_1, t_2 \in Fix(h)$. By Lemma 2.2, there exists an order conjugate $\hat{h}$ of h such that $\tilde{\phi} \circ \hat{h}^{-1} \in N$. Then $\hat{h} \in N$ since $\hat{h} \circ \tilde{\phi} \circ \hat{h}^{-1} \in N$. Which implies $h \in N$. This proves $H \subset N$ whenever there exists $f \in N$ with adjacent fixed points. By Lemma 2.1, any homeomorphisms on $\mathbb{R}$ without fixed points is either order conjugate to $x + 1$ or to $x - 1$. If N contains an element with a unique fixed point then it contains an element without fixed points. Therefore it follows that N always contains an element with atleast two adjacent fixed points. Hence the proof. □

Lemma 2.13. *Let N be a normal subgroup of $IH(\mathbb{R})$. If $f \in N$ such that both $Fix(f)$ and $Fix(f)^c$ are not bounded above, then N contains an element such that its set of all fixed points is bounded above.*

Proof. Let $f \in N$ and $t_0 \in \mathbb{R}$ be a fixed point of f. By Corollary 2.6, $f^*_{t_0} \in N$. Let $\{(a_\alpha, a_{\alpha+1})\}_\alpha$ be the collection of all intervals of $\mathbb{R}$ such that $f^*_{t_0}(a_\alpha) = a_\alpha$, $f^*_{t_0}(a_{\alpha+1}) = a_{\alpha+1}$ and either $f^*_{t_0}(t) > t$ or $f^*_{t_0}(t) < t$ for all $t \in (a_\alpha, a_{\alpha+1})$. Consider a collection of intervals $\{(b_\alpha, b_{\alpha+1})\}_\alpha$ of $\mathbb{R}$ such that $b_\alpha < a_\alpha < b_{\alpha+1} < a_{\alpha+1}$ for all α and an increasing homeomorphism g on $\mathbb{R}$ which is order conjugate to $f^*_{t_0}$ such that $g(b_\alpha) = b_\alpha$, $g(b_{\alpha+1}) = b_{\alpha+1}$ and either $g(t) > t$ or $g(t) < t$ for all $t \in (b_\alpha, b_{\alpha+1})$. This is possible since $IH([a, b])$ is isomorphic to $IH([c, d])$ for any intervals $[a, b]$ and $[c, d]$. Without loss of generality we can assume that $f^*_{t_0}$ and g do not have common fixed points which are greater than t_0. Then $f^*_{t_0} \circ g \in N$ and $Fix(f^*_{t_0} \circ g)$ is bounded above. Hence the proof. □

Remark 2.14. Let N be a normal subgroup of $IH(\mathbb{R})$.

1. If there exists $f \in N$ such that $Fix(f)$ and $Fix(f)^c$ are not bounded above. Then $H_r \subset N$ by Proposition 2.7 and Lemma 2.13.
2. If there exists $f \in N$ with $Fix(f)$ and $Fix(f)^c$ are not bounded below then $H_l \subset N$. This follows by considering analogous arguments involved in the proof of Lemma 2.13 and by Remark 2.8.

Now we are ready to prove our main theorems:

Theorem 2.15. *The group $IH(\mathbb{R})$ has exactly five normal subgroups. They are:*

1. *The whole group $IH(\mathbb{R})$*
2. *The trivial group $\{1\}$*
3. H_l
4. H_r
5. $H = H_l \cap H_r$.

Proof. Let N be a non-trivial normal subgroup of the group $IH(\mathbb{R})$.

Suppose that the function $x+1$ is in N. We claim that $N = IH(\mathbb{R})$. Consider a function $f \in IH(\mathbb{R})$, and let $g = (f \vee (x-1)) \wedge (x+1)$, where $\vee$, $\wedge$ denote the maximum, and minimum of functions respectively. Then $g \in IH(\mathbb{R})$ and it is at a distance ≤ 1 from the diagonal. Then $Fix(f) = Fix(g)$, and $f(x) - x$ and $g(x) - x$ have the same sign between any two adjacent fixed points. Hence g is order conjugate to f. Now it is enough to prove $g \in N$. As $g(x) \geq x-1$, $\forall x \in \mathbb{R}$, we have $g(x) + 2 > x$ $\forall x \in \mathbb{R}$ and therefore the function $g(x) + 2$ is order conjugate to the function $\phi(x) = x+1$, since any two elements of $IH(\mathbb{R})$ whose graphs are above the diagonal are order conjugate. Then $g + 2 \in N$ and hence $\phi^{-1} \circ \phi^{-1} \circ (g + 2) = g \in N$, where $(g + 2)(x) = g(x) + 2$ for $x \in \mathbb{R}$. But f is order conjugate to g. Therefore $f \in N$. Thus $N = IH(\mathbb{R})$. Similarly, if $x - 1$ is in N then also we can prove that $N = IH(\mathbb{R})$. If there is an element f in N without fixed point then $N = IH(\mathbb{R})$. This is because f is order conjugate to either $x - 1$ or $x + 1$ by Lemma 2.1.

Now, let $\phi \in N$ be such that it has arbitrarily large fixed points and arbitrarily large non-fixed points (that is, $Fix(\phi)$ and $Fix(\phi)^c$ are not bounded above). Then by Lemma 2.13 and by Remark 2.14, $H_r \subset N$. Analogues to the above claim, if there exists $\phi \in N$ such that $Fix(\phi)$ and $Fix(\phi)^c$ are not bounded below then $H_l \subset N$ by Remark 2.14. Now suppose $Fix(\phi)^c$ is bounded below for every $\phi \in N$. Then $N \subset H_r$. Also if $Fix(\phi)^c$ is bounded above for every $\phi \in N$ then $N \subset H_l$.

Therefore from the following table we conclude that either $N \subset H_l$ or $H_r \subset N$.

(1)	There exists $\psi \in N$ such that $Fix(\psi)$ is bounded above	Then $H_r \subset N$
(2)	There exists $\phi \in N$ such that neither $Fix(\phi)$ nor $Fix(\phi)^c$ is bounded above	Then (1) by Lemma 2.13
(3)	For every $\phi \in N$, $Fix(\phi)^c$ is bounded above	Then $N \subset H_l$

Similarly by considering the following table analogues to the above table, we can show that $N \subset H_r$ or $H_l \subset N$.

(1)	There exists $\psi \in N$ such that $Fix(\psi)$ is bounded below	Then $H_l \subset N$
(2)	There exists $\phi \in N$ such that neither $Fix(\phi)$ nor $Fix(\phi)^c$ is bounded below	Then (1) by Remark 2.14
(3)	For every $\phi \in N$, $Fix(\phi)^c$ is bounded below	Then $N \subset H_r$

Now there are only four possibilities for a non-trivial normal subgroup N of $IH(\mathbb{R})$:

Case: 1 $N \subset H_l$ and $N \subset H_r$

In Case: 1, $N \subset H_l \cap H_r = H$. Therefore $N = H$.

Case: 2 $N = H_l$

Case: 3 $N = H_r$

Case: 4 $H_r \subset N$ and $H_l \subset N$

In Case 4, $H_r \cup H_l \subset N$. Let $f(x) := \begin{cases} x & \text{if } x \geq 0 \\ \frac{1}{2}x & \text{if } x < 0 \end{cases}$ and $g(x) := \begin{cases} 2x & \text{if } x \geq 1 \\ \frac{3x+1}{2} & \text{if } -1 \leq x \leq 1 \\ x & \text{if } x \leq -1 \end{cases}$. Then $f \in H_l$ and $g \in H_r$. Hence $g \circ f \in N$ and it has no fixed points. Therefore $x + 1 \in N$. Hence $N = IH(\mathbb{R})$.

By Lemma 2.12, H is the smallest normal subgroup contained in $IH(\mathbb{R})$. Hence the other two cases $H_r \subset N \subset H_l$ and $H_l \subset N \subset H_r$ are not possible. This completes the proof. □

Remark 2.16. Let N be a non-trivial normal subgroup of the group $IH(\mathbb{R})$ and suppose that there is an element in N such that it has only two fixed points. Let ϕ be in N such that it has only two fixed points namely a and b, $a < b$. Consider $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$ with order topology. If $\phi \in IH(\mathbb{R}^*)$ fixes only $-\infty$ and ∞ then $\phi|_{\mathbb{R}} \in IH(\mathbb{R})$ and it has no other fixed points. Let N^* be a normal subgroup of $IH(\mathbb{R}^*)$ containing a map, fixing only $-\infty$ and ∞. Consider $N_* = \{f|_{\mathbb{R}} : f \in N^*\}$. Then N_* is a normal subgroup of $IH(\mathbb{R})$ and $\phi|_{\mathbb{R}} \in N_*$. Then $\phi|_{\mathbb{R}}$ has no fixed points and hence $N_* = IH(\mathbb{R})$. Therefore $N^* = IH(\mathbb{R}^*)$. Then N contains an element without fixed points since $IH(\mathbb{R})$, $IH(\mathbb{R}^*)$ and $IH([a,b])$ are isomorphic. Hence $N = IH(\mathbb{R})$. In this case, note that $H_l \cup H_r \subset N$, and therefore N becomes $IH(\mathbb{R})$ since there is an element in N without fixed points.

For a subgroup H of a group G, we denote $H \leq G$.

Theorem 2.17. *For $H(\mathbb{R})$ there are exactly four normal subgroups. They are:*

1. *The whole group $H(\mathbb{R})$.*
2. *The trivial group $\{1\}$.*
3. $H = \{f \in IH(\mathbb{R}) : Fix(f)^c$ *is bounded*$\}$.
4. $IH(\mathbb{R})$.

Proof. Let G be a group and $K \leq N \leq G$. If K is normal in G then K is normal in N also. The subgroups H_l and H_r are not normal in $H(\mathbb{R})$. Hence by Theorem 2.15, if $N = IH(\mathbb{R})$ and $G = H(\mathbb{R})$, then either $K = \{1\}$ or $K = H$ or $K = IH(\mathbb{R})$. Next suppose $IH(\mathbb{R}) \leq K \leq H(\mathbb{R})$. Then the index $[H(\mathbb{R}) : K] \leq [H(\mathbb{R}) : IH(\mathbb{R})] = 2$. Therefore either $K = H(\mathbb{R})$ or $K = IH(\mathbb{R})$. Finally, suppose there is a normal subgroup N of $H(\mathbb{R})$ such that $N = A \cup B$ with A is a proper subgroup of $IH(\mathbb{R})$ and $\varnothing \neq B \subset H(\mathbb{R}) \setminus IH(\mathbb{R})$. Which implies $IH(\mathbb{R}) \cup B$ is a normal subgroup of $H(\mathbb{R})$. But there is no such normal subgroup for $H(\mathbb{R})$ other than $H(\mathbb{R})$ itself. Hence the proof. □

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