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Strongly path-confluent mappings

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Abstract

In this paper, we introduce a new class of path-confluent mapping, called strongly path-confluent maps. We discuss and study some characterizations and some basic properties of this class of mappings. Relations between this class and some other existing classes of mappings are also obtained. Also we study some operations on this class of mappings, such as: composition property, composition factor property, component restriction property and path-component restriction property.

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1. Introduction and preliminaries

Throughout this work a space will always mean a topological space on which no separation axiom is assumed unless explicitly stated and all mappings are assumed to be continuous. In this paper, we obtain a new kind of path-confluent mapping called, strongly path-confluent. A subset K of a space X is said to be a continuum if K is connected and compact. Using this idea of a continuum, Charatonik introduced and studied the concept of confluent mapping between topological spaces [1] as follows: a mapping $f : X \longrightarrow Y$ is said to be confluent provided that for each subcontinuum K of Y and for each component C of $f^{-1}(K)$ we have $f(C) = K$. Motivated by Charatonik's work, we have introduced the notions of quasi-confluent and path-confluent mappings in [9, 10] and studied their basic properties. Recall that space X is said to be connected between two subsets A and B if there is no closed-open set F such that $A \subset F$ and $B \cap F = \phi$. The connectedness of a space X between points is an equivalence relation on X. The equivalence classes of this relation are called quasi-components of the space X, that is, a quasi-component of a space X containing a point $p \in X$ is the set of all points $x \in X$ such that the space X is connected between $\{p\}$ and $\{x\}$. In other words, a quasi-component of a space X containing a point $p \in X$ is the intersection of all closed and open subsets of X containing p (see [7]). Moreover a mapping $f : X \longrightarrow Y$ is said to be quasi-confluent provided that for each continuum K subset of Y and for each quasi-component QC of $f^{-1}(K)$ we have $f(QC) = K$. The notion of path-component of a point $p \in X$ is a maximal path-connected subset of X containing p and it is denoted by $PC(X, p)$ (see [5]). By using this notion we introduced the appropriate definition for path-confluent mapping as follows: A mapping $f : X \longrightarrow Y$ is said to be path-confluent provided that for each continuum K subset of Y and for each path-component PC of $f^{-1}(K)$ we have $f(PC) = K$.

In this paper we are interested in further generalization of the work of Charatonik in the context of path-components and connectedness. In Section 2 we introduce the notion of strongly path-confluent mapping and study some characterizations and some basic properties of this class of mappings. Also we study its relation with other known classes of generalized confluent mappings, namely the classes of confluent, quasi-confluent, path-confluent, and strongly confluent mappings. In Section 3 we study the composition property and composition factor property for this class. In Section 4 we study the notion of path-component restriction property for the class of strongly path-confluent mappings.

We denote by C, QC(or QT), and PC(or PT) the components, quasi-components, and path-components of any topological spaces X at any point $x \in X$, respectively, and the symbol $\mathbb{N}$ is used for the set of natural numbers.

Now we recall some known notions, definitions which will be used in this work.

Definition 1.1 ([4])**.** A mapping $f : X \longrightarrow Y$ is said to be strongly confluent provided for each connected non-empty subset K of Y and for each component C of $f^{-1}(K)$ we have $f(C) = K$.

Definition 1.2 ([11])**.** A mapping $f : X \longrightarrow Y$ is a local homeomorphism if for each point $x \in X$ there exists an open neighborhood U of x such that $f(U)$ is an open neighborhood of $f(x)$ and the restriction mapping $f \mid_U : U \longrightarrow f(U)$ is a homeomorphism.

Definition 1.3 ([2])**.** A class $\mathfrak{M}$ of mappings between topological spaces is said to have the component restriction property provided that for each mapping $f : X \longrightarrow Y \in \mathfrak{M}$ and for each B subset of Y, if $A \subset X$ is the union of some components of $f^{-1}(B)$, the restriction mapping $f \mid_A : A \longrightarrow f(A) \in \mathfrak{M}$.

2. Strongly path-confluent mappings

In this section we introduce the concept of strongly path-confluent mapping and discuss some of its interesting properties and its relations with other known mappings.

Definition 2.1. A mapping $f : X \longrightarrow Y$ is said to be strongly path-confluent provided that for each connected K subset of Y and for each path-component PC of $f^{-1}(K)$ we have $f(PC) = K$.

Proposition 2.2. *The following statements are true:*

(1) *every strongly path-confluent mapping is path-confluent;*
(2) *every strongly path-confluent mapping is confluent;*
(3) *every strongly path-confluent mapping is strongly confluent;*
(4) *every strongly path-confluent mapping is quasi-confluent.*

Proof. The proof comes directly from the definitions. □

The following diagram follows immediately from the definitions in which none of these implications is reversible as shown by the following example.

$$\begin{tikzcd}
\text{strongly path-confluent} \arrow[rr] \arrow[dd] \arrow[dr] & & \text{strongly confluent} \arrow[dl] \arrow[dd] \\
& \text{quasi-confluent} & \\
\text{path-confluent} \arrow[rr] \arrow[ur] & & \text{confluent} \arrow[ul]
\end{tikzcd}$$

Diagram 2.1

Example 2.3. Let $X = \{(x, y) : x = 0, \quad \text{and} \quad y \in [-1, 1]\} \cup \{(x, y) : y = \sin\frac{\pi}{x} : x \in (0, 1]\} \subset \mathbb{R}^2$ with usual topology and $Y = X/R$, where R the equivalence relation in X, given by $R = \{((0, y), (0-y)) : y \in [-1, 1]\} \cup \{(p, p) : p \in X\}$. Then the natural projection $f : X \longrightarrow Y$ is:

(1) confluent;
(2) quasi-confluent;
(3) strongly confluent.

But, it is neither path-confluent nor strongly path-confluent mappings.

Proposition 2.4. *Let $f : X \longrightarrow Y$ be a mapping. If Y is totally disconnected space, then the following items are equivalent:*

(1) *f is strongly path-confluent;*
(2) *f is path-confluent.*

Proof. (1) $\Longrightarrow$ (2): Follows immediately from the Proposition 2.2.
(2) $\Longrightarrow$ (1): The proof comes directly from the fact that in totally disconnected space the classes of connected and continua are the same. □

Recall that a space X is called almost discrete if every open subset of X is closed; equivalently, if every closed subset of X is open (see [6, 7]).

Proposition 2.5. *Let $f : X \longrightarrow Y$ be a mapping of locally path-connected space X into totally disconnected space Y. If X is almost discrete and Y is Hausdorff, then the following items are equivalent:*

(1) *f is strongly path-confluent;*
(2) *f is strongly confluent;*
(3) *f is path-confluent;*
(4) *f is confluent;*
(5) *f is quasi-confluent.*

Proof. The proof of the implication (1) $\Longrightarrow$ (2) is obvious.
(2) $\Longrightarrow$ (3): Suppose that mapping $f : X \longrightarrow Y$ is a strongly confluent. Let K be any continuum subset of Y and PC be any path-component of $f^{-1}(K)$. Then, K is connected subset of Y. Since Y is totally disconnected and Hausdorff space, then K is closed singleton set. Since, X is locally path-connected and almost discrete space, then the set $f^{-1}(K)$ is locally path-connected. Hence, its components and path-component are the same. Thus, $f(PC) = K$ by assumption. Therefore f is path-confluent mapping.
The proofs of the implications (3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (5) are obvious.
(5) $\Longrightarrow$ (1): Assume that mapping $f : X \longrightarrow Y$ is a quasi-confluent. Let K be any connected subset of Y and PC be any path-component of $f^{-1}(K)$. Since Y is totally disconnected and Hausdorff space, then K is closed singleton set. Thus, K is continuum in Y. Since, X is locally path-connected and almost discrete space, then the set $f^{-1}(K)$ is locally path-connected. Hence, its quasi-components and path-components are the same. Thus $f(PC) = K$ by assumption. Therefore f is path-confluent mapping. □

Proposition 2.6. *Let $f : X \longrightarrow Y$ be a mapping of space X into compact space Y. If every connected subset of Y is closed, then the following items are equivalent:*

(1) *f is strongly path-confluent;*
(2) *f is path-confluent.*

Proof. (1) $\Longrightarrow$ (2) :Obvious.
(2) $\Longrightarrow$ (1): Let $K \subseteq Y$ be an arbitrary connected, and PC be an arbitrary path-component of $f^{-1}(K)$. Then $K \subseteq Y$ is closed by the assumption. So, K is compact subset of Y. Thus, K is continuum subset of Y. Then $f(K) = PC$ by assumption. Therefore, f is strongly path-confluent. □

Proposition 2.7. *If $f : X \longrightarrow Y$ is a mapping of space X into compact Hausdorff space Y, then the following statements are equivalent:*

(1) *f is path-confluent;*
(2) *for each closed connected $K \subseteq Y$, the path-components of $f^{-1}(K)$ are mapped into K under f.*

Proof. (1) $\Longrightarrow$ (2): Let $K \subseteq Y$ be any closed connected, and PC be any path-component $f^{-1}(K)$. Since, Y is compact space, then K is compact. Thus, K is continuum in Y. So, $f(PC) = K$ by the path-confluence of f.
(2) $\Longrightarrow$ (1): Let K be any continuum in Y , and PC be any path-component of $f^{-1}(K)$. Since, Y is Hausdorff space, then K is closed. So, K is closed connected subset of Y. Thus, $f(PC) = K$. Hence, f is path-confluent mapping. □

Recall that a connected space X is said to be σ-connected provided that it can not be decomposed into countably many mutually separated non-empty subsets, (see[3, 8]). Also a space X is said to be hereditarily σ-connected provided that it is connected and each connected subset of it is σ-connected (see[4]).

The following theorem shows the hereditarily σ-connected property is strongly path-confluent property.

Theorem 2.8. *A surjective strongly path-confluent mapping preserves hereditarily σ-connected spaces.*

Proof. Assume that mapping $f : X \longrightarrow Y$ be a surjective strongly path-confluent such that X is a hereditarily σ-connected space, and suppose on the contrary that Y is not hereditarily σ-connected. Let K be any connected subset of Y such that

$$K = \bigcup_{i=1}^{\infty} A_i,$$

where A_i and A_j are non-empty mutually separated sets for $i \neq j$ and $i, j \in \mathbb{N}$. Then $f^{-1}(A_i)$ and $f^{-1}(A_j)$ are non-empty mutually separated for $i \neq j$ and $i, j \in \mathbb{N}$. Let PC be a path-component of $f^{-1}(K)$. Since, f is strongly path-confluent, then we have $f(PC) = K$. So, we infer that

$$PC \cap f^{-1}(A_i) \neq \phi \quad \text{for} \quad i \in \mathbb{N}$$

Since, $PC \subset f^{-1}(K) = f^{-1}(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} f^{-1}(A_i)$ then,

$$PC = \bigcup_{i=1}^{\infty}(f^{-1}(A_i) \cap PC),$$

where $f^{-1}(A_i) \cap PC$ and $f^{-1}(A_j) \cap PC$ are non-empty mutually separated sets for $i \neq j$ and $i, j \in \mathbb{N}$. But this contradicts the fact that PC is σ-connected set. Thus, K is σ-connected. Hence, Y is hereditarily σ-connected space. □

Theorem 2.9. *Let $f : X \longrightarrow Y$ be a mapping between topological spaces X and Y such that $Y = Y_1 \cup Y_2$ is a decomposition of Y into connected subsets. If the following properties hold:*

(1) *either $Y_1 \cap Y_2 \neq \phi$ or Y_1, and Y_2 are separated;*
(2) *the intersection of any two connected subsets of Y is connected;*
(3) *$f \mid_{f^{-1}(Y_i)}$ is strongly path-confluent mapping for $i = 1, 2$.*

Then f is strongly path-confluent.

Proof. Let K be an arbitrary connected subset of Y, and PC be any path-component of $f^{-1}(K)$. Assume that $Y = Y_1 \cup Y_2$, if $Y_1 \cap Cl(Y_2) = \phi = Cl(Y_1) \cap Y_2$, then either $K \subseteq Y_1$ or $K \subseteq Y_2$. So that by condition (3) we infer that $f(PC) = K$. Therefore, f is strongly path-confluent mapping. Suppose that $Y_1 \cap Y_2 \neq \phi$ and that $K - Y_1 \neq \phi \neq K - Y_2$. Let $x \in PC$ such that $f(x) \in Y_1$. Since, $K \cap Y_1$ is a connected subset of Y_1, if PC_1 is a path-component of x in $f^{-1}(K \cap Y_1)$, then by the condition (3), we have

$$f(PC_1) = K \cap Y_1, \quad \text{and} \quad PC_1 \subseteq PC. \tag{2.1}$$

Also, let $x' \in PC_1 \cap f^{-1}(y)$ such that $y \in K \cap Y_1 \cap Y_2 \neq \phi$. Since, $K \cap Y_2$ is a connected subset of Y_2, if PC_2 is a path-component of x' in $f^{-1}(K \cap Y_2)$, then by the condition (3), we have

$$f(PC_2) = K \cap Y_2, \quad \text{and} \quad PC_2 \subseteq PC. \tag{2.2}$$

By (2.1) and (2.2) we obtain

$$K = K \cap (Y_1 \cup Y_2) = (K \cap Y_1) \cup (K \cap Y_2) = f(PC_1) \cup f(PC_2) \subseteq f(PC).$$

But, we always have $f(PC) \subseteq K$. So, $f(PC) = K$. Hence, f is strongly path-confluent mapping. □

The following corollary is a generalization of the Theorem 2.9.

Corollary 2.10. *If $f : X \longrightarrow Y$ be a mapping, and $Y_1, Y_2, ..., Y_k$ are connected subsets of Y such that $Y = Y_1 \cup ... \cup Y_k$. If the following properties hold:*

(1) *either $Y_i \cap Y_j \neq \phi$ or Y_i, Y_j are separated, for each $i \neq j$ and $i, j \in \{1, ..., k\}$;*
(2) *the intersection of any two connected subsets of Y is connected;*
(3) *$f \mid_{f^{-1}(Y_i)}$ is strongly path-confluent mapping for $i \in \{1, ..., k\}$.*

Then f is strongly path-confluent.

3. The composition properties

We say that a class $\mathfrak{M}$ of mappings has the composition property provided that for any two mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ belonging to $\mathfrak{M}$ then their composition gof belongs to $\mathfrak{M}$. Also, we say that a class $\mathfrak{M}$ of mappings has the composition factor property provided that for any two mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ such that gof belongs to $\mathfrak{M}$, then g belongs to $\mathfrak{M}$.

Before we prove the main results in this section, we need to introduce the following lemma.

Lemma 3.1. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two mappings. If f is a strongly path-confluent and $h = gof$, then for each connected K subset of Z, and each path-component PC of $h^{-1}(K)$, we have $f(PC)$ is a path-component of $g^{-1}(K)$.*

Proof. Let K be any connected in Z, and PC be any path-component of $h^{-1}(K)$. Since $PC \subseteq h^{-1}(K) = (gof)^{-1}(K) = f^{-1}(g^{-1}(K))$, then $PC \subseteq f^{-1}(g^{-1}(K))$. So, $f(PC) \subseteq g^{-1}(K)$. It is obviously that $f(PC) \subseteq PT$ for some path-component PT of $g^{-1}(K)$. Then, $PC \subseteq f^{-1}(PT)$. Since, $f(PC) \subseteq PT \subseteq g^{-1}(K)$, then $PC \subseteq f^{-1}(PT) \subseteq f^{-1}(g^{-1}(K)) = (gof)^{-1}(K) = h^{-1}(K)$. Thus, PC is the path-component of $f^{-1}(PT)$. Now, let Q be any connected in PT. So, $f^{-1}(Q) \subseteq f^{-1}(PT)$. Since PC is a path-component of $f^{-1}(PT)$, then, $PC \cap f^{-1}(Q)$ is the path-component of $f^{-1}(Q)$. But, f is strongly path-confluent mapping. Thus,

$$f(PC \cap f^{-1}(Q)) = Q = f(PC) \cap Q.$$

Which implies that $Q \subseteq f(PC) \subseteq g^{-1}(K)$. Hence, $f(PC) = PT$. Therefore, $f(PC)$ is a path-component of $g^{-1}(K)$. □

The following theorem shows that the class of strongly path-confluent mappings has the composition property.

Theorem 3.2. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two strongly path-confluent mappings. Then $h = gof$ is strongly path-confluent mapping.*

Proof. Let $K \subseteq Z$ be a connected and PC be any path-component of $h^{-1}(K)$. Since, f is a strongly path-confluent mapping, then $f(PC)$ is a path-component of $g^{-1}(K)$ by the Lemma 3.1. Then by the strongly path-confluence of g, we infer that $h(PC) = g(f(PC)) = K$. Hence, $h = gof$ is strongly path-confluent mapping. □

Proposition 3.3. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two mappings. If f is a homeomorphism and g is a strongly path-confluent mapping, then gof is a strongly path-confluent mapping.*

Proof. Let K be a connected subset of Z, and PC be the path-component of the inverse image $(gof)^{-1}(K)$. We want to prove that $gof(PC) = K$. Obviously $PC \subseteq (gof)^{-1}(K) = f^{-1}g^{-1}(K)$. So, $f(PC) \subseteq g^{-1}(K)$. Since, f is a homeomorphism, then $f(PC)$ is the path-component of $g^{-1}(K)$. Since, g is strongly path-confluent, then $g(f(PC)) = K$. Therefore, gof is strongly path-confluent mapping. □

The following theorem shows that the class of strongly path-confluent mappings has the composition factor property.

Theorem 3.4. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two mappings. If $h = gof$ is strongly path-confluent mapping, then g is also strongly path-confluent mapping.*

Proof. Let K be a connected subset of Z and PC be any path-component of $g^{-1}(K)$. Then

$$g(PC) \subseteq K. \tag{3.1}$$

On the other hand, since, h is a strongly path-confluent we have that for each $x \in f^{-1}(PC)$

$$h(PC(h^{-1}(K), x)) = K.$$

Then $f(PC(h^{-1}(K), x)) \subseteq PC$. So, we get

$$K = gf(PC(h^{-1}(K), x)) \subseteq g(PC). \tag{3.2}$$

Then from (3.1) and (3.2), we get $g(PC) = K$. Hence, g is strongly path-confluent. □

Remark 3.5. If $h = gof$ is strongly path-confluent mapping, then f is not necessarily strongly path-confluent mapping as shown by the following example.

Example 3.6. Let $X = \{a, b, c, d, e\}$, $Y = \{\ell, m, n, o\}$ and $Z = \{p, q, r\}$ with topologies $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{c, d, e\}, \{c, d, a, e\}\}, \sigma = \{\phi, Y, \{n\}, \{n, o\}\}$, and $\gamma = \{\phi, Z\}\}$ defined on X, Y and Z, respectively. Let $f : X \longrightarrow Y$ be a mapping defined by:

$$f(a) = \ell, f(b) = m, f(c) = f(d) = n, \quad \text{and} \quad f(e) = o.$$

Also let $g : Y \longrightarrow Z$ be a mapping defined by:

$$g(\ell) = g(m) = p, g(n) = r, \quad \text{and} \quad g(o) = q.$$

Assume that $h = g \circ f : X \longrightarrow Z$ which is defined by:

$$h(a) = h(b) = p, h(c) = h(d) = r, \quad \text{and} \quad h(e) = q.$$

Then the mappings h and g are strongly path-confluent, but f is not strongly path-confluent. Note that, if we take the subcontinuum $K = \{\ell, n\}$ of Y. Then the path-components of $f^{-1}(K) = \{a, c, d\}$ are $PC = \{a\}$ and $PT = \{c, d\}$ and $f(PC) = \ell \neq K$ and $f(PT) = n \neq K$. Hence, f is not strongly path-confluent mapping.

Now, the following theorem clarifies under certain conditions, the mapping f will be strongly path-confluent. Let a mapping $f : X \longrightarrow Y$ be given. Recall that a subset $A \subset X$ is said to be an inverse set under f provided that $A = f^{-1}(f(A))$, (see [1]).

Theorem 3.7. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two mappings. If $h = g \circ f$ is strongly path-confluent mapping, and if every set in Y is an inverse set, then f is strongly path-confluent mapping.*

Proof. Let K be any connected set in Y, and let PC be any path-component of $f^{-1}(K)$. Then, $g(K)$ will be connected in Z. Since $g^{-1}(g(K)) \subset Y$, and since, every set in Y is an inverse set, then K is an inverse set and, thus $g^{-1}(g(K)) = K$. Then $f^{-1}g^{-1}(g(K)) = f^{-1}(K)$, which implies that $h^{-1}(g(K)) = (gof)^{-1}(g(K)) = f^{-1}(K)$. That is, PC is a path-component of $h^{-1}(g(K))$. Since h is a strongly path-confluent mapping. So, $h(PC) = g(K)$. Therefore, $f(PC) = K$. Hence, f is a strongly path-confluent mapping. □

Corollary 3.8. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two mappings. If $h = g \circ f$ is strongly path-confluent mapping, and g is a homeomorphism then f is strongly path-confluent mapping.*

Proof. Since, g is a homeomorphism, then every set in Y is an inverse set. Then, by Theorem 3.7, the mapping f is strongly path-confluent. □

4. The path-component restriction property

In this section we study the path-component restriction property for the class of strongly path-confluent mappings.

Definition 4.1 ([10])**.** A class $\mathfrak{M}$ of mappings between topological spaces is said to have the path-component restriction property provided that for each mapping $f : X \longrightarrow Y \in \mathfrak{M}$ and for each B subset of Y, if $A \subset X$ is the union of some path-components of $f^{-1}(B)$, the restriction mapping $f \mid_A : A \longrightarrow f(A) \in \mathfrak{M}$.

The following theorem shows that the class of strongly path-confluent mappings has the path-component restriction property.

Theorem 4.2. *Let $f : X \longrightarrow Y$ be a strongly path-confluent mapping and let $B \subseteq Y$ and $A \subset X$ is the union of some path-components of $f^{-1}(B)$. Then the restriction mapping $f \mid_A : A \longrightarrow f(A)$ is a strongly path-confluent.*

Proof. Assume that the mapping $f : X \longrightarrow Y$ is a strongly path-confluent. Take $B \subseteq Y$, and A is the union of some path-components of $f^{-1}(B)$. Suppose that K be any connected subset of $f(A)$, and let PC and PT be the path-components of $(f\,|_A)^{-1}(K)$ and $f^{-1}(K)$, respectively. Since,

$$(f\,|_A)^{-1}(K) = A \cap f^{-1}(K), \quad \text{then}$$

$$PC \subset PT. \tag{4.1}$$

Since, $PC \subset A$, then $\phi \neq PC = A \bigcap PC \subset A \bigcap PT$. But, $K \subset f(A) \subset B$. So, $f^{-1}(K) \subset f^{-1}(f(A)) \subset f^{-1}(B)$. According to the assumption on A, we get $PT \subset f^{-1}(K) \subset A$, whence $PT \subset (f\,|_A)^{-1}(K) \subset f^{-1}(K)$. Which implies that

$$PT \subset PC. \tag{4.2}$$

Then from (4.1) and (4.2), we get $PC = PT$, and $f \mid_A(PC) = f(PC) = f(PT) = K$. Therefore the restriction mapping $f \mid_A$ is a strongly path-confluent. □

The following corollary is a particular case of the Theorem 4.2.

Corollary 4.3. *Let $f : X \longrightarrow Y$ be a strongly path-confluent mapping. Let $A \subset X$ be an inverse set under f, then the restriction mapping $f \mid_A : A \longrightarrow f(A)$ is also strongly path-confluent.*

Proposition 4.4. *Let $f : X \longrightarrow Y$ be a strongly path-confluent mapping. If f is local homeomorphism, then restriction mapping $f \mid_U : U \longrightarrow f(U)$ is also strongly path-confluent for every open subset U of X.*

Proof. Since, f is local homeomorphism, then the restriction mapping $f \mid_U : U \longrightarrow f(U)$ is a homeomorphism. Which means that $U = f^{-1}(f(U))$. Hence, $U \subseteq X$ is an inverse set. By the Corollary 4.3, we infer that $f \mid_U$ is strongly path-confluent mapping. $\square$

The following proposition shows that if the mapping $f : X \longrightarrow Y$ has the component restriction property, then it also has the path-component restriction property, which means that the path-component restriction property is weaker notion than the notion of component restriction property.

Proposition 4.5. *Let $\mathfrak{M}$ denote the class of mappings. If $\mathfrak{M}$ has the component restriction property, then it has also the path-component restriction property.*

Proof. Assume that class $\mathfrak{M}$ of mappings has the component restriction property and let $f : X \longrightarrow Y \in \mathfrak{M}$. Take B be a subset of Y, and $A \subset X$ is the union of some components of $f^{-1}(B)$. Then the restriction mapping $f \mid_A : A \longrightarrow f(A) \in \mathfrak{M}$. We need to show that $A \subset X$ is the union of some path-components of the set $f^{-1}(B)$. Now, let $A = \bigcup_{\alpha\in\Delta} C_\alpha$, for some components C_α of the inverse set $f^{-1}(B)$, where Δ be the index set. Since, each component is a disjoint union of path-components, then we can put $C_\alpha = \bigcup_{\beta\in \mathrm{I}} PC_\beta$ with $\bigcap_{\beta\in \mathrm{I}} PC_\beta = \phi$, for some path-components PC_β of the inverse set $f^{-1}(B)$, where I be the index set. Hence, we get $A = \bigcup_{\alpha\in\Delta} C_\alpha = \bigcup_{\alpha\in\Delta} (\bigcup_{\beta\in \mathrm{I}} PC_\beta)$. Then by the Definition 4.1, the class $\mathfrak{M}$ of mappings has the path-component restriction property. $\square$

Proposition 4.6. *The classes of strongly path-confluent mappings has the component restriction property.*

Proof. Let $\mathfrak{M}$ be the class of strongly path-confluent mappings, and let $f : X \longrightarrow Y \in \mathfrak{M}$. Take $B \subseteq Y$, and A is the union of some components of $f^{-1}(B)$. Let $K \subset f(A)$ be a subcontinuum, and C and PC be the component and path-component of $(f \mid_A)^{-1}(K) = A \cap f^{-1}(K)$. Thus, C is contained in a component T of $f^{-1}(K)$. Let PT be the path-component of $f^{-1}(K)$. Obviously, $PC \subset C$ and $PT \subset T$. So, $PC \subset C \subset T$. Since $C \subset A$, it follows that $\phi \neq C = A \cap C \subset A \cap T$. Further $K \subset f(A) \subset B$, implies that $T \subset f^{-1}(B)$. According to the assumptions on A, we infer that $T \subset A$, whence $PT \subset T \subseteq (f \mid_A)^{-1}(K)$. Which implies that $T = C$. Thus $PC = PT$ and consequently $(f \mid_A)(PC) = f(PC) = f(PT) = K$ by Theorem 4.2. Therefore, $f \mid_A$ is strongly path-confluent mapping. $\square$

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