Applied General Topology
© Universidad Politécnica de Valencia
Volume 13, no. 1, 2012
pp. 21-25

Extending maps between pre-uniform spaces

Adalberto García-Máynez and Rubén Mancio-Toledo

Abstract

We give sufficient conditions on a uniformly continuous map $f\colon (X,U) \to (Y,V)$ between completable T_1-pre-uniform spaces (X,U), (Y,V) to have a continuous or a uniformly continuous extension $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ between the corresponding completions.

2010 MSC: *54A20, 54E15*

Keywords: *Minimal, round, pre-uniform, completion, extension*

1. Preliminary results

The basic concepts used in this paper: pre-uniformity bases, Cauchy or minimal filters, round, weakly round or strongly round filters and completion conditions are given in [1]. The concept of pre-uniform basis appeared in 1970 under the name of *structure* [3]. However, non Hausdorff pre-uniform spaces were very seldom considered in Harris monography.

T_1-pre-uniform spaces have an important property: Every Cauchy filter contains a unique weakly round filter and every neighborhood filter is weakly round. The set of weakly round filters $\widehat{X}$ of a T_1-pre-uniform space has a complete T_1-pre-uniform basis $\widehat{U}$ such that the map $h\colon (X,U) \to (\widehat{X},\widehat{U})$ which assigns to each $x \in X$ its neighborhood filter is a uniform embedding. Hence, any uniformly continuous map $\varphi\colon (X,U) \to (Y,V)$ between T_1-pre-uniform spaces induces a map $\widehat{\varphi}\colon (\widehat{X},\widehat{U}) \to (\widehat{Y},\widehat{V})$ which sends every weakly round filter $\xi \in \widehat{X}$ into the unique weakly round filter $\mathcal{N}$ in $\widehat{Y}$ which is contained in the Cauchy filter

$$\varphi(\xi) = \{\varphi(L) \mid L \in \xi\}^+ .$$

(For every subfamily $\mathcal{G}$ of the power set of a set Z, we define $\mathcal{G}^+ = \{L \subseteq Z \mid \text{for some } G \in \mathcal{G}, G \subseteq L\}$). If $k\colon (Y,V) \to (\widehat{Y},\widehat{V})$ is the canonical uniform embedding, *i.e.* $k(y)$ = neighborhood filter of y, we have the relation $\widehat{\varphi} \circ h = k \circ \varphi$. In this paper, we find conditions on φ, U and V which insure that $\widehat{\varphi}$ is continuous or uniformly continuous.

2. Main results

We start this section with a lemma.

Lemma 2.1. *Let (X, U) be a T_1-pre-uniform space and suppose (X, τ_U) is a T_1-space. Then every Cauchy filter ξ in (X, U) contains a unique minimal filter ξ'.*

Proof. We know $\xi' = \{{S_T}^{**}(\xi, \alpha) \mid \alpha \in U\}^+$ is U-minimal and is contained in ξ, where

$$ {S_T}^{**}(\xi, \alpha) = \bigcup \{L \mid L \in \alpha \cap \xi\} \ . $$

Suppose $\mathcal{N} \subseteq \xi$ is another U-minimal filter.

Therefore, $\mathcal{N}' = \mathcal{N} \subseteq \xi'$. The minimal property of ξ' implies that $\mathcal{N}' = \mathcal{N} = \xi'$. □

We give two cases in which $\widehat{\varphi}$ is uniformly continuous.

Lemma 2.2. *Suppose for each $\xi \in \widehat{X} - h(X), \varphi(\xi) = \varphi(\xi)'$. Then $\widehat{\varphi}$ is uniformly continuous.*

Proof. Let $\beta \in V$ and let $\alpha \in U$ be such that $\alpha \leq \varphi^{-1}(\beta)$. We shall prove that $\widehat{\alpha} \leq \widehat{\varphi}^{-1}(\widehat{\beta})$. Let $A \in \alpha$ and $B \in \beta$ be such that $A \subseteq \varphi^{-1}(B)$. We claim that $\widehat{A} \subseteq \widehat{\varphi}^{-1}(\widehat{B})$. Let us take $\xi \in \widehat{A}$. Then $A \in \xi$ and $\varphi(A) \in \varphi(\xi)$. Since $\varphi(A) \subseteq B$, we have also $B \in \varphi(\xi)$. Therefore, $\widehat{\varphi}(\xi) = \varphi(\xi) \in \widehat{B}$ and the proof is complete. □

Lemma 2.3. *If (Y, V) is a semi-uniform space, $\widehat{\varphi}$ is uniformly continuous.*

Proof. Let $\beta \in V$. Since (Y, V) is a semi-uniform space, there exists a cover $\gamma \in V$ which satisfies the following condition:

Su) For each $C \in \gamma$, there exists $\delta_C \in V$ and $B_C \in \beta$ such that $S_T(C, \delta_C) \subseteq B_C$.

Let $\alpha \in U$ be such that $\alpha \leq \varphi^{-1}(\gamma)$. We shall prove that $\widehat{\alpha} \leq \widehat{\varphi}^{\,-1}\widehat{(\beta)}$. If $A \in \alpha$, there exists a set $C \in \gamma$ such that $A \subseteq \varphi^{-1}(C)$. By condition Su), there exist $\delta_C \in V$ and $B_C \in \beta$ such that $S_T(C, \delta_C) \subseteq B_C$. We claim that $\widehat{A} \subseteq \widehat{\varphi}^{-1}(\widehat{B}_C)$. If $\xi \in \widehat{A}$, we have $A \in \xi$. Since $\varphi(A) \subseteq C$, we have $C \in \varphi(\xi)^+$. Therefore, $S_T(C, \delta_C) \in \varphi(\xi)' = \widehat{\varphi}(\xi)$. Since $S_T(C, \delta_C) \subseteq B_C$, we conclude that $B_C \in \widehat{\varphi}(\xi)$ and $\widehat{\varphi}(\xi) \in \widehat{B}_C$. □

Lemma 2.4. *Let X, Y be T_2-spaces and let U, V, respectively, be the families of densely finite covers of X, Y. Let $\varphi \colon X \to Y$ be continuous, open and surjective. Then φ is uniformly continuous as a map from (X, U) onto (Y, V).*

Proof. Let β be a densely finite cover of Y. Then we can find a finite subfamily $\{B_1, B_2, \cdots, B_n\} \subseteq \beta$ such that $B_1^- \cup B_2^- \cup \cdots \cup B_n{}^- = Y$. If $B = B_1 \cup B_2 \cup \cdots \cup B_n$ and $\alpha = f^{-1}(\beta)$, the hypotheses imply that

$$\{A_1, A_2, \ldots, A_n\} \subseteq \alpha$$

where $A_i = f^{-1}(B_i)$ for $i = 1, 2, \ldots, n$, satisfies

$$A_1^- \cup A_2^- \cup \cdots \cup A_n^- = f^{-1}(B^-) = X\,.$$

Hence α is a densely finite cover of X and $\alpha \leq f^{-1}(\beta)$ (In fact, $\alpha = f^{-1}(\beta)$). Then f is uniformly continuous. ☐

Lemma 2.5. *Keep the hypotheses of* (2.4). *Then* $\widehat{\varphi}\colon \widehat{X} \to \widehat{Y}$ *is continuous and surjective.*

Proof. Let $\mathcal{N} \in \widehat{X}$ and let T be an open set in Y such that $\widehat{\varphi}(\mathcal{N}) = \varphi(\mathcal{N})' \in \widehat{T}$. Then $T \in \varphi(\mathcal{N})'$. Therefore, there exists a cover $\gamma \in V$ such that $T \supseteq S_T{}^{**}(\varphi(\mathcal{N}), \gamma)$. Since $\varphi\colon (X, U) \to (Y, V)$ is uniformly continuous (2.4), the filter $\varphi(\mathcal{N})$ is Cauchy in (Y, V). Select an element $N_0 \in \gamma \cap \varphi(\mathcal{N})$. Then $\varphi^{-1}(N_0) \in \varphi^{-1}(\gamma) \cap \mathcal{N}$ and $N_0 \subseteq S_T{}^{**}(\varphi(\mathcal{N}), \gamma) \subseteq T$. Therefore:

$$\varphi^{-1}(N_0) \subseteq S_T{}^{**}(\mathcal{N}, \varphi^{-1}(\gamma)) = \varphi^{-1}(S_T{}^{**}(\varphi(\mathcal{N}), \gamma)) \subseteq \varphi^{-1}(T).$$

We shall prove that $\widehat{\varphi}(S_T(\mathcal{N}, \varphi^{-1}(\gamma)^{\widehat{}}\,) \subseteq \widehat{T}$ and the continuity of $\widehat{\varphi}$ will follow.

Let $\mathcal{M} \in S_T(\mathcal{N}, \varphi^{-1}(\gamma)^{\widehat{}}\,)$. Then there exists an element $C \in \gamma$ such that $\varphi^{-1}(C) \in \mathcal{M} \cap \mathcal{N}$. Hence, $\varphi^{-1}(C) \subseteq S_T{}^{**}(\mathcal{N}, \varphi^{-1}(\gamma)) \subseteq \varphi^{-1}(T)$ and $C \subseteq S_T{}^{**}(\varphi(\mathcal{N}), \gamma) \subseteq T$. We also have

$$C \subseteq S_T{}^{**}(\varphi(\mathcal{M}), \gamma) \in \varphi(\mathcal{M})^{'} = \widehat{\varphi}(\mathcal{M}).$$

Then $T \in \widehat{\varphi}(\mathcal{M})$ and $\widehat{\varphi}(\mathcal{M}) \in \widehat{T}$. ☐

Before we prove $\widehat{\varphi}$ is surjective, we need a lemma.

Lemma 2.6. *A non-adherent filter* $\mathcal{T}$ *in* (X, U) *is* U*-round if and only if* $\mathcal{T}$ *has as a basis an ultrafilter of open sets.*

Proof. Suppose $\mathcal{T}$ is a non-adherent round filter in (X, U). Let $\mathcal{G}$ be the family of open sets in $\mathcal{T}$ and take an open set V such that $V \cap G \neq \varnothing$ for every $G \in \mathcal{G}$. We have to prove that $V \in \mathcal{T}$ and that will convert $\mathcal{G}$ into an ultrafilter of open sets.

Since $\mathcal{T}$ is non-adherent, the family $\{X - F^- \,|\, F \in \mathcal{T}\}$ is an open cover of X. Hence.

$$\alpha = \{V, X - V^-\} \cup \{X - F^- \,|\, F \in \mathcal{T}\}$$

is a densely finite cover of X. Since $\mathcal{T}$ is U-Cauchy, we have $V \in \mathcal{T}$ or $X - V^- \in \mathcal{T}$. If we had $X - V^- \in \mathcal{T}$, we use the roundness of $\mathcal{T}$ and find a cover $\beta \in U$ such that $X - V^- \supseteq S_T{}^*(\mathcal{T}, \beta) = \cup\{B \in \beta \mid B \cap F \neq \varnothing$ for every $F \in \mathcal{T}\}$. If $G \in \beta \cap \mathcal{T}$, we have $G \subseteq X - V^-$ and hence $V \cap G = \varnothing$, a contradiction. Therefore we must have $V \in \mathcal{T}$ and $\mathcal{G}$ is an ultrafilter of open sets.

Conversely, suppose $\mathcal{G}$ is an ultrafilter of open sets. We have to prove that $\mathcal{T}$ is $\mathcal{U}$-round. We prove first that $\mathcal{T}$ is $\mathcal{U}$-Cauchy. Let $\alpha \in \mathcal{U}$. If $\mathcal{T} \cap \alpha = \varnothing$, then $A \notin \mathcal{T} \cap \tau$ for every $A \in \alpha$. Let $\{A_1, A_2, \dots, A_n\} \subseteq \alpha$ be such that $X = {A_1}^{-} \cup {A_2}^{-} \cup \cdots \cup {A_n}^{-}$. Since $A_i \notin \mathcal{T} \cap \tau$ and $\mathcal{T} \cap \tau$ is an ultrafilter of open sets, we can find elements $G_i \in \mathcal{T} \cap \tau$ such that $A_i \cap G_i = \varnothing$ $(i = 1, 2, \dots, n)$. Hence $(G_1 \cap G_2 \cap \cdots \cap G_n) \cap (A_1 \cup A_2 \cup \cdots \cap A_n) = \varnothing$. But $A_1 \cup A_2 \cup \cdots \cup A_n$ is dense in X. Hence $G_1 \cap G_2 \cap \cdots \cap G_n = \varnothing$, a contradiction. We finally prove that $\mathcal{T}$ is $\mathcal{U}$-round. Pick any element $F_0 \in \mathcal{T}$ and consider the cover $\alpha = \{F_0\} \cup \{X - F^{-} \mid F \in \mathcal{T}\}$. Clearly $S_T{}^*(\mathcal{T}, \alpha) = F_0$ and hence $\mathcal{T}$ is $\mathcal{U}$-round. □

In [4] it is proved that every Cauchy filter in $(X,\mathcal{U})$, where $\mathcal{U}$ is the family of densely finite covers of the Hausdorff space (X, τ), contains an $\mathcal{U}$-round filter and by [1],$(X,\mathcal{U})$ has a completion $(\widehat{X},\widehat{\mathcal{U}}\,)$ where every $\widehat{\mathcal{U}}$-round filter is convergent and the topology $\tau_{\widehat{\mathcal{U}}}$ is Hausdorff closed. Besides the completion $(\widehat{X},\widehat{\mathcal{U}}\,), (X, \tau)$ has the Katetov extension kX, which is also Hausdorff closed. In this volume we prove that in general, the extensions $\widehat{X}$ and kX are non-equivalent.

3. Applications

Proposition 3.1. *Let X be a separable, metrizable, dense in itself, 0-dimensional space and let Z be a compact, Hausdorff, separable space. Then there exists a surjective continuous map $g\colon \widehat{X} \to Z$, where $\widehat{X}$ is the completion of the pre-uniformity basis of X consisting of all densely finite covers of X.*

Proof. The hypothesis imply the existence of mutually disjoint non-empty open sets $L_1, L_2, \dots$ such that $X = \bigcup\limits_{n=1}^{\infty} L_n$. The map $\varphi\colon X \to \mathbb{N}$ where $L_n = \varphi^{-1}(n)$ for each $n \in \mathbb{N}$, is continuous, open and surjective. By 2.4, there exists a continuous surjective extension $\widehat{\varphi}\colon \widehat{X} \to \widehat{\mathbb{N}}$. But $\widehat{\mathbb{N}}$ coincides with the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ (because a cover α of $\mathbb{N}$ is densely finite if and only if it is finite). On the other hand, by the universal property of $\beta\mathbb{N}$, there exists a continuous surjective map $\psi\colon \beta\mathbb{N} \to Z$. Hence, $g = \psi \circ \widehat{\varphi}$ is a continuous surjective map from $\widehat{X}$ onto Z. □

Proposition 3.2. *Let X be a non-empty completely metrizable separable space. Then there exists a continuous surjective map $\psi\colon (\mathbb{N}^{w})\widehat{\ } \to \widehat{X}$.*

Proof. $\mathbb{N}^{w}$ may be identified with the set of irrational numbers and this space satisfies the conditions of (3.1). On the other hand, there exists a continuous open surjective map $\varphi\colon \mathbb{N}^{w} \to X$ (see 5.15 in [2]). Using 2.4, we complete the proof. □

Corollary 3.3. *If Z is a Tychonoff separable space which is either compact or completely metrizable, then there exists a continuous surjective map $\psi\colon (\mathbb{N}^{w})\widehat{\ } \to \widehat{Z}$.*

We finish this paper with a problem:

Problem 3.4. Is every Čech-complete separable space a continuous image of $(\mathbb{N}^w)\widehat{}$?

References

[1] A. García-Máynez and R. Mancio Toledo, *Completions of pre-uniform spaces*, Appl. Gen. Topol. **8**, no. 2 (2007), 213–221.
[2] A. García-Máynez and A. Tamariz, *Topología General*, Ed. Porrúa, México, 1988.
[3] D. Harris, *Structures in Topology*, Memoirs of The American Mathematical Society, No. 115, Providence, Rhode Island, AMS, 1971.
[4] R. Mancio Toledo, *Los espacios pre-uniformes y sus completaciones*, Ph. D. Thesis, UNAM, 2006.

(Received January 2010 – Accepted January 2011)

A. García-Máynez (agmaynez@matem.unam.mx)
Instituto de Matemáticas, Universidad Nacional Autónoma de México, Área de la Investigación Científica, Circuito Exterior, Ciudad Universitaria, 04510 México, D.F. México

Rubén Mancio-Toledo (rmancio@esfm.ipn.mx)
Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, Unidad Profesional Adolfo López Mateos, Col. Lindavista, 07738 México, D.F.