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Universal elements for some classes of spaces

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Abstract

In the paper [4] two dimensions, denoted by dm and Dm, are defined in the class of all Hausdorff spaces. The dimension Dm does not have the universality property in the class of separable metrizable spaces because the family of all such spaces X with $Dm(X) \leq 0$ coincides with the family of all totally disconnected spaces in which there are no universal elements (see [5]). In [3] we gave the dimension-like functions $dm_{\mathbb{E}}^{\mathbb{K},\mathbb{B}}$ and $Dm_{\mathbb{E}}^{\mathbb{K},\mathbb{B}}$, where $\mathbb{K}$ is a class of subsets, $\mathbb{E}$ a class of spaces and $\mathbb{B}$ a class of bases and we proved that in the families $\mathbb{P}(dm_{\mathbb{E}}^{\mathbb{K},\mathbb{B}} \leq \kappa)$ and $\mathbb{P}(Dm_{\mathbb{E}}^{\mathbb{K},\mathbb{B}} \leq \kappa)$ of all spaces X for which $dm_{\mathbb{E}}^{\mathbb{K},\mathbb{B}}(X) \leq \kappa$ and $Dm_{\mathbb{E}}^{\mathbb{K},\mathbb{B}}(X) \leq \kappa$, respectively there exist universal elements. In this paper, we give some new dimension-like functions and define using these definitions classes of spaces in which there are universal elements.

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1. Introduction and preliminaries

Agreement. All spaces are assumed to be T_0-spaces of weight $\leq \tau$, where τ is a fixed infinite cardinal. The set of all finite subsets of τ is denoted by $\mathcal{F}$ and the first infinite cardinal is denoted by ω. The cardinality of a set X is denoted by $|X|$. The class of all ordinals is denoted by $\mathcal{O}$. We also consider two symbols: -1 and ∞. It is assumed that $-1 < \alpha < \infty$ for every $\alpha \in \mathcal{O}$.

In the proof of the main results of this paper widely we use notions and notations from [2]. For this reason we start given some of them.

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We shall use the symbol " $\equiv$ " in order to introduce new notations without mention this fact. If " $\sim$ " is an equivalence relation on a non-empty set X, then the set of all equivalence classes of $\sim$ is denoted by $\mathrm{C}(\sim)$.

Let $\mathbf{S}$ be an indexed collection of spaces. An indexed collection

$$\mathbf{M} \equiv \{\{U_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\} \tag{1}$$

where $\{U_\delta^X : \delta \in \tau\}$ is an indexed base for X, is called *a co-mark of* $\mathbf{S}$. The co-mark $\mathbf{M}$ of $\mathbf{S}$ is said to be *a co-extension* of a co-mark

$$\mathbf{M}^+ \equiv \{\{V_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$$

of $\mathbf{S}$ if there exists a one-to-one mapping θ of τ into itself such that for every $X \in \mathbf{S}$ and for every $\delta \in \tau$, $V_\delta^X = U_{\theta(\delta)}^X$. The corresponding mapping θ is called *an indicial mapping from* $\mathbf{M}^+$ *to* $\mathbf{M}$.

Let

$$\mathrm{R}_1 \equiv \{\sim_1^s : s \in \mathcal{F}\}$$

and

$$\mathrm{R}_0 \equiv \{\sim_0^s : s \in \mathcal{F}\}$$

be two indexed families of equivalence relations on $\mathbf{S}$. It is said that R_1 is *a final refinement of* R_0 if for every $s \in \mathcal{F}$ there exists $t \in \mathcal{F}$ such that $\sim_1^t \subseteq \sim_0^s$.

An indexed family $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ of equivalence relations on $\mathbf{S}$ is said to be *admissible* if the following conditions are satisfied: (a) $\sim^\varnothing = \mathbf{S} \times \mathbf{S}$, (b) for every $s \in \mathcal{F}$ the number of $\sim^s$-equivalence classes is finite, and (c) $\sim^s \subseteq \sim^t$, if $t \subseteq s$. We denote by $\mathrm{C}(\mathrm{R})$ the set $\cup\{\mathrm{C}(\sim^s) : s \in \mathcal{F}\}$. The minimal ring of subsets of $\mathbf{S}$ containing $\mathrm{C}(\mathrm{R})$ is denoted by $\mathrm{C}^\diamondsuit(\mathrm{R})$.

Consider the co-mark (1) of $\mathbf{S}$. We denote by

$$\mathrm{R}_\mathbf{M} \equiv \{\sim_\mathbf{M}^s : s \in \mathcal{F}\}$$

the indexed family of equivalence relations $\sim_\mathbf{M}^s$ on $\mathbf{S}$ defined as follows: for every $X, Y \in \mathbf{S}$ we set $X \sim_\mathbf{M}^s Y$ if and only if there exists an isomorphism i of the algebra of subsets of X generated by the set $\{U_\delta^X : \delta \in s\}$ onto the algebra of subsets of Y generated by the set $\{U_\delta^Y : \delta \in s\}$ such that $i(U_\delta^X) = U_\delta^Y$, for every $\delta \in s$. Also, we set $\sim_\mathbf{M}^\varnothing = \mathbf{S} \times \mathbf{S}$. An admissible family R of equivalence relations on $\mathbf{S}$ is said to be $\mathbf{M}$-*admissible* if R is a final refinement of $\mathrm{R}_\mathbf{M}$.

Let $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ be an $\mathbf{M}$-admissible family of equivalence relations on $\mathbf{S}$. On the set of all pairs (x, X), where $X \in \mathbf{S}$ and $x \in X$, we consider an equivalence relation, denoted by $\sim_\mathrm{R}^\mathbf{M}$, as follows: $(x, X) \sim_\mathrm{R}^\mathbf{M} (y, Y)$ if and only if $X \sim^s Y$ for every $s \in \mathcal{F}$, and either $x \in U_\delta^X$ and $y \in U_\delta^Y$ or $x \notin U_\delta^X$ and $y \notin U_\delta^Y$ for every $\delta \in \tau$. The set of all equivalence classes of the relation $\sim_\mathrm{R}^\mathbf{M}$ is denoted by $\mathrm{T}(\mathbf{M}, \mathrm{R})$ or simply by T.

For every $\mathbf{H} \in \mathrm{C}^\diamondsuit(\mathrm{R})$ the set of all $\mathbf{a} \in \mathrm{T}(\mathbf{M}, \mathrm{R})$ for which there exists an element $(x, X) \in \mathbf{a}$ such that $X \in \mathbf{H}$ is denoted by $\mathrm{T}(\mathbf{H})$. For every $\delta \in \tau$ and $\mathbf{H} \in \mathrm{C}^\diamondsuit(\mathrm{R})$ we denote by $U_\delta^\mathrm{T}(\mathbf{H})$ the set of all $\mathbf{a} \in \mathrm{T}(\mathbf{M}, \mathrm{R})$ for which there exists an element $(x, X) \in \mathbf{a}$ such that $X \in \mathbf{H}$ and $x \in U_\delta^X$.

For every subset κ of τ and $\mathbf{L} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$ we set

(1) $\mathrm{B}^{\mathrm{T}}_{\diamondsuit} \equiv \{U^{\mathrm{T}}_{\delta}(\mathbf{H}) : \delta \in \tau \text{ and } \mathbf{H} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})\}$.
(2) $\mathrm{B}^{\mathrm{T}}_{\diamondsuit,\kappa} \equiv \{U^{\mathrm{T}}_{\delta}(\mathbf{H}) : \delta \in \kappa \text{ and } \mathbf{H} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})\}$.
(3) $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\kappa} \equiv \{U^{\mathrm{T}}_{\delta}(\mathbf{H}) \in B^{T}_{\diamondsuit,\kappa} : \mathbf{H} \subseteq \mathbf{L}\}$.

Under some simple (set-theoretical) conditions on R the set $\mathrm{B}^{\mathrm{T}}_{\diamondsuit}$ is a base for a topology on the set $\mathrm{T}(\mathbf{M}, \mathrm{R})$ such that the corresponding space is a T_0-space of weight $\leq \tau$. Moreover, if for every $X \in \mathbf{S}$ the set $\{U^{X}_{\delta} : \delta \in \kappa\}$ is a base for X, then the set $\mathrm{B}^{\mathrm{T}}_{\diamondsuit,\kappa}$ is a base for the same topology on $\mathrm{T}(\mathbf{M}, \mathrm{R})$. Therefore, the family $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\kappa}$ is a base for $\mathrm{T}(\mathbf{L})$. (See Corollary 1.2.8 and Proposition 1.2.9 in [2]).

For every element X of $\mathbf{S}$ there exists a natural embedding i^{X}_{T} of X into the space $\mathrm{T}(\mathbf{M}, \mathrm{R})$ defined as follows: for every $x \in X$, $i^{X}_{\mathrm{T}}(x) = \mathbf{a}$, where $\mathbf{a}$ is the element of $\mathrm{T}(\mathbf{M}, \mathrm{R})$ containing the pair (x, X). Thus, we have constructed containing space $\mathrm{T}(\mathbf{M}, \mathrm{R})$ for $\mathbf{S}$ of weight $\leq \tau$.

Suppose that for every $X \in \mathbf{S}$ a subset Q^X of X is given. The set

$$\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\} \tag{2}$$

is called *a restriction* of $\mathbf{S}$. Let $\mathbb{F}$ be a class of subsets. A restriction $\mathbf{Q}$ of an indexed collection $\mathbf{S}$ of spaces is said to be *a* $\mathbb{F}$*-restriction* if $(Q^X, X) \in \mathbb{F}$ for every $X \in \mathbf{S}$.

Consider the restriction (2) of $\mathbf{S}$. The *trace on* $\mathbf{Q}$ *of the co-mark* $\mathbf{M}$ of $\mathbf{S}$ is the co-mark

$$\mathbf{M}|_{\mathbf{Q}} \equiv \{\{U^{X}_{\delta} \cap Q^X : \delta \in \tau\} : Q^X \in \mathbf{Q}\}$$

of $\mathbf{Q}$. The *trace on* $\mathbf{Q}$ *of an equivalence relation* $\sim$ on $\mathbf{S}$ is the equivalence relation on $\mathbf{Q}$ denoted by $\sim|_{\mathbf{Q}}$ and defined as follows: $Q^X \sim|_{\mathbf{Q}} Q^Y$ if and only if $X \sim Y$. Let $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ be an indexed family of equivalence relations on $\mathbf{S}$. The *trace on* $\mathbf{Q}$ *of the family* R is the family $\mathrm{R}|_{\mathbf{Q}} \equiv \{\sim^s|_{\mathbf{Q}} : s \in \mathcal{F}\}$ of equivalence relations on $\mathbf{Q}$. The *trace on* $\mathbf{Q}$ *of an element* $\mathbf{H}$ of $\mathrm{C}^{\diamondsuit}(\mathrm{R})$ is the element

$$\mathbf{H}|_{\mathbf{Q}} \equiv \{Q^X \in \mathbf{Q} : X \in \mathbf{H}\}$$

of $\mathrm{C}^{\diamondsuit}(\mathrm{R}|_{\mathbf{Q}})$.

The $\mathbf{M}$-admissible family R of equivalence relations on $\mathbf{S}$ is said to be $(\mathbf{M}, \mathbf{Q})$-*admissible* if $\mathrm{R}|_{\mathbf{Q}}$ is an $\mathbf{M}|_{\mathbf{Q}}$-admissible family of equivalence relations on $\mathbf{Q}$.

If R is an $(\mathbf{M}, \mathbf{Q})$-admissible family of equivalence relations on $\mathbf{S}$, then we can consider the containing space $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}}, \mathrm{R}|_{\mathbf{Q}})$ for the indexed collection $\mathbf{Q}$ corresponding to the co-mark $\mathbf{M}|_{\mathbf{Q}}$ and the $\mathbf{M}|_{\mathbf{Q}}$-admissible family $\mathrm{R}|_{\mathbf{Q}}$. The containing space $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}}, \mathrm{R}|_{\mathbf{Q}})$ is denoted briefly by $\mathrm{T}|_{\mathbf{Q}}$. There exists a natural embedding of $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}}, \mathrm{R}|_{\mathbf{Q}})$ into $\mathrm{T}(\mathbf{M}, \mathrm{R})$. So we can consider the containing space $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}}, \mathrm{R}|_{\mathbf{Q}})$ as a subspace of the space $\mathrm{T}(\mathbf{M}, \mathrm{R})$. The subsets of this form will be called *specific subsets* of $\mathrm{T}(\mathbf{M}, \mathrm{R})$.

A class $\mathbb{P}$ of spaces is said to be *saturated* if for every indexed collection $\mathbf{S}$ of spaces belonging to $\mathbb{P}$ there exists a co-mark $\mathbf{M}^{+}$ of $\mathbf{S}$ satisfying the following condition: for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$ there exists an $\mathbf{M}$-admissible family R^+ of equivalence relations on $\mathbf{S}$ such that for every admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and for every $\mathbf{L} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$ the space $\mathrm{T}(\mathbf{L})$ belongs to $\mathbb{P}$.

The co-mark $\mathbf{M}^+$ is said to be *an initial co-mark of* $\mathbf{S}$ *corresponding to the class* $\mathbb{P}$ and the family R is said to be *an initial family of* $\mathbf{S}$ *corresponding to the co-mark* $\mathbf{M}$ *and the class* $\mathbb{P}$.

Agreement. In what follows we denote by ν a fixed cardinal greater than ω and less than or equal to τ.

Notation. For every dimension-like function df_ν, with as domain the class of all spaces and as range the class $\mathcal{O} \cup \{-1, \infty\}$; and for every $\alpha \in \{-1\} \cup \mathcal{O}$, we denote by $\mathbb{P}(df_\nu \leq \alpha)$ the class of all spaces X with $df_\nu(X) \leq \alpha$.

2. The dimension-like functions: $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ and $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$

In this section we give some new dimension-like functions and define using these definitions classes of spaces in which there are universal elements. The proofs of these results are similar to the proofs of the results in [3], for this reason are omitted.

Definition 2.1 (see [1])**.** Let A and B be two disjoint subsets of a space X. We say that a subset L of X *separates* A and B if there exist two open subsets U and W of X such that: (a) $A \subseteq U$, $B \subseteq W$, (b) $U \cap W = \varnothing$, and (c) $X \setminus L = U \cup W$.

Definition 2.2 (see [3])**.** A class $\mathbb{E}$ of spaces is said to be $\mathbb{B}$-*hereditary-separated*, where $\mathbb{B}$ is a class of bases, if for every element X of $\mathbb{E}$ there exists a $\mathbb{B}$-base $B^X \equiv \{U_\delta : \delta \in \tau\}$ for X such that for every two elements U_{δ_1} and U_{δ_2} of B^X with $\mathrm{Cl}(U_{\delta_1}) \cap \mathrm{Cl}(U_{\delta_2}) = \varnothing$ there exists a subset L of X separating the sets $\mathrm{Cl}(U_{\delta_1})$ and $\mathrm{Cl}(U_{\delta_2})$ and belonging to $\mathbb{E}$.

We note that if $\mathbb{E}$ is $\mathbb{B}$-hereditary-separated, then $\varnothing \in \mathbb{E}$. This follows by the fact that the empty set is the unique subset of X separating the elements $\varnothing$ and X of B^X.

Definition 2.3. Let $\mathbb{B}$ be a class of bases, $\mathbb{E}$ a $\mathbb{B}$-hereditary-separated class of spaces, and $\mathbb{K}$ a class of subsets with $(X, X) \in \mathbb{K}$ for every space X. We denote by $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ and $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ the *dimension-like functions* with as domain the class of all spaces and as range the class $\mathcal{O} \cup \{-1, \infty\}$ satisfying the following conditions:

(1) $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = -1$ if and only if $X \in \mathbb{E}$.

(2) $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$, $\alpha \in \mathcal{O}$, if and only if there exists a $\mathbb{B}$-base $B^X \equiv \{U_\delta : \delta \in \tau\}$ for X such that for every two elements U_{δ_1}, U_{δ_2} of B^X with $\mathrm{Cl}(U_{\delta_1}) \cap \mathrm{Cl}(U_{\delta_2}) = \varnothing$ there exists a subset L of X separating $\mathrm{Cl}(U_{\delta_1})$ and $\mathrm{Cl}(U_{\delta_2})$ with $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L) < \alpha$.

(3) $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$, $\alpha \in \mathcal{O}$, if and only if $X = \cup\{S_i : i \in \nu\}$ such that: (a) the subset S_i of X is closed, (b) $(S_i, X) \in \mathbb{K}$, and (c) $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(S_i) \leq \alpha$, $i \in \nu$.

Therefore, $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \infty$ (respectively, $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \infty$) if and only if the inequality $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$ (respectively, $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$) is not true for every $\alpha \in \mathcal{O}$.

Remark 2.4.

(1) In order that the above definition to be well defined we need to show that if for a space X we have $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = -1$, then $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq 0$ and $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq 0$.

For dimension-like function $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ this follows immediately by the fact that $X \in \mathbb{E}$ and the class $\mathbb{E}$ is $\mathbb{B}$-hereditary-separated.

For dimension-like function $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$, we have (a) $X = \{S_i : i \in \nu\}$, where $S_i = X$, (b) $(X, X) \in \mathbb{K}$, and (c) $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = -1 \leq 0$, which means that $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq 0$.

(2) For $\nu = \omega$ the dimension-like functions $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ and $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ coincide with the dimension-like functions $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E}}$ and $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E}}$, respectively which are defined in [3].

Proposition 2.5. *For every space X we have*

$$dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X).$$

Proposition 2.6. *For every space X, $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \in \{-1, \infty\} \cup \tau^+$ and, therefore, $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \in \{-1, \infty\} \cup \tau^+$.*

Theorem 2.7. *Let $\mathbb{B}$ be a saturated class of bases, $\mathbb{E}$ a saturated $\mathbb{B}$-hereditary-separated class of spaces, and $\mathbb{K}$ a saturated class of subsets with $(X, X) \in \mathbb{K}$ for every space X. Then, for every $\kappa \in \{-1\} \cup \omega$ the classes $\mathbb{P}(dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ and $\mathbb{P}(Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ are saturated.*

Corollary 2.8. *For every $\kappa \in \omega$ in the classes*

$$\mathbb{P}(dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa) \quad \textit{and} \quad \mathbb{P}(Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$$

there exist universal elements.

Corollary 2.9. *Let $\mathbb{P}$ be one of the following classes*

(a) *the class of all (completely) regular spaces of weight $\leq \tau$,*
(b) *the class of all (completely) regular countable-dimensi-onal spaces of weight $\leq \tau$,*
(c) *the class of all (completely) regular strongly countable-dimensional spaces of weight $\leq \tau$,*
(d) *the class of all (completely) regular locally finite-dimensional spaces of weight $\leq \tau$, and*

(e) *the class of all (completely) regular spaces X of weight $\leq \tau$ such that* $\text{ind}(X) \leq \alpha \in \tau^+$.

Then, for every $\kappa \in \omega$ in the classes

$$\mathbb{P}(dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa) \cap \mathbb{P} \quad \textit{and} \quad \mathbb{P}(Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa) \cap \mathbb{P}$$

there exist universal elements.

3. The dimension-like functions: $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ and $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$

Definition 3.1. A class $\mathbb{E}$ of spaces is said to be $\mathbb{B}$-*weakly-hereditary-separated*, where $\mathbb{B}$ is a class of bases, if for every element X of $\mathbb{E}$ there exists a $\mathbb{B}$-base $B^X \equiv \{U_\delta : \delta \in \tau\}$ for X such that for every two elements U_{δ_1} and U_{δ_2} of B^X with $\text{Cl}(U_{\delta_1}) \cap U_{\delta_2} = \varnothing$ there exists a subset L of X separating the sets $\text{Cl}(U_{\delta_1})$ and U_{δ_2} and belonging to $\mathbb{E}$.

We note that if $\mathbb{E}$ is $\mathbb{B}$-weakly-hereditary-separated, then $\varnothing \in \mathbb{E}$. This follows by the fact that the empty set is the unique subset of X separating the elements $\varnothing$ and X of B^X.

Definition 3.2. Let $\mathbb{B}$ be a class of bases, $\mathbb{E}$ a $\mathbb{B}$-weakly-hereditary-separated class of spaces, and $\mathbb{K}$ a class of subsets with $(X, X) \in \mathbb{K}$ for every space X. We denote by $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ and $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ the *dimension-like functions* with as domain the class of all spaces and as range the class $\mathcal{O} \cup \{-1, \infty\}$ satisfying the following conditions:

(1) $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = -1$ if and only if $X \in \mathbb{E}$.

(2) $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$, where $\alpha \in \mathcal{O}$, if and only if there exists a $\mathbb{B}$-base $B^X \equiv \{U_\delta : \delta \in \tau\}$ for X such that for every two elements U_{δ_1}, U_{δ_2} of B^X with $\text{Cl}(U_{\delta_1}) \cap U_{\delta_2} = \varnothing$ there exists a subset L of X separating $\text{Cl}(U_{\delta_1})$ and U_{δ_2} with $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L) < \alpha$.

(3) $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$, $\alpha \in \mathcal{O}$, if and only if $X = \cup\{S_i : i \in \nu\}$ such that: (a) the subset S_i of X is closed, (b) $(S_i, X) \in \mathbb{K}$, and (c) $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(S_i) \leq \alpha$, $i \in \nu$.

Therefore, $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \infty$ (respectively, $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \infty$) if and only if the inequality $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$ (respectively, $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$) is not true for every $\alpha \in \mathcal{O}$.

Remark 3.3. In order that the above definition to be well defined we need to show that if for a space X we have $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = -1$, then $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq 0$ and $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq 0$.

For dimension-like function $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ this follows immediately by the fact that $X \in \mathbb{E}$ and the class $\mathbb{E}$ is $\mathbb{B}$-weakly-hereditary-separated.

For dimension-like function $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$, we have (a) $X = \{S_i : i \in \nu\}$, where $S_i = X$, (b) $(X, X) \in \mathbb{K}$, and (c) $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = -1 \leq 0$, which means that $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq 0$.

Proposition 3.4. *For every space X we have*

$$w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X). \quad (3)$$

Proof. Let $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \alpha \in \{-1,\infty\} \cup \mathcal{O}$. The inequality (3) is clear if $\alpha = -1$ or $\alpha = \infty$. Suppose that $\alpha \in \mathcal{O}$. We have $X = \cup\{S_i : i \in \nu\}$, where $S_i = X$. Since $(S_i, X) = (X, X) \in \mathbb{K}$ and $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(S_i) = w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$, the condition (3) of Definition 3.2 implies that $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$. □

Proposition 3.5. *For every space X, $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \in \{-1,\infty\} \cup \tau^+$, and, therefore $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \in \{-1,\infty\} \cup \tau^+$.*

Proof. Suppose that the proposition is not true. Let α be the minimal element of $\mathcal{O} \setminus \tau^+$ such that there exists a space X with $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \alpha$. Let $B^X = \{U_\delta : \delta \in \tau\}$ be the $\mathbb{B}$-base for X mentioned in condition (2) of Definition 3.2.

Denote by P the set of all pairs $(\delta_1, \delta_2) \in \tau \times \tau$ with

$$\mathrm{Cl}(U_{\delta_1}) \cap U_{\delta_2} = \varnothing.$$

For every $(\delta_1, \delta_2) \in P$ let $L(\delta_1, \delta_2)$ be a subset of X separating the sets $\mathrm{Cl}(U_{\delta_1})$ and U_{δ_2} with

$$w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L(\delta_1, \delta_2)) = \beta(\delta_1, \delta_2) < \alpha.$$

First we suppose that $\beta(\delta_1, \delta_2) < \tau^+$ for every $(\delta_1, \delta_2) \in P$. Since $|P| \leq \tau$ there exists an ordinal $\beta \in \tau^+$ such that $\beta(\delta_1, \delta_2) < \beta$ for every $(\delta_1,\ \delta_2) \in P$. Then, $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L(\delta_1, \delta_2)) < \beta$ and, by condition (2) of Definition 3.2, $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \beta$, which is a contradiction.

Now, we suppose that there exists $(\delta_1, \delta_2) \in P$ such that $\tau^+ \leq \beta(\delta_1, \delta_2)$. Since $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L(\delta_1, \delta_2)) = \beta(\delta_1, \delta_2)$, there exist closed subsets $S_i^{L(\delta_1,\delta_2)}$ of $L(\delta_1, \delta_2)$, $i \in \nu$, such that:

(*a*) $L(\delta_1, \delta_2) = \cup\{S_i^{L(\delta_1,\delta_2)} : i \in \nu\}$,

(*b*) $(S_i^{L(\delta_1,\delta_2)}, L(\delta_1, \delta_2)) \in \mathbb{K}$, and

(*c*) $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(S_i^{L(\delta_1,\delta_2)}) = \beta_i \leq \beta(\delta_1, \delta_2) < \alpha$.

If $\beta_i < \tau^+$ for all $i \in \nu$, then there exists an ordinal $\beta \in \tau^+$ such that $\beta_i \leq \beta$, which means that $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(S_i^{L(\delta_1,\delta_2)}) \leq \beta$. Therefore,

$$w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L(\delta_1, \delta_2)) \leq \beta < \tau^+ \leq \beta(\delta_1, \delta_2),$$

which is a contradiction. Thus, there exists $i \in \nu$ such that

$$\tau^+ \leq w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(S_i^{L(\delta_1,\delta_2)}) < \alpha.$$

The last relation contradicts the choice of the ordinal α completing the proof of the proposition. □

Theorem 3.6. *Let $\mathbb{B}$ be a saturated class of bases, $\mathbb{E}$ a saturated $\mathbb{B}$-weakly-hereditary-separated class of spaces, and $\mathbb{K}$ a saturated class of subsets such that $(X, X) \in \mathbb{K}$ for every space X. Then, for every $\kappa \in \{-1\} \cup \omega$ the classes $\mathbb{P}(w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ and $\mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ are saturated.*

Proof. We prove the theorem by induction on κ. Let $\kappa = -1$. Then, a space X belongs to $\mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq -1)$ if and only if X belongs to $\mathbb{E}$, that is

$$\mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq -1) = \mathbb{E}.$$

Therefore, $\mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq -1)$ is a saturated class of spaces. Similarly, the class $\mathbb{P}(w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq -1)$ is saturated.

Let $\kappa \in \omega$. Suppose that the classes $\mathbb{P}(w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq m)$ and $\mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq m)$ are saturated, $m \in \{-1\} \cup \kappa$. We prove that the classes $\mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ and $\mathbb{P}(w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ are also saturated. First we prove that $\mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ is a saturated class.

Let $\mathbf{S}$ be an indexed collection of elements of $\mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$. For every $X \in \mathbf{S}$ let $B^X \equiv \{V^X_\varepsilon : \varepsilon \in \tau\}$ be an indexed $\mathbb{B}$-base for X satisfying condition (2) of Definition 3.2. Then, there exist

(a) an indexed set $\{L^X_\eta : \eta \in \tau\}$ of subsets of X,
(b) two indexed sets $\{W^X_\eta : \eta \in \tau\}$ and $\{O^X_\eta : \eta \in \tau\}$ of open subsets of X, and
(c) a one-to-one mapping φ of $\tau \times \tau$ onto τ
such that

(1) For every $\varepsilon_1, \varepsilon_2 \in \tau$ and $\eta = \varphi(\varepsilon_1, \varepsilon_2)$ we have

(d) $\mathrm{Cl}(V^X_{\varepsilon_1}) \subseteq W^X_\eta$, $V^X_{\varepsilon_2} \subseteq O^X_\eta$,
(e) $W^X_\eta \cap O^X_\eta = \varnothing$, and
(f) $X \setminus L^X_\eta = W^X_\eta \cup O^X_\eta$,
in the case, where $\mathrm{Cl}(V^X_{\varepsilon_1}) \cap V^X_{\varepsilon_2} = \varnothing$, and $L^X_\eta = \varnothing$ in the case, where $\mathrm{Cl}(V^X_{\varepsilon_1}) \cap V^X_{\varepsilon_2} \neq \varnothing$.

(2) For every $\eta \in \tau$, $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L^X_\eta) \leq \kappa - 1$.

For every $\eta \in \tau$ we set

$$\mathbf{L}_\eta = \{L^X_\eta : X \in \mathbf{S}\},$$

$$\mathbf{W}_\eta = \{W^X_\eta : X \in \mathbf{S}\}, \text{ and}$$

$$\mathbf{O}_\eta = \{O^X_\eta : X \in \mathbf{S}\}.$$

By the above property (2), $\mathbf{L}_\eta$ is an indexed collection of elements of the class $\mathbb{P}_{\kappa-1} \equiv \mathbb{P}(w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa - 1)$. By inductive assumption the class $\mathbb{P}_{\kappa-1}$ is saturated. Therefore, there exists an initial co-mark $\mathbf{M}^+_{\mathbf{L}_\eta}$ of $\mathbf{L}_\eta$ corresponding to the class $\mathbb{P}_{\kappa-1}$. Denote by $\mathbf{M}_\eta$ a co-mark of $\mathbf{S}$ such that its trace on $\mathbf{L}_\eta$ is a co-extension of the co-mark $\mathbf{M}^+_{\mathbf{L}_\eta}$. The existence of such a co-mark is easily proved.

Consider the co-indication

$$\mathbf{N} \equiv \{\{V^X_\varepsilon : \varepsilon \in \tau\} : X \in \mathbf{S}\}$$

of the $\mathbb{B}$-co-base $\mathbf{B} \equiv \{B^X : X \in \mathbf{S}\}$ of $\mathbf{S}$. Since $\mathbb{B}$ is a saturated class of bases there exists an initial co-mark $\mathbf{M}^+_{\mathbb{B}}$ of $\mathbf{S}$ corresponding to the co-indication $\mathbf{N}$ of $\mathbf{B}$ and the class $\mathbb{B}$. In particular, $\mathbf{M}^+_{\mathbb{B}}$ is a co-extension of $\mathbf{N}$.

By Lemma 2.1.2 of [2], there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$, which a co-extension of the co-marks $\mathbf{M}^+_{\mathbb{B}}$ and $\mathbf{M}_\eta$ for every $\eta \in \tau$. In particular, $\mathbf{M}^+$ is a co-extension of $\mathbf{N}$. We show that $\mathbf{M}^+$ is an initial co-mark of $\mathbf{S}$ corresponding to the class $\mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$.

Indeed, let

$$\mathbf{M} \equiv \{\{U^X_\delta : \delta \in \tau\} : X \in \mathbf{S}\}$$

be an arbitrary co-extension of $\mathbf{M}^+$. Then, $\mathbf{M}$ is a co-extension of the co-marks $\mathbf{M}^+_{\mathbb{B}}$, $\mathbf{N}$, and $\mathbf{M}_\eta$ for every $\eta \in \tau$. Denote by ϑ an indicial mapping from $\mathbf{N}$ to $\mathbf{M}$. Then, for every $X \in \mathbb{E}$, $V^X_\varepsilon = U^X_{\vartheta(\varepsilon)}$, $\varepsilon \in \tau$. Obviously, the co-mark $\mathbf{M}|_{\mathbf{L}_\eta}$ is a co-extension of the co-mark $\mathbf{M}^+_{\mathbf{L}_\eta}$ of $\mathbf{L}_\eta$.

Let $\mathrm{R}^+_{\mathbb{B}}$ be an initial family of equivalence relations on $\mathbf{S}$ corresponding to the co-mark $\mathbf{M}$, the co-indication $\mathbf{N}$ of $\mathbf{B}$, and the class $\mathbb{B}$. Let also $\mathrm{R}^+_{\mathbf{L}_\eta}$ be an initial family of equivalence relations on $\mathbf{L}_\eta$ corresponding to the co-mark $\mathbf{M}|_{\mathbf{L}_\eta}$ and the class $\mathbb{P}_{\kappa-1}$. Denote by R_η the family of equivalence relations on $\mathbf{S}$ such that the trace on $\mathbf{L}_\eta$ of R_η is the family $\mathrm{R}^+_{\mathbf{L}_\eta}$.

By Lemma 2.1.1 of [2], there exists an admissible family R^+ of equivalence relations on $\mathbf{S}$, which is a final refinement of the families $\mathrm{R}^+_{\mathbb{B}}$ and R_η for every $\eta \in \tau$. In particular, R^+ is $\mathbf{M}$-admissible. Without loss of generality, we can suppose that R^+ is $(\mathbf{M}, \mathbf{W}_\eta)$-admissible, $(\mathbf{M}, \mathbf{O}_\eta)$-admissible, $(\mathbf{M}, \mathbf{Co}(\mathbf{W}_\eta))$-admissible, and $(\mathbf{M}, \mathbf{Co}(\mathbf{O}_\eta))$-admissible. We prove that R^+ is an initial family of $\mathbf{S}$ corresponding to the co-mark $\mathbf{M}$ of $\mathbf{S}$ and the class $\mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$. For this purpose we consider an arbitrary admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and prove that for every $\mathbf{L} \in \mathrm{C}^\diamondsuit(\mathrm{R})$ the space $\mathrm{T}(\mathbf{L})$ belongs to $\mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$. Let $\mathbf{L} \in \mathrm{C}^\diamondsuit(\mathrm{R})$. Since $\mathbb{B}$ is a saturated class, we have $(\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\vartheta(\tau)}, \mathrm{T}(\mathbf{L})) \in \mathbb{B}$. We show that the base $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\vartheta(\tau)}$ of $\mathrm{T}(\mathbf{L})$ satisfies condition (2) of Definition 3.2, that is for every $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1)$ and $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2)$ of $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\vartheta(\tau)}$ (where $\mathbf{H}_1$, $\mathbf{H}_2 \subseteq \mathbf{L}$), with

$$\mathrm{Cl}_{\mathrm{T}(\mathbf{L})}(U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1)) \cap U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2) = \varnothing \tag{4}$$

there exists a subset L of $\mathrm{T}(\mathbf{L})$ separating $\mathrm{Cl}_{\mathrm{T}(\mathbf{L})}(U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1))$ and $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2)$ such that $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L) \leq \kappa - 1$.

Consider two elements $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1)$, $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2)$ of $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\vartheta(\tau)}$ satisfying relation (4). First we suppose that $\mathbf{H}_1 \cap \mathbf{H}_2 = \varnothing$. Then,

(g) $\mathrm{Cl}_{\mathrm{T}(\mathbf{L})}(U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1)) \subseteq \mathrm{T}(\mathbf{H}_1)$, $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2) \subseteq \mathrm{T}(\mathbf{L} \setminus \mathbf{H}_1)$,
(h) $\mathrm{T}(\mathbf{H}_1) \cap \mathrm{T}(\mathbf{L} \setminus \mathbf{H}_1) = \varnothing$, and
(i) $\mathrm{T}(\mathbf{L}) = \mathrm{T}(\mathbf{H}_1) \cup \mathrm{T}(\mathbf{L} \setminus \mathbf{H}_1)$.

Therefore, the empty set separates the sets $\mathrm{Cl}_{\mathrm{T}(\mathbf{L})}(U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1))$ and $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2)$. Since $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\varnothing) = -1 < \kappa$, we have $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\mathrm{T}(\mathbf{L})) \leq \kappa$.

Now, we suppose that $\mathbf{H}_1 \cap \mathbf{H}_2 \neq \varnothing$. Let $\mathbf{H} = \mathbf{H}_1 \cap \mathbf{H}_2$, $\vartheta^{-1}(\delta_1) = \varepsilon_1$, $\vartheta^{-1}(\delta_2) = \varepsilon_2$, and $\eta = \varphi(\varepsilon_1, \varepsilon_2)$. We prove that $\mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta})$ separates the sets $\mathrm{Cl}_{\mathrm{T}(\mathbf{L})}(U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1))$ and $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2)$, and $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta})) \leq \kappa - 1 < \kappa$.

Since $\mathbb{P}_{\kappa-1}$ is a saturated class of spaces, the subspace $\mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta})$ of $\mathrm{T}(\mathbf{M}|_{\mathbf{L}_\eta}, \mathrm{R}|_{\mathbf{L}_\eta})$ belongs to $\mathbb{P}_{\kappa-1}$. Hence,

$$w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta})) \leq \kappa - 1 < \kappa.$$

We prove that the subset $\mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta})$ of $\mathrm{T}(\mathbf{L})$ separates $\mathrm{Cl}_{\mathrm{T}(\mathbf{L})}(U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1))$ and $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2)$. Suppose that $X \in \mathbf{H}$. Since the subsets $\mathrm{Cl}(V^X_{\varepsilon_1})$ and $V^X_{\varepsilon_2}$ of X are disjoint, by condition (1) we have

(k) $\mathrm{Cl}(V^X_{\varepsilon_1}) \subseteq W^X_\eta$, $V^X_{\varepsilon_2} \subseteq O^X_\eta$,
(l) $W^X_\eta \cap O^X_\eta = \varnothing$, and
(m) $X \setminus L^X_\eta = W^X_\eta \cup O^X_\eta$.

The above relations imply that

(n) $\mathrm{Cl}_{\mathrm{T}(\mathbf{L})}(U^{\mathrm{T}}_{\delta_1}(\mathbf{H})) \subseteq \mathrm{T}(\mathbf{H}|_{\mathbf{W}_\eta}) = \mathrm{T}|_{\mathbf{W}_\eta} \cap \mathrm{T}(\mathbf{H})$,
$U^{\mathrm{T}}_{\delta_2}(\mathbf{H}) \subseteq \mathrm{T}(\mathbf{H}|_{\mathbf{O}_\eta}) = \mathrm{T}|_{\mathbf{O}_\eta} \cap \mathrm{T}(\mathbf{H})$,
(o) $\mathrm{T}(\mathbf{H}|_{\mathbf{W}_\eta}) \cap \mathrm{T}(\mathbf{H}|_{\mathbf{O}_\eta}) = \varnothing$, and
(p) $\mathrm{T}(\mathbf{H}) \setminus \mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta}) = \mathrm{T}(\mathbf{H}|_{\mathbf{W}_\eta}) \cup \mathrm{T}(\mathbf{H}|_{\mathbf{O}_\eta})$.

Since the restriction $\mathbf{W}_\eta$ of $\mathbf{S}$ is open and the family R is $(\mathbf{M}, \mathbf{Co}(\mathbf{W}_\eta))$-admissible, by Lemma 1.4.7 of [2], the subset $\mathrm{T}|_{\mathbf{W}_\eta}$ of T is open. Similarly, the subset $\mathrm{T}|_{\mathbf{O}_\eta}$ of T is open. Also, since the subset $\mathrm{T}(\mathbf{H})$ of T is open and $\mathrm{T}(\mathbf{H}) \subseteq \mathrm{T}(\mathbf{L})$, the sets $\mathrm{T}(\mathbf{H}|_{\mathbf{W}_\eta})$ and $\mathrm{T}(\mathbf{H}|_{\mathbf{O}_\eta})$ are open in $\mathrm{T}(\mathbf{L})$.

Setting

$$W = \mathrm{T}(\mathbf{H}_1 \setminus \mathbf{H}) \cup \mathrm{T}(\mathbf{H}|_{\mathbf{W}_n}) \quad \text{and} \quad O = \mathrm{T}(\mathbf{L} \setminus \mathbf{H}_1) \cup \mathrm{T}(\mathbf{H}|_{\mathbf{O}_n})$$

we have

(q) $\mathrm{Cl}_{\mathrm{T}(\mathbf{L})}(U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1)) \subseteq W$, $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2) \subseteq O$,
(r) $W \cap O = \varnothing$, and
(s) $\mathrm{T}(\mathbf{L}) \setminus \mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta}) = W \cup O$.

Therefore, the subset $\mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta})$ of $\mathrm{T}(\mathbf{L})$ separates the sets $\mathrm{Cl}_{\mathrm{T}(\mathbf{L})}(U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1))$ and $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_2)$. Thus, the class $\mathbb{P}(w - Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ is saturated.

Now, we prove that the class $\mathbb{P}(w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ is saturated. Let $\mathbf{S}$ be a indexed collection of elements of $\mathbb{P}(w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$. For every $X \in \mathbf{S}$ there exists an indexed set $\{Q^X_i : i \in \nu\}$ of subsets of X such that

(3) $X = \cup\{Q_i^X : i \in \nu\}$.
(4) For every $i \in \nu$, the subset Q_i^X of X is closed and $(Q_i^X, X) \in \mathbb{K}$.
(5) For every $i \in \nu$, $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(Q_i^X) \leq \kappa$.

We set $\mathbf{Q}_i = \{Q_i^X : X \in \mathbf{S}\}$, $i \in \nu$. By the preceding, the class $\mathbb{P} \equiv \mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ is saturated. By property (5), $\mathbf{Q}_i$ is an indexed collection of elements of the class $\mathbb{P}$. Therefore, there exists an initial co-mark $\mathbf{M}^+_{\mathbf{Q}_i}$ of $\mathbf{Q}_i$ corresponding to the class $\mathbb{P}$. Denote by $\mathbf{M}_i$ a co-mark of $\mathbf{S}$ such that its trace on $\mathbf{Q}_i$ is a co-extension of the co-mark $\mathbf{M}^+_{\mathbf{Q}_i}$.

By property (4), the restriction $\mathbf{Q}_i$ of $\mathbf{S}$ is a $\mathbb{K}$-restriction. Since $\mathbb{K}$ is a saturated class of subsets, for every $i \in \nu$ there exists an initial co-mark $\mathbf{M}^+_{\mathbb{K},i}$ of $\mathbf{S}$ corresponding to the $\mathbb{K}$-restriction $\mathbf{Q}_i$.

By Lemma 2.1.2 of [2], there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$, which a co-extension of the co-marks $\mathbf{M}_i$ and $\mathbf{M}^+_{\mathbb{K},i}$ for every $i \in \nu$. We show that $\mathbf{M}^+$ is an initial co-mark of $\mathbf{S}$ corresponding to the class $\mathbb{P}(w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$.

Indeed, let

$$\mathbf{M} \equiv \{\{U_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$$

be an arbitrary co-extension of $\mathbf{M}^+$. Then, $\mathbf{M}$ is a co-extension of the co-marks $\mathbf{M}_i$ and $\mathbf{M}^+_{\mathbb{K},i}$ and the co-mark $\mathbf{M}|_{\mathbf{Q}_i}$ is a co-extension of the co-mark $\mathbf{M}^+_{\mathbf{Q}_i}$ of $\mathbf{Q}_i$, $i \in \nu$.

Let $\mathrm{R}^+_{\mathbf{Q}_i}$ be an initial family of equivalence relations on $\mathbf{Q}_i$ corresponding to the co-mark $\mathbf{M}|_{\mathbf{Q}_i}$ and the class $\mathbb{P}$. Denote by R_i the family of equivalence relations on $\mathbf{S}$ such that the trace on $\mathbf{Q}_i$ of R_i is the family $\mathrm{R}^+_{\mathbf{Q}_i}$. Let also $\mathrm{R}^+_{\mathbb{K},i}$ be an initial family of equivalence relations on $\mathbf{S}$ corresponding to the co-mark $\mathbf{M}$ and the $\mathbb{K}$-restriction $\mathbf{Q}_i$.

By Lemma 2.1.1 of [2], there exists an admissible family R^+ of equivalence relations on $\mathbf{S}$, which is a final refinement of the families R_i and $\mathrm{R}^+_{\mathbb{K},i}$, $i \in \nu$. Therefore, R^+ is an $\mathbf{M}$-admissible family.

We prove that R^+ is an initial family of $\mathbf{S}$ corresponding to the co-mark $\mathbf{M}$ of $\mathbf{S}$ and the class $\mathbb{P}(w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$. For this purpose, we consider an arbitrary admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+. Then, R is a final refinement of the families R_i and $\mathrm{R}^+_{\mathbb{K},i}$ for every $i \in \nu$. We need to prove that for every $\mathbf{L} \in \mathrm{C}^\diamondsuit(\mathrm{R})$, $\mathrm{T}(\mathbf{L}) \in \mathbb{P}(w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$. Let $\mathbf{L} \in \mathrm{C}^\diamondsuit(\mathrm{R})$. It suffices to show that $\mathrm{T}(\mathbf{L}) = \cup\{\mathrm{T}_i(\mathbf{L}) : i \in \nu\}$ such that

(t) the subset $\mathrm{T}_i(\mathbf{L})$ of $\mathrm{T}(\mathbf{L})$ is closed,
(u) $(\mathrm{T}_i(\mathbf{L}), \mathrm{T}(\mathbf{L})) \in \mathbb{K}$, and
(v) $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\mathrm{T}_i(\mathbf{L})) \leq \kappa$, $i \in \nu$.

We set $\mathrm{T}_i(\mathbf{L}) = \mathrm{T}(\mathbf{L}|_{\mathbf{Q}_i})$, $i \in \nu$. It is easy to verify that the subset $\mathrm{T}(\mathbf{L}|_{\mathbf{Q}_i})$ of $\mathrm{T}(\mathbf{L})$ is closed and $\mathrm{T}(\mathbf{L}) = \cup\{\mathrm{T}(\mathbf{L}|_{\mathbf{Q}_i}) : i \in \nu\}$. Since $\mathbb{K}$ is a saturated class of subsets, $(\mathrm{T}(\mathbf{L}|_{\mathbf{Q}_i}), \mathrm{T}(\mathbf{L})) \in \mathbb{K}$. Since $\mathbb{P}$ is a saturated class, the subspace $\mathrm{T}(\mathbf{L}|_{\mathbf{Q}_i})$ of $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}_i}, \mathrm{R}|_{\mathbf{Q}_i})$ belongs to $\mathbb{P}$. Hence, $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\mathrm{T}(\mathbf{L}|_{\mathbf{Q}_i})) \leq \kappa$.

Thus, by condition (3) of Definition 3.2, $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\mathrm{T}(\mathbf{L})) \leq \kappa$ proving that the class $\mathbb{P}(w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ is saturated. □

Corollary 3.7. *For every $\kappa \in \omega$ in the classes*

$$\mathbb{P}(w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa) \ \ \textit{and} \ \ \mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$$

there exist universal elements.

Corollary 3.8. *Let $\mathbb{P}$ be one of the following classes*

(a) *the class of all (completely) regular spaces of weight $\leq \tau$,*
(b) *the class of all (completely) regular countable-dimensi-onal spaces of weight $\leq \tau$,*
(c) *the class of all (completely) regular strongly countable-dimensional spaces of weight $\leq \tau$,*
(d) *the class of all (completely) regular locally finite-dimensional spaces of weight $\leq \tau$, and*
(e) *the class of all (completely) regular spaces X of weight $\leq \tau$ such that* $\mathrm{ind}(X) \leq \alpha \in \tau^+$.

Then, for every $\kappa \in \omega$ in the classes

$$\mathbb{P}(w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa) \cap \mathbb{P} \ \ \textit{and} \ \ \mathbb{P}(w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa) \cap \mathbb{P}$$

there exist universal elements.

4. The dimension-like functions: $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ and $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$

Definition 4.1. A class $\mathbb{E}$ of spaces is said to be $\mathbb{B}$-*strong-hereditary-separated*, where $\mathbb{B}$ is a class of bases, if for every element X of $\mathbb{E}$ there exists a $\mathbb{B}$-base $B^X \equiv \{U_\delta : \delta \in \tau\}$ for X such that for every two elements U_{δ_1} and U_{δ_2} of B^X with $U_{\delta_1} \cap U_{\delta_2} = \varnothing$ there exists a subset L of X separating the sets U_{δ_1} and U_{δ_2} and belonging to $\mathbb{E}$.

We note that if $\mathbb{E}$ is $\mathbb{B}$-strong-hereditary-separated, then $\varnothing \in \mathbb{E}$. This follows by the fact that the empty set is the unique subset of X separating the elements $\varnothing$ and X of B^X.

Definition 4.2. Let $\mathbb{B}$ be a class of bases, $\mathbb{E}$ a $\mathbb{B}$-strong-hereditary-separated class of spaces, and $\mathbb{K}$ a class of subsets with $(X, X) \in \mathbb{K}$ for every space X. We denote by $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ and $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ the *dimension-like functions* with as domain the class of all spaces and as range the class $\mathcal{O} \cup \{-1, \infty\}$ satisfying the following conditions:

(1) $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = -1$ if and only if $X \in \mathbb{E}$.
(2) $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$, where $\alpha \in \mathcal{O}$, if and only if there exists a $\mathbb{B}$-base $B^X \equiv \{U_\delta : \delta \in \tau\}$ for X such that for every two elements U_{δ_1}, U_{δ_2} of

B^X with $U_{\delta_1} \cap U_{\delta_2} = \varnothing$ there exists a subset L of X separating U_{δ_1} and U_{δ_2} with $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L) < \alpha$.

(3) $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$, $\alpha \in \mathcal{O}$, if and only if $X = \cup\{S_i : i \in \nu\}$ such that: (a) the subset S_i of X is closed, (b) $(S_i, X) \in \mathbb{K}$, and (c) $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(S_i) \leq \alpha$, $i \in \nu$.

Therefore, $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \infty$ (respectively, $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \infty$) if and only if the inequality $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$ (respectively, $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$) is not true for every $\alpha \in \mathcal{O}$.

Remark 4.3. In order that the above definition to be well defined we need to show that if for a space X we have $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X)$=$s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = -1$, then $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq 0$ and $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq 0$.

For dimension-like function $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ this follows immediately by the fact that $X \in \mathbb{E}$ and the class $\mathbb{E}$ is $\mathbb{B}$-strong-hereditary-separated.

For dimension-like function $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$, we have (a) $X = \{S_i : i \in \nu\}$, where $S_i = X$, (b) $(X, X) \in \mathbb{K}$, and (c) $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = -1 \leq 0$, which means that $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq 0$.

Proposition 4.4. *For every space X we have*

$$s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X). \tag{5}$$

Proof. Let $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \alpha \in \{-1, \infty\} \cup \mathcal{O}$. The inequality (5) is clear if $\alpha = -1$ or $\alpha = \infty$. Suppose that $\alpha \in \mathcal{O}$. We have $X = \cup\{S_i : i \in \nu\}$, where $S_i = X$. Since $(S_i, X) = (X, X) \in \mathbb{K}$ and $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(S_i) = s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$, the condition (3) of Definition 4.2 implies that $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$. □

Proposition 4.5. *For every space X, $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \in \{-1, \infty\} \cup \tau^+$, and, therefore $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \in \{-1, \infty\} \cup \tau^+$.*

Proof. Suppose that the proposition is not true. Let α be the minimal element of $\mathcal{O} \setminus \tau^+$ such that there exists a space X with $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \alpha$. Let $B^X = \{U_\delta : \delta \in \tau\}$ be the $\mathbb{B}$-base for X mentioned in condition (2) of Definition 4.2.

Denote by P the set of all pairs $(\delta_1, \delta_2) \in \tau \times \tau$ with

$$U_{\delta_1} \cap U_{\delta_2} = \varnothing.$$

For every $(\delta_1, \delta_2) \in P$ let $L(\delta_1, \delta_2)$ be a subset of X separating the sets U_{δ_1} and U_{δ_2} with

$$s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L(\delta_1, \delta_2)) = \beta(\delta_1, \delta_2) < \alpha.$$

First we suppose that $\beta(\delta_1, \delta_2) < \tau^+$ for every $(\delta_1, \delta_2) \in P$. Since $|P| \leq \tau$ there exists an ordinal $\beta \in \tau^+$ such that $\beta(\delta_1, \delta_2) < \beta$ for every $(\delta_1, \delta_2) \in P$. Then,

$s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L(\delta_1,\delta_2)) < \beta$ and, by condition (2) of Definition 4.2, $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \beta$, which is a contradiction.

Now, we suppose that there exists $(\delta_1,\delta_2) \in P$ such that $\tau^+ \leq \beta(\delta_1,\delta_2)$. Since $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L(\delta_1,\delta_2)) = \beta(\delta_1,\delta_2)$, there exist closed subsets $S_i^{L(\delta_1,\delta_2)}$ of $L(\delta_1,\delta_2)$, $i \in \nu$, such that:

(a) $L(\delta_1,\delta_2) = \cup\{S_i^{L(\delta_1,\delta_2)} : i \in \nu\}$,

(b) $(S_i^{L(\delta_1,\delta_2)}, L(\delta_1,\delta_2)) \in \mathbb{K}$, and

(c) $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(S_i^{L(\delta_1,\delta_2)}) = \beta_i \leq \beta(\delta_1,\delta_2) < \alpha$.

If $\beta_i < \tau^+$ for all $i \in \nu$, then there exists an ordinal $\beta \in \tau^+$ such that $\beta_i \leq \beta$, which means that $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(S_i^{L(\delta_1,\delta_2)}) \leq \beta$. Therefore,

$$s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L(\delta_1,\delta_2)) \leq \beta < \tau^+ \leq \beta(\delta_1,\delta_2),$$

which is a contradiction. Thus, there exists $i \in \nu$ such that

$$\tau^+ \leq s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(S_i^{L(\delta_1,\delta_2)}) < \alpha.$$

The last relation contradicts the choice of the ordinal α completing the proof of the proposition. □

Theorem 4.6. *Let $\mathbb{B}$ be a saturated class of bases, $\mathbb{E}$ a saturated $\mathbb{B}$-strong-hereditary-separated class of spaces, and $\mathbb{K}$ a saturated class of subsets such that $(X,X) \in \mathbb{K}$ for every space X. Then, for every $\kappa \in \{-1\} \cup \omega$ the classes $\mathbb{P}(s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ and $\mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ are saturated.*

Proof. We prove the theorem by induction on κ. Let $\kappa = -1$. Then, a space X belongs to $\mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq -1)$ if and only if X belongs to $\mathbb{E}$, that is

$$\mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq -1) = \mathbb{E}.$$

Therefore, $\mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq -1)$ is a saturated class of spaces. Similarly, the class $\mathbb{P}(s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq -1)$ is saturated.

Let $\kappa \in \omega$. Suppose that the classes $\mathbb{P}(s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq m)$ and $\mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq m)$ are saturated, $m \in \{-1\} \cup \kappa$. We prove that the classes $\mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ and $\mathbb{P}(s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ are also saturated. First we prove that $\mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ is a saturated class.

Let $\mathbf{S}$ be an indexed collection of elements of $\mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$. For every $X \in \mathbf{S}$ let $B^X \equiv \{V^X_\varepsilon : \varepsilon \in \tau\}$ be an indexed $\mathbb{B}$-base for X satisfying condition (2) of Definition 4.2. Then, there exist

(a) an indexed set $\{L^X_\eta : \eta \in \tau\}$ of subsets of X,
(b) two indexed sets $\{W^X_\eta : \eta \in \tau\}$ and $\{O^X_\eta : \eta \in \tau\}$ of open subsets of X, and
(c) a one-to-one mapping φ of $\tau \times \tau$ onto τ
such that

(1) For every $\varepsilon_1, \varepsilon_2 \in \tau$ and $\eta = \varphi(\varepsilon_1, \varepsilon_2)$ we have

(d) $V^X_{\varepsilon_1} \subseteq W^X_\eta$, $V^X_{\varepsilon_2} \subseteq O^X_\eta$,
(e) $W^X_\eta \cap O^X_\eta = \varnothing$, and
(f) $X \setminus L^X_\eta = W^X_\eta \cup O^X_\eta$,
in the case, where $V^X_{\varepsilon_1} \cap V^X_{\varepsilon_2} = \varnothing$, and $L^X_\eta = \varnothing$ in the case, where $V^X_{\varepsilon_1} \cap V^X_{\varepsilon_2} \neq \varnothing$.

(2) For every $\eta \in \tau$, $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L^X_\eta) \leq \kappa - 1$.

For every $\eta \in \tau$ we set

$$\mathbf{L}_\eta = \{L^X_\eta : X \in \mathbf{S}\},$$
$$\mathbf{W}_\eta = \{W^X_\eta : X \in \mathbf{S}\}, \text{ and}$$
$$\mathbf{O}_\eta = \{O^X_\eta : X \in \mathbf{S}\}.$$

By the above property (2), $\mathbf{L}_\eta$ is an indexed collection of elements of the class $\mathbb{P}_{\kappa-1} \equiv \mathbb{P}(s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa - 1)$. By inductive assumption the class $\mathbb{P}_{\kappa-1}$ is saturated. Therefore, there exists an initial co-mark $\mathbf{M}^+_{\mathbf{L}_\eta}$ of $\mathbf{L}_\eta$ corresponding to the class $\mathbb{P}_{\kappa-1}$. Denote by $\mathbf{M}_\eta$ a co-mark of $\mathbf{S}$ such that its trace on $\mathbf{L}_\eta$ is a co-extension of the co-mark $\mathbf{M}^+_{\mathbf{L}_\eta}$. The existence of such a co-mark is easily proved.

Consider the co-indication

$$\mathbf{N} \equiv \{\{V^X_\varepsilon : \varepsilon \in \tau\} : X \in \mathbf{S}\}$$

of the $\mathbb{B}$-co-base $\mathbf{B} \equiv \{B^X : X \in \mathbf{S}\}$ of $\mathbf{S}$. Since $\mathbb{B}$ is a saturated class of bases there exists an initial co-mark $\mathbf{M}^+_{\mathbb{B}}$ of $\mathbf{S}$ corresponding to the co-indication $\mathbf{N}$ of $\mathbf{B}$ and the class $\mathbb{B}$. In particular, $\mathbf{M}^+_{\mathbb{B}}$ is a co-extension of $\mathbf{N}$.

By Lemma 2.1.2 of [2], there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$, which a co-extension of the co-marks $\mathbf{M}^+_{\mathbb{B}}$ and $\mathbf{M}_\eta$ for every $\eta \in \tau$. In particular, $\mathbf{M}^+$ is a co-extension of $\mathbf{N}$. We show that $\mathbf{M}^+$ is an initial co-mark of $\mathbf{S}$ corresponding to the class $\mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$.

Indeed, let

$$\mathbf{M} \equiv \{\{U^X_\delta : \delta \in \tau\} : X \in \mathbf{S}\}$$

be an arbitrary co-extension of $\mathbf{M}^+$. Then, $\mathbf{M}$ is a co-extension of the co-marks $\mathbf{M}^+_{\mathbb{B}}$, $\mathbf{N}$, and $\mathbf{M}_\eta$ for every $\eta \in \tau$. Denote by ϑ an indicial mapping from $\mathbf{N}$ to $\mathbf{M}$. Then, for every $X \in \mathbb{E}$, $V^X_\varepsilon = U^X_{\vartheta(\varepsilon)}$, $\varepsilon \in \tau$. Obviously, the co-mark $\mathbf{M}|_{\mathbf{L}_\eta}$ is a co-extension of the co-mark $\mathbf{M}^+_{\mathbf{L}_\eta}$ of $\mathbf{L}_\eta$.

Let $\mathrm{R}^+_{\mathbb{B}}$ be an initial family of equivalence relations on $\mathbf{S}$ corresponding to the co-mark $\mathbf{M}$, the co-indication $\mathbf{N}$ of $\mathbf{B}$, and the class $\mathbb{B}$. Let also $\mathrm{R}^+_{\mathbf{L}_\eta}$ be an initial family of equivalence relations on $\mathbf{L}_\eta$ corresponding to the co-mark $\mathbf{M}|_{\mathbf{L}_\eta}$ and the class $\mathbb{P}_{\kappa-1}$. Denote by R_η the family of equivalence relations on $\mathbf{S}$ such that the trace on $\mathbf{L}_\eta$ of R_η is the family $\mathrm{R}^+_{\mathbf{L}_\eta}$.

By Lemma 2.1.1 of [2], there exists an admissible family R^+ of equivalence relations on $\mathbf{S}$, which is a final refinement of the families $\mathrm{R}^+_{\mathbb{B}}$ and R_η for every

$\eta \in \tau$. In particular, R^+ is $\mathbf{M}$-admissible. Without loss of generality, we can suppose that R^+ is $(\mathbf{M}, \mathbf{W}_\eta)$-admissible, $(\mathbf{M}, \mathbf{O}_\eta)$-admissible, $(\mathbf{M}, \mathbf{Co}(\mathbf{W}_\eta))$-admissible, and $(\mathbf{M}, \mathbf{Co}(\mathbf{O}_\eta))$-admissible. We prove that R^+ is an initial family of $\mathbf{S}$ corresponding to the co-mark $\mathbf{M}$ of $\mathbf{S}$ and the class $\mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$. For this purpose we consider an arbitrary admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and prove that for every $\mathbf{L} \in \mathrm{C}^\diamondsuit(\mathrm{R})$ the space $\mathrm{T}(\mathbf{L})$ belongs to $\mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$. Let $\mathbf{L} \in \mathrm{C}^\diamondsuit(\mathrm{R})$. Since $\mathbb{B}$ is a saturated class, we have $(\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\vartheta(\tau)}, \mathrm{T}(\mathbf{L})) \in \mathbb{B}$. We show that the base $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\vartheta(\tau)}$ of $\mathrm{T}(\mathbf{L})$ satisfies condition (2) of Definition 4.2, that is for every $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1)$ and $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2)$ of $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\vartheta(\tau)}$ (where $\mathbf{H}_1$, $\mathbf{H}_2 \subseteq \mathbf{L}$), with

$$U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1) \cap U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2) = \varnothing \tag{5}$$

there exists a subset L of $\mathrm{T}(\mathbf{L})$ separating $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1)$ and $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2)$ such that $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L) \leq \kappa - 1$.

Consider two elements $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1)$, $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2)$ of $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\vartheta(\tau)}$ satisfying relation (5). First we suppose that $\mathbf{H}_1 \cap \mathbf{H}_2 = \varnothing$. Then,

(g) $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1) \subseteq \mathrm{T}(\mathbf{H}_1)$, $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2) \subseteq \mathrm{T}(\mathbf{L} \setminus \mathbf{H}_1)$,
(h) $\mathrm{T}(\mathbf{H}_1) \cap \mathrm{T}(\mathbf{L} \setminus \mathbf{H}_1) = \varnothing$, and
(i) $\mathrm{T}(\mathbf{L}) = \mathrm{T}(\mathbf{H}_1) \cup \mathrm{T}(\mathbf{L} \setminus \mathbf{H}_1)$.

Therefore, the empty set separates the sets $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1)$ and $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2)$. Since $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\varnothing) = -1 < \kappa$, we have $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\mathrm{T}(\mathbf{L})) \leq \kappa$.

Now, we suppose that $\mathbf{H}_1 \cap \mathbf{H}_2 \neq \varnothing$. Let $\mathbf{H} = \mathbf{H}_1 \cap \mathbf{H}_2$, $\vartheta^{-1}(\delta_1) = \varepsilon_1$, $\vartheta^{-1}(\delta_2) = \varepsilon_2$, and $\eta = \varphi(\varepsilon_1, \varepsilon_2)$. We prove that $\mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta})$ separates the sets $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1)$ and $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2)$, and $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta})) \leq \kappa - 1 < \kappa$.

Since $\mathbb{P}_{\kappa-1}$ is a saturated class of spaces, the subspace $\mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta})$ of $\mathrm{T}(\mathbf{M}|_{\mathbf{L}_\eta}, \mathrm{R}|_{\mathbf{L}_\eta})$ belongs to $\mathbb{P}_{\kappa-1}$. Hence,

$$s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta})) \leq \kappa - 1 < \kappa.$$

We prove that the subset $\mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta})$ of $\mathrm{T}(\mathbf{L})$ separates $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1)$ and $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2)$. Suppose that $X \in \mathbf{H}$. Since the subsets $V^X_{\varepsilon_1}$ and $V^X_{\varepsilon_2}$ of X are disjoint, by condition (1) we have

(k) $V^X_{\varepsilon_1} \subseteq W^X_\eta$, $V^X_{\varepsilon_2} \subseteq O^X_\eta$,
(l) $W^X_\eta \cap O^X_\eta = \varnothing$, and
(m) $X \setminus L^X_\eta = W^X_\eta \cup O^X_\eta$.

The above relations imply that

(n) $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}) \subseteq \mathrm{T}(\mathbf{H}|_{\mathbf{W}_\eta}) = \mathrm{T}|_{\mathbf{W}_\eta} \cap \mathrm{T}(\mathbf{H})$, $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}) \subseteq \mathrm{T}(\mathbf{H}|_{\mathbf{O}_\eta}) = \mathrm{T}|_{\mathbf{O}_\eta} \cap \mathrm{T}(\mathbf{H})$,
(o) $\mathrm{T}(\mathbf{H}|_{\mathbf{W}_\eta}) \cap \mathrm{T}(\mathbf{H}|_{\mathbf{O}_\eta}) = \varnothing$, and
(p) $\mathrm{T}(\mathbf{H}) \setminus \mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta}) = \mathrm{T}(\mathbf{H}|_{\mathbf{W}_\eta}) \cup \mathrm{T}(\mathbf{H}|_{\mathbf{O}_\eta})$.

Since the restriction $\mathbf{W}_\eta$ of $\mathbf{S}$ is open and the family R is $(\mathbf{M}, \mathbf{Co}(\mathbf{W}_\eta))$-admissible, by Lemma 1.4.7 of [2], the subset $\mathrm{T}|_{\mathbf{W}_\eta}$ of T is open. Similarly, the subset $\mathrm{T}|_{\mathbf{O}_\eta}$ of T is open. Also, since the subset $\mathrm{T}(\mathbf{H})$ of T is open and $\mathrm{T}(\mathbf{H}) \subseteq \mathrm{T}(\mathbf{L})$, the sets $\mathrm{T}(\mathbf{H}|_{\mathbf{W}_\eta})$ and $\mathrm{T}(\mathbf{H}|_{\mathbf{O}_\eta})$ are open in $\mathrm{T}(\mathbf{L})$.

Setting

$$W = \mathrm{T}(\mathbf{H}_1 \setminus \mathbf{H}) \cup \mathrm{T}(\mathbf{H}|_{\mathbf{W}_n}) \text{ and } O = \mathrm{T}(\mathbf{L} \setminus \mathbf{H}_1) \cup \mathrm{T}(\mathbf{H}|_{\mathbf{O}_n})$$

we have

(q) $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1) \subseteq W$, $U^{\mathrm{T}}_{\delta_2}(\mathbf{H}_2) \subseteq O$,
(r) $W \cap O = \varnothing$, and
(s) $\mathrm{T}(\mathbf{L}) \setminus \mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta}) = W \cup O$.

Therefore, the subset $\mathrm{T}(\mathbf{H}|_{\mathbf{L}_\eta})$ of $\mathrm{T}(\mathbf{L})$ separates the sets $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_1)$ and $U^{\mathrm{T}}_{\delta_1}(\mathbf{H}_2)$. Thus, the class $\mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ is saturated.

Now, we prove that the class $\mathbb{P}(s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ is saturated. Let $\mathbf{S}$ be a indexed collection of elements of $\mathbb{P}(s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$. For every $X \in \mathbf{S}$ there exists an indexed set $\{Q^X_i : i \in \nu\}$ of subsets of X such that

(3) $X = \cup\{Q^X_i : i \in \nu\}$.
(4) For every $i \in \nu$, the subset Q^X_i of X is closed and $(Q^X_i, X) \in \mathbb{K}$.
(5) For every $i \in \nu$, $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(Q^X_i) \leq \kappa$.

We set $\mathbf{Q}_i = \{Q^X_i : X \in \mathbf{S}\}$, $i \in \nu$. By the preceding, the class $\mathbb{P} \equiv \mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ is saturated. By property (5), $\mathbf{Q}_i$ is an indexed collection of elements of the class $\mathbb{P}$. Therefore, there exists an initial co-mark $\mathbf{M}^+_{\mathbf{Q}_i}$ of $\mathbf{Q}_i$ corresponding to the class $\mathbb{P}$. Denote by $\mathbf{M}_i$ a co-mark of $\mathbf{S}$ such that its trace on $\mathbf{Q}_i$ is a co-extension of the co-mark $\mathbf{M}^+_{\mathbf{Q}_i}$.

By property (4), the restriction $\mathbf{Q}_i$ of $\mathbf{S}$ is a $\mathbb{K}$-restriction. Since $\mathbb{K}$ is a saturated class of subsets, for every $i \in \nu$ there exists an initial co-mark $\mathbf{M}^+_{\mathbb{K},i}$ of $\mathbf{S}$ corresponding to the $\mathbb{K}$-restriction $\mathbf{Q}_i$.

By Lemma 2.1.2 of [2], there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$, which a co-extension of the co-marks $\mathbf{M}_i$ and $\mathbf{M}^+_{\mathbb{K},i}$ for every $i \in \nu$. We show that $\mathbf{M}^+$ is an initial co-mark of $\mathbf{S}$ corresponding to the class $\mathbb{P}(s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$.

Indeed, let

$$\mathbf{M} \equiv \{\{U^X_\delta : \delta \in \tau\} : X \in \mathbf{S}\}$$

be an arbitrary co-extension of $\mathbf{M}^+$. Then, $\mathbf{M}$ is a co-extension of the co-marks $\mathbf{M}_i$ and $\mathbf{M}^+_{\mathbb{K},i}$ and the co-mark $\mathbf{M}|_{\mathbf{Q}_i}$ is a co-extension of the co-mark $\mathbf{M}^+_{\mathbf{Q}_i}$ of $\mathbf{Q}_i$, $i \in \nu$.

Let $\mathrm{R}^+_{\mathbf{Q}_i}$ be an initial family of equivalence relations on $\mathbf{Q}_i$ corresponding to the co-mark $\mathbf{M}|_{\mathbf{Q}_i}$ and the class $\mathbb{P}$. Denote by R_i the family of equivalence relations on $\mathbf{S}$ such that the trace on $\mathbf{Q}_i$ of R_i is the family $\mathrm{R}^+_{\mathbf{Q}_i}$. Let also $\mathrm{R}^+_{\mathbb{K},i}$ be an initial family of equivalence relations on $\mathbf{S}$ corresponding to the co-mark $\mathbf{M}$ and the $\mathbb{K}$-restriction $\mathbf{Q}_i$.

By Lemma 2.1.1 of [2], there exists an admissible family R^+ of equivalence relations on $\mathbf{S}$, which is a final refinement of the families R_i and $\mathrm{R}^+_{\mathbb{K},i}$, $i \in \nu$. Therefore, R^+ is an $\mathbf{M}$-admissible family.

We prove that R^+ is an initial family of $\mathbf{S}$ corresponding to the co-mark $\mathbf{M}$ of $\mathbf{S}$ and the class $\mathbb{P}(s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$. For this purpose, we consider an arbitrary admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+. Then, R is a final refinement of the families R_i and $\mathrm{R}^+_{\mathbb{K},i}$ for every $i \in \nu$. We need to prove that for every $\mathbf{L} \in \mathrm{C}^\diamondsuit(\mathrm{R})$, $\mathrm{T}(\mathbf{L}) \in \mathbb{P}(s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$. Let $\mathbf{L} \in \mathrm{C}^\diamondsuit(\mathrm{R})$. It suffices to show that $\mathrm{T}(\mathbf{L}) = \cup\{\mathrm{T}_i(\mathbf{L}) : i \in \nu\}$ such that

(t) the subset $\mathrm{T}_i(\mathbf{L})$ of $\mathrm{T}(\mathbf{L})$ is closed,
(u) $(\mathrm{T}_i(\mathbf{L}), \mathrm{T}(\mathbf{L})) \in \mathbb{K}$, and
(v) $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\mathrm{T}_i(\mathbf{L})) \leq \kappa$, $i \in \nu$.

We set $\mathrm{T}_i(\mathbf{L}) = \mathrm{T}(\mathbf{L}|_{\mathbf{Q}_i})$, $i \in \nu$. It is easy to verify that the subset $\mathrm{T}(\mathbf{L}|_{\mathbf{Q}_i})$ of $\mathrm{T}(\mathbf{L})$ is closed and $\mathrm{T}(\mathbf{L}) = \cup\{\mathrm{T}(\mathbf{L}|_{\mathbf{Q}_i}) : i \in \nu\}$. Since $\mathbb{K}$ is a saturated class of subsets, $(\mathrm{T}(\mathbf{L}|_{\mathbf{Q}_i}), \mathrm{T}(\mathbf{L})) \in \mathbb{K}$. Since $\mathbb{P}$ is a saturated class, the subspace $\mathrm{T}(\mathbf{L}|_{\mathbf{Q}_i})$ of $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}_i}, \mathrm{R}|_{\mathbf{Q}_i})$ belongs to $\mathbb{P}$. Hence, $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\mathrm{T}(\mathbf{L}|_{\mathbf{Q}_i})) \leq \kappa$.

Thus, by condition (3) of Definition 4.2, $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(\mathrm{T}(\mathbf{L})) \leq \kappa$ proving that the class $\mathbb{P}(s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$ is saturated. □

Corollary 4.7. *For every $\kappa \in \omega$ in the classes*

$$\mathbb{P}(s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa) \quad \textit{and} \quad \mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa)$$

there exist universal elements.

Corollary 4.8. *Let $\mathbb{P}$ be one of the following classes*

(a) *the class of all (completely) regular spaces of weight $\leq \tau$,*
(b) *the class of all (completely) regular countable-dimensi-onal spaces of weight $\leq \tau$,*
(c) *the class of all (completely) regular strongly countable-dimensional spaces of weight $\leq \tau$,*
(d) *the class of all (completely) regular locally finite-dimensional spaces of weight $\leq \tau$, and*
(e) *the class of all (completely) regular spaces X of weight $\leq \tau$ such that* $\mathrm{ind}(X) \leq \alpha \in \tau^+$.

Then, for every $\kappa \in \omega$ in the classes

$$\mathbb{P}(s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa) \cap \mathbb{P} \quad \textit{and} \quad \mathbb{P}(s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq \kappa) \cap \mathbb{P}$$

there exist universal elements.

5. Questions

Question 5.1. *Does there exists a universal element in the class of all spaces X with $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$ or in the class of all spaces X with $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$, where α is an ordinal.*

Question 5.2. *Does there exists a universal element in the class of all spaces X with* $w\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$ *or in the class of all spaces X with* $w\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$*, where* α *is an ordinal.*

Question 5.3. *Does there exists a universal element in the class of all spaces X with* $s\text{-}dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$ *or in the class of all spaces X with* $s\text{-}Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$*, where* α *is an ordinal.*

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