Applied General Topology
© Universidad Politécnica de Valencia
Volume 12, no. 2, 2011
pp. 221-225

Density of κ-Box-Products and the existence of generalized independent families

Stefan Ottmar Elser

Abstract

In this paper we will prove a slight generalisation of the Hewitt-Marczewski-Pondiczery theorem (theorem 2.3 below) concerning the density of κ-box-products. With this result we will prove the existence of generalized independent families of big cardinality (corollary 2.5 below) which were introduced by Wanjun Hu.

2010 MSC: *54A25, 54B10; Secondary: 03E05.*

Keywords: *κ-box-product, generalized independent family.*

1. Introduction

Let $d(X)$ denote the density and $w(X)$ the weight of the topological space X.

Definition 1.1. Let μ, κ be two cardinals with $\aleph_0 \leq \kappa \leq \mu$ and $\{X_i\}_{i\in\mu}$ be a family of topological spaces.
$\square^{\kappa}_{i\in\mu} X_i$ denotes the κ-*box-product* which is induced on the full cartesian product $\prod_{i\in\mu} X_i$ by the canonical base

$$\mathfrak{B} = \left\{ \bigcap_{i\in I} pr_i^{-1}(U_i); I \in P_{<\kappa}(\mu) \text{ and } U_i \text{ is open in } X_i \right\}$$

where $P_{<\kappa}(\mu) := \{I \subseteq \mu; |I| < \kappa\}$.

For $\kappa = \aleph_0$ the κ-box-product is the usual Tychonoff-product [8] and for $\kappa^+ = \mu$ the κ-box-product is the full box-product mentioned by Kelley [5] and Bourbaki [1].

In this paper we will discuss the density of κ-box-products and the connection with infinite combinatorics. The classical Hewitt-Marczewski-Pondiczery theorem states:

$$d\left(\square_{i\in 2^\mu}^{\aleph_0} X_i\right) \le \mu \text{ for all spaces } X_i \text{ with } d(X_i) \le \mu$$

This has been proven for separable spaces by E. Marczewski [6] in 1941. In 1944 E. S. Pondiczery [7] proved a slighty weaker version for Hausdorff spaces and in 1947 E. Hewitt [3] proved the general version as stated above.
In theorem 2.4 we will prove:

$$d\left(\square_{i\in 2^\mu}^{\kappa} X_i\right) \le \mu^{<\kappa} \text{ for all spaces } X_i \text{ with } d(X_i) \le \mu$$

2. Density of κ-Box-Products

In this section we will prove a generalisation of Theorem 1 in [2]. To do so we start with the following definition and proposition:

Definition 2.1. Let κ, μ be two infinite cardinals with $\mu \ge \kappa$, $\{X_i\}_{i\in I}$ a family of topological spaces and for all $i \in I$ let $\mathfrak{B}_i$ be a base of the topology on X_i. $W \subseteq \prod_{i\in I} X_i$ is called a μ-cube if for every $i \in I$ there exists $\mathfrak{W}_i \subseteq \mathfrak{B}_\mathfrak{i}$ with $W = \prod_{i\in I}\left(\bigcap \mathfrak{W}_i\right)$.

Proposition 2.2. *Let X be a set, $\mu \ge \kappa$ two infinite cardinals, $\{X_i\}_{i\in I}$ a family of topological spaces, $\{f_i : X \to X_i\}_{i\in I}$ a family of functions and let W be a subset of $\prod_{i\in I} X_i$ which is a union of μ-cubes.*
For every cardinal $\lambda < \kappa$ and every tuple $\left\langle \{x_i\}_{i\in\lambda}; \{J_i\}_{i\in\lambda}\right\rangle$ of families $\{x_i\}_{i\in\lambda} \subseteq X$ and $\{J_i\}_{i\in\lambda} \subseteq P(I)$, where all J_i are pairwise disjunct and not empty, there exists a subset $Q \subseteq W$ of cardinality less or equal to $\mu^{<\kappa}$ so that for all families $\{j_i; j_i \in J_i\}_{i\in\lambda}$ the following holds:

$$\left(W \cap \bigcap_{i\in\lambda} pr_{j_i}^{-1}\left(f_{j_i}\left(x_{j_i}\right)\right) \neq \varnothing\right) \Rightarrow \left(Q \cap \bigcap_{i\in\lambda} pr_{j_i}^{-1}\left(f_{j_i}\left(x_{j_i}\right)\right) \neq \varnothing\right).$$

Proof. For every tuple $\left\langle \{x_i\}_{i\in\lambda}; \{J_i\}_{i\in\lambda}\right\rangle$ with $|\{i \in \lambda; |J_i| > 1\}| = 0$ the claim is pretty obvious.
So we assume that the proposition is valid for cardinals less than ν and let $\left\langle \{x_i\}_{i\in\lambda}; \{J_i\}_{i\in\lambda}\right\rangle$ be a tuple with $|\{i \in \lambda; |J_i| > 1\}| = \nu$.
Without loss of generality we may assume that $|J_i| > 1$ for all $i \in \nu$ and $|J_i| = 1$ for all other $i \ge \nu$ and that there exists at least one family $\{j_i; j_i \in J_i\}_{i\in\lambda}$ with $W \cap \bigcap_{i\in\lambda} pr_{j_i}^{-1}\left(f_{j_i}\left(x_{j_i}\right)\right) \neq \varnothing$.
Let $p \in W$ be an point so that $pr_{j_i}(p) \in f_{j_i}(x_i)$ for all $\nu \le i \in \lambda$.
Then there exists an $J \in P_{\le\mu}(I)$ with

$$\left\{q \in \prod_{i\in I} X_i; \forall j \in J : pr_j(q) = pr_j(p)\right\} \subseteq W.$$

We choose for all $i \in \nu$ and $j_i \in (J_i - J)$ a point $q_{j_i} \in f_{j_i}(x_i)$ and we define a point $q \in W$ as follows:

$$pr_i(q) := \begin{cases} pr_i(p) & \text{, if } i \in (I - \bigcup_{l\in\nu}(J_l - J)) \\ q_{j_l} & \text{, if } i = j_l \text{ and } j_l \in (J_l - J) \end{cases}$$

By the definition of q we have $q \in \left(W \cap \bigcap_{i\in\lambda} pr_{j_i}^{-1}(f_{j_i}(x_i))\right)$ for every family $\{j_i; j_i \in J_i\}_{i\in\lambda}$ such that for all $i \in \nu$: $j_i \in (J_i - J)$.

Now we have to consider families $\{j_i; j_i \in J_i\}_{i\in\lambda}$ with $j_i \in (J_i \cap J)$ for at least one $i \in \lambda$.
We define

$$\Sigma := \left\{\{J_i^*\}_{i\in\nu}\, ; |\{i \in \kappa; J_i^* = J_i\}| < \nu \wedge (J_i^* \neq J_i \Rightarrow J_i^* \in P_1(J_i \cap J))\right\}.$$

$\Rightarrow |\Sigma| \leq \mu^\nu \leq \mu^\lambda \leq \mu^{<\kappa}$
For all $\sigma = \{J_i^*\}_{i\in\nu} \in \Sigma$ we define a family $\{J_i^\sigma\}_{i\in\lambda}$ as follows:

$$J_i^\sigma := \begin{cases} J_i^* & \text{, if } i \in \nu \\ J_i & \text{, if } i \geq \nu \end{cases}$$

For all these $\{J_i^\sigma\}_{i\in\lambda}$ the proposition already holds, so we can choose a set $Q_\sigma \subseteq W$ with $|Q_\sigma| \leq \mu^{<\kappa}$ and for all families $\{j_i; j_i \in J_i^\sigma\}_{i\in\lambda}$ the following holds:

$$\left(W \cap \bigcap_{i\in\lambda} pr_{j_i}^{-1}\left(f_{j_i}\left(x_{j_i}\right)\right) \neq \varnothing\right) \Rightarrow \left(Q_\sigma \cap \bigcap_{i\in\lambda} pr_{j_i}^{-1}\left(f_{j_i}\left(x_{j_i}\right)\right) \neq \varnothing\right).$$

Let $\sigma = \{j_i; j_i \in J_i\}_{i\in\nu}$ be a family with $W \cap \bigcap_{i\in\lambda} pr_{j_i}^{-1}\left(f_{j_i}\left(x_{j_i}\right)\right) \neq \varnothing$.
Then $\sigma \in \Sigma$ and $Q_\sigma \cap \bigcap_{i\in\lambda} pr_{j_i}^{-1}\left(f_{j_i}\left(x_{j_i}\right)\right) \neq \varnothing$.
We define

$$Q := \{q\} \cup \bigcup_{\sigma\in\Sigma} Q_\sigma$$

and because $|Q| \leq \mu^{<\kappa}$ this is the set we were looking for. □

Theorem 2.3. *Let κ and μ be two infinite cardinals with $\mu \geq \kappa$ and let $\square_{i\in I}^\kappa X_i$ be a κ-box-product with $|I| \leq 2^\mu$ and $w(X_i) \leq \mu$ for all $i \in I$.
Then $d(W) \leq \mu^{<\kappa}$ holds for every subset $W \subseteq \prod_{i\in I} X_i$ which is a union of μ-cubes.*

Proof. Let $|I| = 2^\mu$, so we may assume that $I = 2^\mu$.
Let $\mathfrak{B}^*$ be a base of the κ-box-product $\square_{i\in\mu}^\kappa D$ of the discrete space $D = \{0; 1\}$ with $|\mathfrak{B}^*| = \mu^{<\kappa}$.
For all $i \in 2^\mu$ let $\mathfrak{B}_i$ be a base of the topology on X_i with $|\mathfrak{B}_i| = \mu$, X be a set with $|X| = \mu$, $\{f_i; f_i : X \to \mathfrak{B}_i\}_{i\in 2^\mu}$ be a family of surjective functions and $\psi : 2^\mu \to \prod_{i\in\mu} D$ be a bijection. We define

$$\begin{aligned}\Sigma := \{&\langle \{x_i\}_{i\in\lambda}\, ; \{J_i\}_{i\in\lambda}\rangle\, ; \lambda < \kappa \wedge \forall i, j \in \lambda : \\ &x_i \in X \wedge \varnothing \neq J_i \subseteq 2^\mu \wedge \psi(J_i) \in \mathfrak{B}^* \wedge (i \neq j \Rightarrow J_i \cap J_j = \varnothing)\}\end{aligned}$$

and choose for every $\sigma \in \Sigma$ a set $Q_\sigma \subseteq W$ with all the properties as stated in proposition 2.2. We define $Q := \bigcup_{\sigma\in\Sigma} Q_\sigma$. Because of $|\mathfrak{B}^*| = \mu$ we have $|\Sigma| \leq \mu^{<\kappa}$ and therefore $|Q| \leq \mu^{<\kappa}$. We will now show that Q is dense in W.

Let O be a nonempty open set in W and U an element of the canonical base $\mathfrak{B}$ of $\Box^\kappa_{i\in 2^\mu} X_i$ with $\varnothing \neq U \cap W \subseteq O$. Then there exists a set $\{j_i; i \in \lambda\} \in P_{<\kappa}(2^\mu)$ and a family $\{U_i; U_i \in \mathfrak{B}_i\}_{i\in\lambda}$ with $U = \bigcap_{i\in\lambda} pr^{-1}_{j_i}(U_i)$. We can choose for all $i \in \lambda$ pairwise disjunct open sets $B^*_i \in \mathfrak{B}^*$ with $\psi(j_i) \in B^*_i$ and $x_i \in X$ with $f_{j_i}(x_i) = U_i$.
Obviously $\sigma := \left\langle \{x_i\}_{i\in\lambda}; \{J_i\}_{i\in\lambda} \right\rangle$ is an element of Σ and we have the condition $\varnothing \neq W \cap \bigcap_{i\in\lambda} pr^{-1}_{j_i}(f_{j_i}(x_i))$, thus $Q_\sigma \cap U \neq \varnothing$
$\Rightarrow Q \cap O \supseteq O_\sigma \cap W \cap U = O_\sigma \cap U \neq \varnothing$
Therefore Q is dense in W and we have $d(W) \leq |Q| \leq \mu^{<\kappa}$. □

The following is a slight generalisation of the Hewitt-Marczewski- Pondiczery theorem:

Theorem 2.4. *Let κ and λ be two infinite cardinals with $\mu \geq \kappa$ and let $\Box^\kappa_{i\in I} X_i$ a κ-box-product with $|I| \leq 2^\mu$ and $d(X_i) \leq \mu$ for all $i \in I$.*
Then $d(\Box^\kappa_{i\in I} X_i) \leq \mu^{<\kappa}$.

Proof. Obviously there is a set D which is dense in $\Box^\kappa_{i\in I} X_i$ and $|pr_i(D)| \leq \mu$ for all $i \in I$.

Let $\Box^\kappa_{i\in I} W_i$ be the κ-box-product of discrete spaces W_i with $|W_i| = \mu$ and let $f : \prod_{i\in I} W_i \to D$ be a continuous and surjective function.
Because $\prod_{i\in I} W_i$ itself is an union of μ-cubes and due to theorem 2.3 there is a dense subset Q of W with $|Q| \leq \mu^{<\kappa}$.
Let O be a nonempty open set in $\Box^\kappa_{i\in I} X_i$. Then $D \cap O \neq \varnothing$ and $f^{-1}(D \cap O)$ is open in $\Box^\kappa_{i\in I} W_i$.
So $Q \cap f^{-1}(D \cap O) \neq \varnothing$ and $\varnothing \neq f\left(Q \cap f^{-1}(D \cap O)\right) \subseteq f(Q) \cap O$.
Therefore $f(Q)$ is dense in $\Box^\kappa_{i\in I} X_i$ and $d\left(\Box^\kappa_{i\in I} X_i\right) \leq \mu^{<\kappa}$. □

Following Wanjun Hu we define:

Definition 2.5. Let S be an infinite set, κ, λ and θ be three cardinals with $\kappa \geq \aleph_0$ and $\lambda \geq 2$. A family $\mathfrak{I} = \{\mathfrak{I}_\alpha\}_{\alpha\in\tau}$ of partitions $\mathfrak{I}_\alpha = \left\{I^\beta_\alpha; \beta \in \lambda\right\}$ of S is called a $(\kappa, \theta, \lambda)$*-generalized independent family*, if following holds:

$$\forall J \in P_{<\kappa}(\tau) \forall f : J \to \lambda : \left|\left\{\bigcap I^{f(\alpha)}_\alpha; \alpha \in J\right\}\right| \geq \theta$$

We can now apply 2.4 on this theorem and we receive the following:

Corollary 2.6. *Let κ and λ be two infinite cardinals with $\mu \geq \kappa$.*
On every set with at least $\mu^{<\kappa}$ elements exists a $(\kappa, 1, \mu)$-generalized independent family of cardinality 2^μ.

Proof. Let S be a set of cardinality $\mu^{<\kappa}$.
For every family $\{X_i\}_{i\in\mu}$ of topological spaces with $d(X_i) \leq \lambda$ the following

holds with theorem 2.4:

$$d\left(\square^{\kappa}_{i\in\mu} X_i\right) \leq |S|$$

Wanjun Hu proved in theorem 3.2 in [4] that this is equivalent to the existence of a $(\kappa, 1, \mu)$-generalized independent family of cardinality 2^{μ} on S. $\square$

ACKNOWLEDGEMENTS. *I am very grateful to Prof. Dr. Ulrich Felgner for his support and helpful advice.*

References

[1] N. Bourbaki, *Livre III: Topologie générale. Chapitre 1: Structures topologiques. Chapitre. 2: Structures uniformes.(2ième edition)*, (Hermann & Cie., Paris, 1951) p. 72.

[2] R. Engelking, *Cartesian products and dyadic spaces*, Fund. Math. **57** (1965), 287–304.

[3] E. Hewitt, *A remark on density characters*, Bull. Amer. Math. Soc. **52** (1946), 641–643.

[4] W. Hu, *Generalized independent families and dense sets of Box-Product spaces*, Appl. Gen. Topol. **7**, no. 2 (2006), 203–209.

[5] J. L. Kelley, *General Topology*, New York 1955, p. 107.

[6] E. Marczewski, *Séparabilité et multiplication cartésienne des espaces topologiques*, Fund. Math. **34** (1937), 127–143.

[7] E. S. Pondiczery, *Power problems in abstract spaces*, Duke Math. Journ. **11** (1944), 835–837.

[8] A. Tychonoff, *Über die topologische Erweiterung von Räumen*, Math. Ann. **102** (1930), 544–561.

(Received December 2008 – Accepted October 2009)

STEFAN OTTMAR ELSER *(stefan.elser@web.de)*
Mathematisches Institut, Eberhard Karls Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany