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Remarks on fixed point assertions in digital topology, 5

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ABSTRACT

As in [6, 3, 4, 5], we discuss published assertions concerning fixed points in "digital metric spaces" - assertions that are incorrect or incorrectly proven, or reduce to triviality.

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1. INTRODUCTION

As stated in [3]:

> The topic of fixed points in digital topology has drawn much attention in recent papers. The quality of discussion among these papers is uneven; while some assertions have been correct and interesting, others have been incorrect, incorrectly proven, or reducible to triviality.

Paraphrasing [3] slightly: in [6, 3, 4, 5], we have discussed many shortcomings in earlier papers and have offered corrections and improvements. We continue this work in the current paper.

Authors of many weak papers concerning fixed points in digital topology seek to obtain results in a "digital metric space" (see section 2.1 for its definition). This seems to be a bad idea. We quote [5]:

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- Nearly all correct nontrivial published assertions concerning digital metric spaces use either the adjacency of the digital image or the metric, but not both.
- If X is finite (as in a "real world" digital image) or the metric d is a common metric such as any ℓ_p metric, then (X, d) is uniformly discrete as a topological space, hence not very interesting.
- Many of the published assertions concerning digital metric spaces mimic analogues for subsets of Euclidean $\mathbb{R}^n$. Often, the authors neglect important differences between the topological space $\mathbb{R}^n$ and digital images, resulting in assertions that are incorrect, trivial, or trivial when restricted to conditions that many regard as essential. E.g., in many cases, functions that satisfy fixed point assertions must be constant or fail to be digitally continuous [6, 3, 4].

Since the publication of [5], additional papers concerning fixed points in digital metric spaces have come to our attention. This paper continues the work of [6, 3, 4, 5] in discussing shortcomings of published assertions concerning fixed points in digital metric spaces.

Many of the definitions and assertions we discuss were written with typographical and grammatical errors, and mathematical flaws. We have quoted these by using images of the originals so that the reader can see these errors as they appear in their sources (we have removed or replaced with a different style labels in equations and inequalities in the images to remove confusion with labels in our text).

2. Preliminaries

Much of the material in this section is quoted or paraphrased from [5].

We use $\mathbb{N}$ to represent the natural numbers, $\mathbb{Z}$ to represent the integers, and $\mathbb{R}$ to represent the reals.

A *digital image* is a pair (X, κ), where $X \subset \mathbb{Z}^n$ for some positive integer n, and κ is an adjacency relation on X. Thus, a digital image is a graph. In order to model the "real world," we usually take X to be finite, although there are several papers that consider infinite digital images. The points of X may be thought of as the "black points" or foreground of a binary, monochrome "digital picture," and the points of $\mathbb{Z}^n \setminus X$ as the "white points" or background of the digital picture.

For this paper, we need not specify the details of adjacencies or of digitally continuous functions.

A *fixed point* of a function $f : X \to X$ is a point $x \in X$ such that $f(x) = x$.

2.1. Digital metric spaces.

A *digital metric space* [9] is a triple (X, d, κ), where (X, κ) is a digital image and d is a metric on X. The metric is usually taken to be the Euclidean metric or some other ℓ_p metric. We are not convinced that the digital metric space is a notion worth developing. Typically, assertions

Definition: Let, (X, d, ρ) be any digital metric space and $T: (X, d, \rho) \to (X, d, \rho)$ be a self digital map. If there exists $\lambda \in \left(0, \frac{1}{2}\right)$ such that for all $x, y \in X$,

$$d\big(T(x), T(y)\big) \leq \lambda\Big(d\big(x, T(x)\big) + d\big(y, T(y)\big)\Big),$$

then T is called a Kannan digital contraction map.

Figure 1. Definition of Kannan digital contraction in [11]

Theorem: **(Kannan Contraction principle)** Let (X, d, ρ) be a complete metric space which has a usual Euclidean metric in $\mathbb{Z}^n$. Let, $T: X \to X$ be a Kannan digital contraction map. Then T has a unique fixed point, i.e. there exists a unique $c \in X$ such that $T(c) = c$.

Figure 2. Fixed point assertion for Kannan digital contractions in [11]

in the literature do not make use of both d and κ, so that "digital metric space" seems an artificial notion. E.g., for a discrete topological space X, all functions $f : X \to X$ are continuous, although on digital images, many functions $g : X \to X$ are not digitally continuous (digital continuity is defined in [2], generalizing an earlier definition [15]).

3. Assertions for contractions in [11]

The paper [11] claims fixed point theorems for several types of digital contraction functions. Serious errors in the paper are discussed below.

3.1. **Fixed point for Kannan contraction.** Figure 1 shows the definition appearing in [11] of a *Kannan digital contraction*.

Figure 2 shows a fixed point assertion for Kannan digital contractions in [11]. The "proof" of this assertion has errors discussed below.

- In the fourth and fifth lines of the "proof" is the claim that

  $$\lambda[d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2})] \leq 2\lambda d(x_n, x_{n-1}).$$

  Since $\lambda > 0$, this claim implies

  $$d(x_{n-1}, x_{n-2}) \leq d(x_n, x_{n-1}),$$

  so if any $d(x_n, x_{n-1})$ is positive, the sequence $\{x_n\}$ is not a Cauchy sequence, contrary to a claim that appears later in the argument.

Example: Let $X = [0,2]_{\mathbb{Z}}$ be a digital interval with 2-adjacency. Let, $d(x,y) = \min\{x,y\}$ be the digital metric. Consider the map $T: X \to X$ defined by

$$T(x) = \begin{cases} \frac{x}{2}, & if\ 0 \le x \le 1 \\ 1 + \frac{x}{2}, & if\ 1 < x \le 2 \end{cases}$$

It is clear that T has a unique fixed point i.e., $x = 0$.

FIGURE 3. The example of [11], pp. 10769 - 10770

***Theorem:* (A generalization of Kannan Contraction principle)** Let (X, d, ρ) be a complete metric space which has a usual Euclidean metric d in $\mathbb{Z}^n$ and let $T: X \to X$ be a digital self map. Assume that there exists a right continuous real function

$\Upsilon: [0,u] \to \left[0, \frac{u}{2}\right]$

Where u is sufficiently large real number such that

$\Upsilon(a) < \frac{a}{2}\ if\ a > 0,$ (*i*)

And let f satisfies

$d(T(x_1), T(x_2)) \le \Upsilon\left(d(x_1, T(x_1)) + d(x_2, T(x_2))\right)$ (*ii*)

For all $x_1, x_2 \in (X, d, \rho)$. Then T has a unique fixed point $c \in (X, d, \rho)$ and the sequence $T^n(x)$ converges to c for every $x \in X$.

FIGURE 4. A “theorem” of [11], p. 10770

- Towards the end of the existence argument, the authors claim that a Kannan digital contraction is digitally continuous. This assumption is contradicted by Example 4.1 of [6].

In light of these errors, we must conclude that the assertion of Figure 2 is unproven.

3.2. **Example of pp. 10769 - 10770.** This example is shown in Figure 3. One sees easily the following.

- $d(0,1) = d(0,2) = 0$, so d, contrary to the claim, is not a metric.
- $T(1) = 1/2 \notin X$.

3.3. **Fixed point for generalization of Kannan contraction.** Figure 4 shows an assertion of a fixed point result on p. 10770 of [11].

Note “And let f satisfies” should be “and let T satisfy”.

Let, (X,d,ρ) be any digital metric space and $T:(X,d,\rho)\to(X,d,\rho)$ be a self digital map. If there exists $\lambda\in(0,1)$ such that for all $x,y\in X$,

$$d(T(x),T(y))\leq \lambda\, max\left\{d(x,y),\frac{\{d(x,T(x))+d(y,T(y))\}}{2},\frac{\{d(x,T(y))+d(y,T(x))\}}{2}\right\},$$

then T is called a Zamfirescu digital contraction.

FIGURE 5. Definition 3.2 of [13], used in [11, 14].

More importantly: In the argument offered as proof of this assertion, the authors let $x_0 \in X$, and, inductively,

$$x_{n+1} = T(x_n), \qquad a_{n+1} = d(x_n, x_{n+1}).$$

They claim that by using the statements marked (i) and (ii) in Figure 4, it follows that

$$a_{n+1} = d(x_n, x_{n+1}) \leq \Upsilon(d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}))$$

$$< 2d(x_n, x_{n-1}) = 2a_n, \tag{3.1}$$

which, despite the authors' claim, does not show that the sequence $\{a_n\}$ is decreasing. However, what correctly follows from the statements marked (i) and (ii) in Figure 4 is

$$a_{n+1} = d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n)) \leq$$

$$\Upsilon(d(x_{n-1}, T(x_{n-1})) + d(x_n, T(x_n))) = \Upsilon(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))$$

$$= \Upsilon(a_n + a_{n+1}) < \frac{a_n + a_{n+1}}{2}. \tag{3.2}$$

From this we see that $a_{n+1} < a_n$, so the sequence $\{a_n\}$ is decreasing and bounded below by 0, hence tends to a limit $a \geq 0$.

The authors then claim that if $a > 0$ then $a_{n+1} \leq Y(a_n)$. However, what we showed in (3.2) does not support this conclusion, which is not justified in any obvious way. Since the authors wish to contradict the hypothesis that $a > 0$ in order to derive that the sequence $\{x_n\}$ is a Cauchy sequence, we must regard the assertion shown in Figure 4 as unproven.

3.4. **Fixed point for Zamfirescu contraction.** A *Zamfirescu digital contraction* is defined in Figure 5. This notion is used in [11, 14], and will be discussed in the current section and in section 6.

Figure 6 shows an assertion found on p. 10770 of [11].

The argument offered as "proof" of this assertion considers cases. For an arbitrary $x_0 \in X$, a sequence is inductively defined via $x_{n+1} = T(x_n)$. For

Theorem: **(Zamfirescu Contraction principle)** Let (X, d, ρ) be a complete metric space which has a usual Euclidean metric in $\mathbb{Z}^n$. Let, $f: X \to X$ be a Zamfirescu digital contraction. Then T has a unique fixed point, i.e. there exists a unique $c \in X$ such that $T(c) = c$.

FIGURE 6. Another "theorem" of [11], p. 10770

convenience, let us define

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

The argument considers several cases.

- Case 1 says if $d(x_{n+1}, x_n) = M(x_{n+1}, x_n)$ then, by implied induction,
  $$d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0).$$
  But this argument is based on the unproven assumption that this is also the case for all indices $i < n$; i.e., that $d(x_{i+1}, x_i) = M(x_{i+1}, x_i)$.
- Case 2 says if
  $$\frac{d(x_{n+1}, Tx_{n+1}) + d(x_n, Tx_n)}{2} =$$
  $$\max\left\{\begin{array}{l} d(x_{n+1}, x_n), \frac{d(x_{n+1}, Tx_{n+1})+d(x_n, Tx_n)}{2}, \\ \frac{d(x_{n+1}, Tx_n)+d(x_n, Tx_{n_1})}{2} \end{array}\right\},$$
  then
  $$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \lambda \frac{d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2})}{2} \leq$$
  $$\lambda d(x_n, x_{n-1}).$$
  But in order for the second inequality in this chain to be true, we would need $d(x_{n-1}, x_{n-2}) \leq d(x_n, x_{n-1})$, and no reason is given to believe the latter.
- Case 3 says that if $\frac{d(x_{n+1}, Tx_n)+d(x_n, Tx_{n+1})}{2} = M(x_{n+1}, x_n)$ then
  $$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \lambda \frac{d(x_n, x_{n-2}) + d(x_{n-1}, x_{n-1})}{2}.$$
  The correct upper bound, according to the definition shown in Figure 5, is
  $$\lambda \frac{d(x_{n+1}, Tx_n) + d(x_n, Tx_{n+1})}{2} = \lambda \frac{d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2})}{2} =$$
  $$\lambda \frac{d(x_n, x_{n+2})}{2}.$$

Definition: Let, (X, d, ρ) be any digital metric space and $T: (X, d, \rho) \to (X, d, \rho)$ be a self digital map. If there exists $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$d(T(x), T(y)) \leq \lambda \, max \left\{ d(x,y), \frac{\{d(x, T(x)) + d(y, T(y))\}}{2}, d(x, T(y)), d(y, T(x)) \right\},$$

then T is called a Rhoades digital contraction.

FIGURE 7. Definition of Rhoades digital contraction, [11], p. 10769

Theorem: **(Rhoades Contraction principle)** Let (X, d, ρ) be a complete metric space which has a usual Euclidean metric in $\mathbb{Z}^n$. Let, $T: X \to X$ be a Rhoades digital contraction map. Then T has a unique fixed point, i.e. there exists a unique $c \in X$ such that $T(c) = c$.

FIGURE 8. Fixed point assertion for Rhoades digital contraction, [11], p. 10771

Further, the conclusion reached by the authors for this case, that the distances $d(x_{n+1}, x_n)$ are bounded above by an expression that tends to 0 as $n \to \infty$, depends on the unproven hypothesis that an analog of this case holds for all indices $i < n$.

Thus all three cases considered by the authors are handled incorrectly. We must conclude that the assertion of Figure 6 is unproven.

3.5. **Fixed point for Rhoades contraction.** Figure 7 shows the definition appearing in [11] of a *Rhoades digital contraction*. The paper [11] claims the fixed point result shown in Figure 8 for such functions. The argument offered in "proof" of this assertion has errors that are discussed below.

For convenience, let

$$M(x,y) = \max \left\{ d(x,y), \ \frac{d(x,Tx) + d(y,Ty)}{2}, \ d(x,Ty), \ d(y,Tx) \right\}.$$

The authors' argument considers cases corresponding to which of the embraced expressions above gives the value of $M(x_{n+1}, x_n)$. In each case, the authors assume without proof that the same case is valid for $M(x_{i+1}, x_i)$, for all indices $i < n$.

Additional errors:

- In case 2, the inequality

$$d(Tx_n, Tx_{n-1}) \leq \lambda \frac{d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2})}{2}$$

should be, according to Figure 7,

$$d(x_n, x_{n+1}) = d(Tx_n, Tx_{n-1}) \leq \lambda \frac{d(x_n, Tx_n) + d(x_{n+1}, Tx_{n-1})}{2}$$

$$= \lambda \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_n)}{2} = \lambda d(x_n, x_{n+1}).$$

Note this implies

$$x_n = x_{n+1}, \tag{3.3}$$

which would imply the existence of a fixed point. Also, the authors claim that

$$\lambda \frac{d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2})}{2} \leq \lambda d(x_n, x_{n-1}),$$

which is equivalent to

$$d(x_{n-1}, x_{n-2}) \leq d(x_n, x_{n-1}).$$

No reason is given in support of the latter; further, it undermines the later claim that $\{x_n\}$ is a Cauchy sequence, since the authors did not deduce (3.3).

- In case 3, it is claimed that

$$\lambda d(x_n, Tx_{n-1}) \leq \lambda d(x_n, x_{n-1}).$$

This should be corrected to

$$\lambda d(x_n, Tx_{n-1}) = \lambda d(x_n, x_n) = 0,$$

which would guarantee a fixed point.

- In case 4, we see the claim

$$d(x_{n-1}, Tx_n) = d(x_{n-1}, x_{n-1}) = 0.$$

This should be corrected to

$$d(x_{n-1}, Tx_n) = d(x_{n-1}, x_{n+1}).$$

In view of these errors, we must regard the assertion shown in Figure 8 as unproven.

4. Assertions for weakly compatible maps in [1]

In this section, we show that the assertions of [1] are trivial or incorrect.

4.1. Theorem 3.1 of [1].

Definition 4.1. [8] Let $S, T : X \to X$. Then S and T are *weakly compatible* or *coincidentally commuting* if, for every $x \in X$ such that $S(x) = T(x)$ we have $S(T(x)) = T(S(x))$.

Let (X, d, ρ) is a complete digital metric space, let N be a nonempty closed subset of X. Let $P, Q : N \to N$ and $G, H : N \to X$ be mappings satisfying $Q(N) \subset H(N)$ and for every $x, y \in X$,

$$\Psi(d(Px, Qy)) \leq \varphi\left(d_{G,H}(x,y)\right) - \frac{1}{2}\Psi\left(d_{G,H}(x,y)\right) - \varphi\left(d_{G,H}(x,y)\right)$$

Where $\Psi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\Psi(\rho) = 0$ if and only if $\rho = 0$. $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function such that $\Psi(\rho) = 0$ if and only if $\rho = 0$, and

$$d_{G,H}(x,y) = \max\left\{(Gx, Hy), (Gx, Px), (Hy, Qy), \frac{1}{2}\left((Gx, Qy) + (Hy, Px)\right)\right\}$$

If one of $P(N), Q(N), G(N), HN$ is a closed subset of X, then $\{P, G\}$ and $\{Q, H\}$ have a unique point of coincidence in X. Moreover, if $\{P, G\}$ and $\{Q, H\}$ are weakly compatible, then P, Q, G and H have a unique common fixed point in X.

Figure 9. The assertion stated as Theorem 3.1 of [1]

The assertion stated as Theorem 3.1 in [1] is shown in Figure 9. In this section, we show the assertion is false except in a trivial case.

Note if d is any ℓ_p metric (including the usual Euclidean metric) then the requirements of closed subsets of X are automatically satisfied, since (X, d) is a discrete space.

Theorem 4.1. If functions G, H, P, Q satisfy the hypotheses of Theorem 3.1 of [1] then $G = H = P = Q$. Therefore, each of the pairs (P, G) and (Q, H) has a unique common point of coincidence if and only if N consists of a single point.

Proof. We observe the following.

- The inequality in the assertion simplifies as
$$\Psi(d(Px, Qy)) \leq -\frac{1}{2}\Psi(d_{G,H}(x,y)).$$
Since the function Ψ is non-negative, we have
$$\Psi(d(Px, Qy)) = \Psi(d_{G,H}(x,y)) = 0. \tag{4.1}$$
Therefore, we have
$$P(x) = Q(y) \text{ for all } x, y \in N. \tag{4.2}$$
- In the equation for $d_{G,H}$ in Figure 9, the pairs of points listed on the right side should be understood as having d applied, i.e.,
$$d_{G,H}(x,y) = \max\left\{\begin{array}{c} d(G(x), H(y)), d(G(x), P(x)), d(H(y), Q(y)), \\ \frac{1}{2}[d(G(x), Q(y)) + d(H(y), P(x))] \end{array}\right\}. \tag{4.3}$$
Since $\Psi(x) = 0$ if and only if $x = 0$, we have from (4.1) that $d_{G,H}(x,y) = 0$, so (4.2) and (4.3) imply
$$G(x) = P(x) = Q(x) = H(x) \text{ for all } x \in N.$$

Let (X, d, ρ) is a complete digital metric space, let $X = [4, 40]$ and d be the usual metric on X. Define $P, Q, G, H: X \to X$ as follows: $PX = 4\ for\ each\ X$;

$GX = X\ if\ x \leq 16, and$ $\quad GX = 16\ if\ 16 < x < 22,$ $\quad GX = \frac{x+18}{5}\ if\ 16 \leq x \leq$

$HX = 4\ if\ x = 4\ or\ 12\ and$ $\quad GX = \frac{X+15}{5}\ if\ x > 25;$ $\quad 25$ $\quad HX = 17 + X\ if\ 13 \leq x \leq 14$

$QX = 4\ if x < 8\ or\ x > 12,$ $\quad HX = 24 + X\ if\ 4 < x < 8,$ $\quad QX = 4 + x\ if\ 14 \leq x \leq 15.$

$HX = 16\ if\ 13 \leq x \leq 14;$

Then $P, Q, G\ and\ H$ satisfy all the conditions of the above theorem and have a unique common fixed point $x = 4$. being compatible mappings, all $P, Q, G\ and\ H$ are weakly compatible mappings.

Figure 10. The assertion stated as Example 3.1 of [1]

We conclude that (P, G) and (Q, H) are pairs of functions whose respective common points of coincidence are unique if and only if N consists of a single point. □

4.2. **Example 3.1 of [1].** In Figure 10, we see the assertion presented as Example 3.1 of [1].

Note the following.

- If "$X = [4, 40]$" is meant to mean the real interval from 4 to 40, then X is not a digital image, as all coordinates of members of a digital image must be integers. Perhaps X is meant to be the digital interval $[4, 40]_{\mathbb{Z}} = [4, 40] \cap \mathbb{Z}$.
- The function G is not defined on all of $[4, 40]_{\mathbb{Z}}$, appears to be doubly defined for some values of x, (notice the incomplete inequality at the end of the 3rd line) and is not restricted to integer values.
- The function H is not defined on all of $[4, 40]_{\mathbb{Z}}$ and is doubly defined for $x \in \{13, 14\}$.
- The function Q is not defined for $x \in \{9, 10, 11, 12\}$, and is doubly defined for $x \in \{14, 15\}$.
- The "above theorem" referenced in the last sentence of Figure 10 is the assertion discredited by our Theorem 4.1, which shows that $P = Q = G = H$. Clearly, the assertion shown in Figure 10 fails to satisfy the latter.

Thus, Example 3.1 of [1] is not useful.

4.3. **Corollary 3.2 of [1].** The assertion presented as Corollary 3.2 of [1] is presented in Figure 11. Notice that "weakly mappings" is an undefined phrase. No proof is given in [1] for this assertion, and it is not clear how the assertion might follow from previous assertions of the paper (which, as we have seen above, are also flawed).

Corollary. 3.2. Let P *and* Q be weakly mappings of a complete digital metric space (X, d, ρ) into itself. Suppose $P(X) \subset Q(X)$. if there exists $\alpha \in (0,1)$ and a positive integer k such that $d\big(P^k(x), P^k(y)\big) \leq \alpha d(Q(x), Q(y))$ for all x and y in X, then P and Q have a unique common fixed point.

Figure 11. The assertion stated as Corollary 3.2 of [1]

Theorem 4: Let (L, b, a) be a digital metric space with coefficient $k \geq 1$ and $f, g: D \to L$ the mapping such that

i. $b(fx_x, fy) \geq h.\min\{b(gx, gy), b(fy, gy), \frac{1}{2k}[b(gx, fy) + b(gy, fx)]\}$ where for all $x, y \in Xkh < 1$

ii. Either $f(D)$ or $g(D)$ is complete then f and g have a coincidence point and hence fixed pint.

Figure 12. The assertion presented as Theorem 4 of [12]

Perhaps "weakly mappings" is intended to be "weakly compatible mappings". At any rate, by labeling this assertion as a Corollary, the authors suggest that it follows from the paper's flawed "Theorem" 3.1.

The assertion presented as "Corollary" 3.2 of [1] must be regarded as undefined and unproven.

5. Assertion for coincidence and fixed points in [12]

Figure 12 shows the assertion presented as Theorem 4 of [12]. The assertion as stated is false. Flaws in this assertion include:

- "D" apparently should be "L", and "x_x" apparently should be "x".
- No restriction is stated for the value of h. Therefore, we can take $h = 0$, leaving the inequality in i) as $b(fx, fy) \geq 0$; since b is a metric, this inequality is not a restriction. Thus f and g are arbitrary; they need not have a coincidence point or fixed points.

6. Assertion for Zamfirescu contractions in [14]

Let $f : X \to X$, where (X, d, κ) is a digital metric space. Recall that a *Zamfirescu digital contraction* [13] is defined in Figure 5.

We show the assertion presented as Theorem 4.1 of [14] in Figure 13.

Observe:

- The symbol θ has distinct uses in this assertion. In the first line, θ is introduced as both the metric and the adjacency of X. Since our discussion below does not use an adjacency, we will assume θ is the metric d.

Theorem 4.1 (Zamfirescu Contraction). *Let (B, θ, θ) be a complete digital metric space and* $\mathrm{T} : B \to B$ *be an injective mapping satisfying the condition:*

1. $\theta(Tu, Tv) \leq \alpha\theta(u, v)$
2. $\theta(Tu, Tv) \leq \beta[\theta(u, Tu) + \theta(v, Tv)]$
3. $\theta(Tu, Tv) \leq \gamma[\theta(u, Tv) + \theta(v, Tv)]$

$$\theta(Tu, Tv) < \varphi \operatorname{Max}\left\{d(u, v), \frac{d(u, Tu) + d(v, Tv)}{2}, \frac{d(u, Tv) + d(v, Tu)}{2}\right\} \qquad (4.1)$$

for all $\theta \in [0, 1); \alpha, \beta, \gamma > 0; u, v \in B$ *and* $u \neq v$, *have a fixed point if* $\alpha + \beta + \gamma < 1$ *and moreover a unique point if* $\alpha + \gamma < 1$.

Figure 13. The assertion presented as Theorem 4.1 of [14]

- The symbol φ seems intended to be a real number satisfying some restriction, but no restriction is stated. Alternately, it may be that φ is intended to be a function to be applied to the Max value in the statement, but no description of such a function appears.

Perhaps most important, we have the following.

Theorem 6.1. If Y has more than one point and d is any ℓ_p metric, then no function T satisfies the hypotheses shown in Figure 13.

Proof. Suppose there is such a function T. By choice of d, there exist $u_0, v_0 \in X$ such that
$$d(u_0, v_0) = \min\{d(x, y) \mid x, y \in X, x \neq y\}.$$
By the inequality stated in item 1 of Figure 13, $d(Tu_0, Tv_0) = 0$. This contradicts the hypothesis that T is injective. □

7. Assertion for expansive map in [7]

The paper [7] claims to have a fixed point theorem for digital metric spaces. However, it is not clear what the authors intend to assert, as the paper has many undefined and unreferenced terms and many obscuring typographical errors.

The assertion stated as "Preposition" 2.7 of [7] (and also as an unlabeled Proposition on p. 10769 of [11]) was shown in Example 4.1 of [6] to be false.

The definition of what this paper calls a *Generalized* $(\alpha - \phi)$ - ψ *expansive mapping for random variable* is shown in Figure 14. We observe the following.

- This is not the same as a $\beta - \psi - \phi$ *expansive mapping* defined in [10].
- Notice the ρ that appears intended to be the adjacency of the digital metric space. This is significant in our discussion later.
- The set Ψ is not defined anywhere in the paper. Perhaps it is meant to be the set Ψ of [10].
- The functions d, α, and M all have a third parameter a that appears to be extraneous, since each of these functions is used later with only two parameters.

Definition 3.1: Let (Y, d, ρ) is a DMS, and S:Y → Y be a given mapping. ξ is a measerable selector, we say that S is known as Generalised $(\alpha - \varphi) - \psi-$expansive mapping for random variable, ∃

$\alpha: Y \times Y \to [0, \infty)$ & $\varphi \in \Psi$

we have $(\alpha - \varphi)(d(S\xi u, S\xi v, a)) \geq \alpha(\xi u, \xi v, a) M(\xi u, \xi v, a)$

Where

$$M(\xi u, \xi v, a) = \mu\left\{ d(\xi u, \xi v, a), \frac{d(\xi u, S\xi u, a) + d(\xi v, S\xi v, a)}{2}, \frac{d(\xi u, S\xi v, a) + d(\xi v, S\xi u, a)}{2} \right\} + \omega d(\xi u, \xi v, a) \qquad (3a)$$

For every $\xi u, \xi v \in Y$. $\mu + \omega \geqslant 0$ and there exist a fixed point further if $\mu + \omega < 1$.

Figure 14. The statement presented as Definition 3.1 of [7]

Theorem 3.3 If $(Y, d\rho)$ is a DGMS which is complete, $S : Y \to Y$ is a one- one and onto, generalized $(\alpha - \varphi) - \psi-$expansive mapping, ξ is a measerable *as prilimnary* 2:

1. S^{-1} is a *AAMS*
2. There exist$\xi u_{0,} \in$ Y such that $\alpha(\xi u_{0,} S^{-1} \xi u_{0,}) \geq 1$.
3. S is digital continuous.

Then *u*is an exclusive RFP

Figure 15. The assertion presented as Theorem 3.3 of [7]

- One supposes μ and ω must be non-negative, but this is not stated.

The assertion presented as Theorem 3.3 of [7] is shown in Figure 15.
In the statement of this assertion:

- "DGMS" appears, although it is not defined anywhere in the paper. Perhaps it represents "digital metric space".
- The term "exclusive RFP" is not defined anywhere in the paper. One supposes the "FP" is for "fixed point".

In the argument offered as "Verification" of this assumption, we note the following.

- The second line of the verification contains an undefined operator, $\overset{+}{n}$ which perhaps is meant to be +.
- The same line contains part of the phrase "u_n is a unique point of S." What the authors intend by this is unclear.
- At the start of the long statement (3e), it is claimed that $M(\xi u_n, \xi u_{n+1})$ is the maximum of three expressions. The second term of the expression

for $M(\xi u_n, \xi u_{n+1})$ applies ρ to a numeric expression. This makes no sense, since ρ is the adjacency of Y (see Figure 14). Notice also that Figure 14 shows no such term in its expression for the function M. The use of ρ, as a numeric value that has neither been defined nor restricted to some range of values, propagates through both of the cases considered.

- In the expression for $M(\xi u_n, \xi u_{n+1})$, the third term, $\omega(\xi u_n, \xi u_{n-1})$, should be $\omega d(\xi u_n, \xi u_{n-1})$ according to Figure 14. This error repeats several times in statement (3e).

Other errors are present, but we have established enough to conclude that whatever the authors were trying to prove is unproven.

8. Further remarks

We have shown that nearly every assertion introduced in the papers [11, 1, 12, 14, 7] is incorrect, unproven due to errors in the "proofs," or trivial. These papers are part of a larger body of highly flawed publications devoted to fixed point assertions in digital metric spaces, and emphasize our contention that the digital metric space is not a worthy subject of study.

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