Applied General Topology
© Universidad Politécnica de Valencia
Volume 10, No. 2, 2009
pp. 221-225

On an algebraic version of Tamano's theorem

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Abstract. Let X be a non-paracompact subspace of a linearly ordered topological space. We prove, in particular, that if a Hausdorff topological group G contains closed copies of X and a Hausdorff compactification bX of X then G is not normal. The theorem also holds in the class of monotonically normal spaces.

2000 AMS Classification: 54H11,22A05, 54F05

Keywords: topological group, Hausdorff compactification, normal space, stationary set

1. Introduction.

This note is devoted to analysis of Tamano's characterization [5] of paracompactness in the context of Hausdorff topological groups. The Tamano's argument implies that if a Tychonov space X is not paracompact then $X \times bX$ is not normal for every Hausdorff compactification bX of X. A natural algebraic analysis of this statement leads to the following conjecture:

Conjecture. *Let X be a non-paracompact topological space and bX a Hausdorff compactification of X. If a topological group G contains closed copies of X and bX then G is not normal.*

We believe that the conjecture has a good chance for a positive resolution. In this note we give a proof of this conjecture in the class of generalized ordered spaces (="subspaces of linearly ordered spaces"), or more generally, in the class of monotonically normal spaces. Since Tamano's theorem is a criterion it would be natural to ask if given a paracompact space X one can find a Hausdorff compactification bX and a normal group G such that G contains closed copies of X and bX. The author does not know if such G and bX exist without additional requirements on X besides paracompactness. It is worth to mention, however, that if X^n is Lindelöf for every $n \in \omega$ then the free group $F(X \oplus bX)$ over $X \oplus bX$ is normal (even Lindelöf) and contains closed copies of

X and bX. This fact is well-known and mentioned, in particular, in the recent survey [4]. Also, we would like to mention that in [2] the author proved that if a group G contains closed copies of an uncountable regular cardinal τ and $\tau+1$, then G contains a closed copy of $\tau\times(\tau+1)$, which makes G not normal. While some of the ideas of this result could be used to prove the main theorem of this note we use a different approach, which may be helpful in proving the general conjecture.

All spaces in this note are assumed to be Tychonov. By βX we denote the Čech-Stone compactification of X. The symbol $\star$ is reserved for group operation. A subspace of a linearly ordered topological space will be called a *GO-space*. A point $x\in X$ is a *complete accumulation point* for an infinite set $A\subset X$ if every open neighborhood of x meets A by a subset of cardinality $|A|$.

To prove our main result, we start with four folklore statements, two of which are left without proof.

Fact 1. *Let S be a stationary subset of an uncountable regular cardinal τ and bS a Hausdorff compactification of S. Then there exists a unique point $p\in bS\setminus S$ that is a complete accumulation point for S. Moreover, $S\cup\{p\}$ is naturally homeomorphic to $S\cup\{\tau\}$.*

The point p in the above fact will be always identified with τ.

Fact 2. *Let S be a stationary subset of an uncountable regular cardinal τ, bS a Hausdorff compactification of S, and $c(S\times bS)$ a Hausdorff compactification of $S\times bS$. Then there exists a unique point $p\in c(S\times bS)$ that is the only common and only complete accumulation point for $S\times\{\tau\}$ and for $\{(\alpha,\alpha):\alpha\in S\}$.*

The point p in Fact 2 will be always identified with (τ,τ).

Lemma 1.1. *Let S be a stationary subset of an uncountable regular cardinal τ. Let τ be a limit point for $A\subset\beta S\setminus(S\cup\{\tau\})$ in βS. Then $Cl_{\beta S}(A)\cap S$ is closed and unbounded in S.*

Proof. Let f be the continuous map from βS to $\tau+1$ that is the identity on S. Since τ is the only complete accumulation point for S in βS, $f(A)\subset\tau$. Since τ is a limit point for A, $f(A)$ is unbounded in τ.

Assume the conclusion of Lemma is false. Then we may also assume that $Cl_{\beta S}(A)\cap S=\varnothing$. Since f maps the remainder of S in βS to the remainder of S in $\tau+1$, we have $f(Cl_{\beta S}(A)\setminus\{\tau\})$ is a closed unbounded subset of τ that does not meet S. This contradicts stationarity of S in τ. □

Lemma 1.2. *Let S be a stationary subset of an uncountable regular cardinal τ. If $f:S\to\tau$ is continuous and unbounded then there exists $\lambda\in S$ such that $f(\lambda)=\lambda$.*

Proof. Since f is unbounded and τ is regular we can select $X = \{x_\beta : \beta < \tau\}$ such that

(1) $x_\alpha > \max\{x_\beta, f(x_\beta)\}$ if $\alpha > \beta$;
(2) $f(x_\alpha) > \max\{x_\beta, f(x_\beta)\}$ if $\alpha > \beta$.

Observe that property 1 and regularity of τ imply that X is unbounded in τ. Since S is stationary there exist $\lambda \in S$ and limit $\alpha \in \tau$ such that λ is limit for $\{x_\beta : \beta < \alpha\}$ and $x_\beta < \lambda$ for all $\beta < \alpha$. By 1 and 2 and continuity of f, we have $f(\lambda) = \lambda$. □

For our main result we need the following fundamental theorem.

Theorem (R. Engelking and D. Lutzer [3]). *A GO-space X is paracompact iff no closed subspace of X is homeomorphic to a stationary subset of a regular uncountable cardinal.*

Theorem 1.3. *Let L be a non-paracompact GO-space and bL a Hausdorff compactification of L. If a topological group G contains closed copies of L and bL, then G is not normal.*

Proof. We may assume that L is a closed subset of G and $bL' \subset G$ is a copy of bL, where L' is a copy of L with a fixed homeomorphism $x \leftrightarrow x'$.

Let S be a closed subset of L that is homeomorphic to a stationary subset of an uncountable regular cardinal τ. Such an S exists due to Theorem's hypothesis and Engelking-Lutzer theorem.

As agreed earlier, by τ we denote the only complete accumulation point for S in any Hausdorff compactification and by τ' the only complete accumulation point for S' in bL'.

Let $H = S \times \{\tau'\}$ and $D = \{(\alpha, \alpha') : \alpha \in S\}$. The sets H and D are closed in $S \times bS'$ and not functionally separated. Let $H_G = \star(H) = \{\alpha \star \tau' : \alpha \in S\}$ and $D_G = \star(D) = \{\alpha \star \alpha' : \alpha \in S\}$.

Claim 1: *$\tilde{\star}(\tau, \tau') \notin G$, where $\tilde{\star}$ is the continuous extension of $\star$ over the Čech-Stone compactification.*
To prove the claim, observe that H_G is a closed subset of G homeomorphic to S. This is because multiplication by a constant is a continuous automorphism. By Fact 2 and Fact 1, (τ, τ') is the only complete accumulation point for H in $\beta(G \times G)$). The set H_G does not have a complete accumulation point in G. Therefore $\tilde{\star}(\tau, \tau') \notin G$. The claim is proved.

Put $H^\alpha = \{(\beta, \tau') : \beta \geq \alpha, \beta \in S\}$, $H_G^\alpha = \{\beta \star \tau' : \beta \geq \alpha, \beta \in S\}$, $D^\alpha = \{(\beta, \beta') : \beta \geq \alpha, \beta \in S\}$, and $D_G^\alpha = \{\beta \star \beta' : \beta \geq \alpha, \beta \in S\}$.

Claim 2: *There exists $\lambda < \tau$ such that $H_G^\lambda \cap Cl_G(D_G^\lambda) = \varnothing$.*
To prove the claim assume the contrary. Then for any $\alpha < \tau$ there exists $p_\alpha \in Cl_{\beta(G\times G)}(D^\alpha)$ such that $\tilde{\star}(p_\alpha) \in H_G^\alpha$. By Lemma 1.1, $Cl_{\beta(G\times G)}\{p_\alpha : \alpha < \tau\}$ meets D by a closed subset T of cardinality τ. Since H_G is closed and $\star$ is continuous we have $\star(T) \subset H_G$. Since $|T| = \tau$ we have $\tilde{\star}(\tau, \tau')$ is a complete accumulation point for $\star(T)$ in βG. By Lemma 1.2, there exists $(\gamma, \gamma') \in T$ such that $\star(\gamma, \gamma') = \gamma \star \gamma' = \gamma \star \tau'$. Therefore, $\gamma' = \tau'$, contradicting to the fact that $(\tau, \tau') \notin T$. The claim is proved.

By Claim 2, H_G^λ and $Cl_G(D_G^\lambda)$ are closed and disjoint in G. If G were normal, then H_G^λ and $Cl_G(D_G^\lambda)$ would have been functionally separated and so would H^λ and D^λ in $G \times G$. But H^λ and D^λ are not functionally separated for every $\lambda < \tau$. □

Observe that the proof of the theorem uses only one property of L, namely, the fact that L contains a closed copy of a stationary subset of an uncountable regular cardinal τ. Since the theorem of Engelking and Lutzer holds for monotonically normal spaces as well (proved by Balogh and Rudin [1]) we have the following.

Theorem 1.4. *Let X be non-paracompact and monotonically normal and bX a Hausdorff compactification of X. If a topological group G contains closed copies of X and bX, then G is not normal.*

We would like to finish the paper with two questions (which may have been asked before by other authors) related to the discussion in the beginning of this work.

Question 1. *Is there a paracompact space that cannot be embedded in a normal group as a closed subspace?*

Question 2. *Let X^n be paracompact for every $n \in \omega$. Is $F(X)$ normal?*

References

[1] Z. Balogh and M. E. Rudin, *Monotone normality*, Topology Appl. **47** (1992), no. 2, 115–127.
[2] R. Z. Buzyakova, *Ordinals in topological groups*, Fund. Math. **196** (2007), no. 2, 127–138.
[3] D. J. Lutzer, *Ordered topological spaces*, Surveys in general topology, pp. 247–295, Academic Press, New York-London-Toronto, Ont., 1980.
[4] O. V. Sipachëva, *The topology of free topological group*, Fundam. Prikl. Mat. **9** (2003), no. 2, 99–204; English translation in J. Math. Sci. (N. Y.) **131** (2005), no. 4, 5765–5838.
[5] H. Tamano, *On paracompactness*, Pacific J. Math. **10** (1960) 1043–1047.

Received April 2008

Accepted February 2009

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