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Lifting dynamical properties to hyperspaces and hyperspace suspension

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Abstract

For a dynamical system (X, f), the passage of various dynamical properties such as transitivity, total transitivity, weakly mixing, mixing, topological exactness, topological conjugacy, to the hyperspace C(X) of X consisting of nonempty closed connected subsets of X, and also to the hyperspace suspension HS(X) of X, have been considered and studied.

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1. Introduction

Throughout the paper, (X, f) denotes a dynamical system, where X is a compact metric space and f is a selfmap on X. By a map, we mean a continuous map. The symbol $\mathbb{N}$ denotes the set of positive integers.

Let (X, f) be a dynamical system. Then f is called (i) *transitive* if for a pair of nonempty open sets U, V of X, there exists an $n \in \mathbb{N}$, such that $f^n(U) \cap V \neq \phi$, (ii) *totally transitive* if the map f^k is transitive, for each $k \in \mathbb{N}$, (iii) *weakly mixing* if for pairs of nonempty open sets U_1, U_2 and V_1, V_2 of X, there exists an $n \in \mathbb{N}$, such that $f^n(U_i) \cap V_i \neq \phi$, for $i = 1, 2$, (iv) *mixing* if for a pair of nonempty open sets U, V of X, there exists an $N \in \mathbb{N}$, such that $f^n(U) \cap V \neq \phi$, $n \geq N$, (v) *topologically exact* if for an open set U of X, there exists an $n \in \mathbb{N}$, such that $f^n(U) = X$. It is known that, if f is weakly

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mixing and $U_1, U_2, \ldots, U_n,\ V_1, V_2, \ldots, V_n$ are nonempty open sets of X, then there exists a $k \geq 1$ such that $f^k(U_i) \cap V_i \neq \phi$, for $i = 1, 2, \ldots, n$ [12]. The dynamical system (X, f) is said to possess a *dynamical property* P, if f has P.

Two dynamical systems (X, f) and (Y, g) are said to be *topologically conjugate* if there exists a homeomorphism φ from X to Y such that $g \circ \varphi = \varphi \circ f$.

By 2^X, we denote the set consisting of all nonempty closed subsets of the space X. On it,

(i) the *upper Vietoris topology* τ_U, has $\mathcal{B}_U \equiv \{< U_1, \ldots, U_m >:\ U_1, \ldots, U_m$ are open sets of $X\}$, where $< U_1, \ldots, U_m > = \ \{A \in 2^X : A \subseteq \bigcup_{i=1}^m U_i\}$, as a basis,

(ii) the *lower Vietoris topology* τ_L, has $\mathcal{B}_L \equiv \{< U'_1, \ldots, U'_m >:\ U'_1, \ldots, U'_m$ are open sets of $X\}$, where $< U'_1, \ldots, U'_m > = \ \{A \in 2^X : A \cap U'_i \neq \phi,\ i = 1, \ldots, m\}$, as a basis, and

(iii) the *Vietoris topology* τ, has $\mathcal{B} \equiv \{< V_1, \ldots, V_n >:\ V_1, \ldots, V_n$ are open sets of $X\}$, where $< V_1, \ldots, V_n >= \{A \in 2^X : A \subseteq \bigcup_{i=1}^n V_i,\ A \cap V_i \neq \phi,\ i = 1, \ldots, n\}$, as a basis.

The hyperspace $C(X)$, denotes the subspace of 2^X consisting of nonempty closed connected subsets of X. We shall write $C(X)$ to mean $(C(X), \tau)$. The upper and lower Vietoris topologies on $C(X)$ are specified by writing $(C(X), \tau_U)$ and $(C(X), \tau_L)$, respectively. For nonempty open sets $U_1, \ldots, U_n$ of X and $U = U_i$, for some $i \leq n$, when $m \geq 1$, we may write, as in [1], $< U_1, \ldots, U_n >$ alternatively in the form $< U_1, \ldots, U_n, \underbrace{U, \ldots, U}_{m} >$. For a nonempty basic open set $\mathcal{U} = < U_1, \ldots, U_m >$ of $C(X)$, by $\bigcup \mathcal{U}$ we mean $\bigcup_{i=1}^m U_i$.

Observe that X is embedded into $C(X)$ by the embedding $x \mapsto \{x\}$. Also, every continuous selfmap f on X, induces in a natural way, a selfmap f' on 2^X [2]. The hyperspace $C(X)$ is invariant with respect to f', and hence we obtain a dynamical system $(C(X), \bar{f})$, where $\bar{f} = f'|_{C(X)}$.

For $k \in \mathbb{N}$, the space $F_k(X)$, denotes the collection of those nonempty closed subsets of X that consist of atmost k elements. The set $F_1(X)$ consisting of all singletons in X, is a closed subspace of $C(X)$, and the quotient space $C(X)/F_1(X)$, denoted by $HS(X)$, is known as the *hyperspace suspension* of X [10]. Further, for a dynamical system (X, f), the pair $(HS(X),\ HS(f))$ constitutes a dynamical system with the evolution map $HS(f)$ sending $[A] \in HS(X)$ to $(q_X \circ f \circ q_X^{-1})([A])$, where q_X denotes the cannonical projection from $C(X)$ to $HS(X)$ [5]. That $HS(f)$ is continuous follows from [4, 4.3, p. 146]. It is pertinent to mention that HS taking X to $HS(X)$ and f to $HS(f)$ describes a covariant functor from the category of topological spaces to itself.

A brief exposition of our work is as follows. If (X, f) is transitive, then $\big((C(X), \tau_U), \bar{f}\big)$ is transitive. However, $\big((C(X), \tau_L), \bar{f}\big)$ is transitive if X is pathconnected and f is weakly mixing. The same is the state of affairs so far as the total transitivity and weakly mixing are concerned. It is known that the transitivity and total transitivity do not necessarily lift to $C(X)$ endowed with the Vietoris topology. For a dynamical system (X, f) with f mixing, it is obtained that $\bar{f}$ on $(C(X), \tau_L)$ continues to be mixing provided X is pathconnected. All these constitute the contents of Section 3. Section 4 is devoted to the passage of these dynamical properties to the dynamical system $(HS(X), HS(f))$ from the dynamical system (X, f). It has been obtained that all these properties get lifted smoothly. Finally, it is obtained that topological conjugacy is preserved under the hyperspace suspension.

We begin with the preliminaries in Section 2, wherein, certain results scattered across various references are stated that we subsequently use.

2. Preliminaries

A *topological graph* is a connected compact Hausdorff space G for which there exists a finite collection of subspaces $I_i, i = 1, 2, \ldots, s$ such that $G = \bigcup_{i=1}^{s} I_i$, where each I_i is homeomorphic to a compact interval, and each intersection $I_i \cap I_j$, for $i \neq j$ is finite. Equivalently, a graph is a one-dimensional connected, compact polyhedron. Below we state a result related to the lift of transitivity to the hyperspace $C(G)$ of a dynamical system (G, f), where G is a graph and f is transitive.

Proposition 2.1 ([9, Cor. 29]). *For a dynamical system* (G, f)*, the map* $\bar{f}$ *is never transitive.*

Remark 2.2. Note that the map $\bar{f}$ has been denoted by $\tilde{f}$ in [9].

Denoting by $\hat{f}$ the map induced on $K(X)$ consisting of all nonempty compact subsets of X, we have the following:

Proposition 2.3 ([7, Lemma 5]). *For a dynamical system* (X, f)*, the following are equivalent*:

(1) f *is topologically exact.*
(2) $\hat{f}$ *is topologically exact.*

Remark 2.4. In [7], the map $\hat{f}$ has been denoted by $\bar{f}$, which we have used for the map induced on the hyperspace $C(X)$.

Proposition 2.5 ([9, Lemma 4]). *If dynamical systems* (X, f) *and* (Y, g) *are topologically conjugate, then the same holds for the induced dynamical systems* $(2^X, f')$ *and* $(2^Y, g')$*, and also for* $(C(X), \bar{f})$ *and* $(C(Y), \bar{g})$.

Remark 2.6. In [9], the maps $f', g', \bar{f}$ and $\bar{g}$ have been denoted by $\bar{f}, \bar{g}, \tilde{f}$ and $\tilde{g}$, respectively.

Proposition 2.7 ([8]). *Topological transitivity is preserved under topological conjugacy.*

3. Dynamical properties and $C(X)$

Theorem 3.1. *Let (X, f) be a dynamical system with f transitive. Then the induced map $\bar{f}$ on $(C(X), \tau_U)$ is transitive.*

Proof. Consider a pair of nonempty basic open sets $\mathcal{U}$ and $\mathcal{V}$ of $C(X)$. Then $\bigcup\mathcal{U}$ and $\bigcup\mathcal{V}$ constitute a pair of nonempty open sets in X. Since f is transitive, there is a natural number l such that for some $y \in \bigcup\mathcal{U}$, $f^l(y) \in \bigcup\mathcal{V}$. That $\bar{f}^l(\mathcal{U})$ intersects $\mathcal{V}$ follows by noting that $\{y\} \in \mathcal{U}$ and $\bar{f}^l(\{y\}) \in \mathcal{V}$. □

Proposition 2.1 shows that there is no dearth of dynamical systems whose evolution maps are transitive possessing induced evolution maps which fail to be transitive. Below we provide an example for the sake of illustration.

Example 3.2. The *irrational rotation* $R_\alpha : S^1 \to S^1$ defined by

$$R_\alpha(\theta) = \theta + \alpha, \qquad \theta \in [0, 2\pi),$$

where α is an irrational number, and the multiplicative group S^1 is identified with the additive group $[0, 2\pi)$ mod 2π, is a transitive map [3, 11]. Let $\overline{R}_\alpha$ be the map induced on $C(S^1)$, which is homeomorphic to $D^2 \equiv \{(r, \theta) \mid r \in [0, 1] \text{ and } \theta \in [0, 2\pi)\}$ [2, 6], via the homeomorphism ψ, sending $A \in C(S^1)$ to $(1 - l_A/2\pi, mid\ A) \in D^2$, where l_A and $mid\ A$, denote the arclength and the middle point of A, respectively. The map $g : D^2 \to D^2$ defined by

$$g(r, \theta) = (r, \theta + \alpha), \qquad r \in [0, 1], \text{ and } \theta \in [0, 2\pi),$$

is such that $g \circ \psi = \psi \circ \overline{R}_\alpha$, making the systems $(C(S^1), \overline{R}_\alpha)$ and (D^2, g) topologically conjugate. Since the orbit of no element of D^2 under g is dense in it, the map g is not transitive [8], and hence, by Proposition 2.7, $\overline{R}_\alpha$ is also not transitive.

Theorem 3.3. *Let (X, f) be a dynamical system with f totally transitive. Then the induced map $\bar{f}$ on $(C(X), \tau_U)$ is totally transitive.*

Proof. Since for a natural number n, $\overline{f^n} = \bar{f}^n$, the result follows from Theorem 3.1. □

Theorem 3.4. *Let (X, f) be a dynamical system with f weakly mixing. Then the induced map $\bar{f}$ on $(C(X), \tau_U)$ is weakly mixing.*

Proof. Consider pairs of nonempty basic open sets $\mathcal{U}_1, \mathcal{V}_1$ and $\mathcal{U}_2, \mathcal{V}_2$ of $C(X)$. Then $\bigcup\mathcal{U}_1, \bigcup\mathcal{V}_1$ and $\bigcup\mathcal{U}_2, \bigcup\mathcal{V}_2$ are pairs of nonempty open sets of X. Since f is weakly mixing, there are $n \in \mathbb{N}$, $y \in \bigcup\mathcal{U}_1$ and $y' \in \bigcup\mathcal{U}_2$ such that $f^n(y) \in \bigcup\mathcal{V}_1$, and $f^n(y') \in \bigcup\mathcal{V}_2$. That $\bar{f}^n(\mathcal{U}_1) \cap \mathcal{V}_1 \neq \phi$ and $\bar{f}^n(\mathcal{U}_2) \cap \mathcal{V}_2 \neq \phi$, follows by observing that $\{y\} \in \mathcal{U}_1$, $\bar{f}^n(\{y\}) \in \mathcal{V}_1$, $\{y'\} \in \mathcal{U}_2$ and $\bar{f}^n(\{y'\}) \in \mathcal{V}_2$. □

Theorem 3.5. *Let (X, f) be a dynamical system with f weakly mixing. If X is pathconnected, then the induced map $\bar{f}$ on $(C(X), \tau_L)$ is weakly mixing.*

Proof. Consider pairs $< U_1, \dots, U_m >, < V_1, \dots, V_m >$, and $< U'_1, \dots, U'_n >, < V'_1, \dots, V'_n >$ of nonempty basic open sets of $C(X)$. Then the pairs $(U_i, V_i),\ i = 1, \dots, m$, and $(U'_j, V'_j),\ j = 1, \dots, n$, consist of nonempty open sets of X. Since f is weakly mixing, there is a $k \in \mathbb{N}$, such that $f^k(U_i) \cap V_i \neq \phi,\ i = 1, \dots, m$, and $f^k(U'_j) \cap V'_j \neq \phi,\ j = 1, \dots, n$. Thus there are elements $x_i \in U_i,\ x'_j \in U'_j$ such that $f^k(x_i) \in V_i$, and $f^k(x'_j) \in V'_j,\ i = 1, \dots, m,\ j = 1, \dots, n$. Let A and B be paths containing all x_i's, and all x'_j's, respectively. Then $A \in < U_1, \dots, U_m >$, $B \in < U'_1, \dots, U'_n >$, and $\bar{f}^k(A) \in < V_1, \dots, V_m >$, $\bar{f}^k(B) \in < V'_1, \dots, V'_n >$. Hence, the result. □

Because, for a given dynamical system (X, f), the weakly mixing of f implies that f is transitive and also totally transitive, we have the following:

Corollary 3.6. *Let (X, f) be a dynamical system with f weakly mixing. If X is pathconnected, then the induced map $\bar{f}$ on $(C(X), \tau_L)$ is transitive and also totally transitive.*

Theorem 3.7. *Let (X, f) be a dynamical system with f mixing. Then the induced map $\bar{f}$ on $(C(X), \tau_U)$ is mixing.*

Proof. For a pair $\mathcal{U}, \mathcal{V}$ of nonempty basic open sets of $C(X)$, $\bigcup \mathcal{U}, \bigcup \mathcal{V}$ constitute a pair of nonempty open sets of X. Since f is mixing, there is an $N \in \mathbb{N}$, such that for $n \geq N$, $f^n(\bigcup \mathcal{U})$ intersects $\bigcup \mathcal{V}$, which implies that $\bar{f}^n(\mathcal{U})$ intersects $\mathcal{V}$, for $n \geq N$. Hence, the result. □

Theorem 3.8. *Let (X, f) be a dynamical system with f mixing. If X is pathconnected, then the induced map $\bar{f}$ on $(C(X), \tau_L)$ is mixing.*

Proof. Consider a pair $< U_1, \dots, U_m >, < V_1, \dots, V_m >$ of nonempty basic open sets of $C(X)$. Since f is mixing, for each pair $(U_i, V_i),\ i = 1, \dots, m$, of open sets of X, there exist $N_i \in \mathbb{N}$ such that $f^k(U_i) \cap V_i \neq \phi$, whenever $k \geq N_i$. Set $N = \max\ \{N_i : i = 1, \dots, m\}$. Then $f^k(U_i) \cap V_i \neq \phi$, for $i = 1, \dots, m$, and $k \geq N$. Thus for $i = 1, \dots, m,\ k \geq N$, there are $x_{i_k} \in U_i$ such that $f^k(x_{i_k}) \in V_i$. Let A_k be a path containing all x_{i_k}'s. Since $\bar{f}^k(A_k)$ lies in $\bar{f}^k(< U_1, \dots, U_m >)$ as well as in $< V_1, \dots, V_m >$, the result follows. □

4. Dynamical properties and $HS(X)$

We begin with the following:

Lemma 4.1. *For $n \in \mathbb{N}$, $(HS(f))^n \equiv (q_X \circ f^n \circ q_X^{-1})$.*

Proof. The result follows by induction on n. □

Theorem 4.2. *Let (X, f) be a dynamical system such that $(C(X), \bar{f})$ is transitive. Then the dynamical system $(HS(X), HS(f))$ is also transitive.*

Proof. For a pair U, V of nonempty open sets of $HS(X)$, $q_X^{-1}(U)$ and $q_X^{-1}(V)$ constitute a pair of nonempty open sets of $C(X)$. Since $\bar{f}$ is transitive, there is an $n \in \mathbb{N}$ such that $f^n(q_X^{-1}(U)) \cap q_X^{-1}(V) \neq \phi$. Applying q_X and using Lemma 4.1, the result follows. □

Theorem 4.3. *Let (X, f) be a dynamical system such that $(C(X), \bar{f})$ is totally transitive. Then the dynamical system $(HS(X), HS(f))$ is also totally transitive.*

Proof. For $n \in \mathbb{N}$, Theorem 4.2 and Lemma 4.1 provide the transitivity of $(HS(f))^n$. Thus, $HS(f)$ is totally transitive. □

Theorem 4.4. *Let (X, f) be a dynamical system such that $(C(X), \bar{f})$ is weakly mixing. Then the dynamical system $(HS(X), HS(f))$ is also weakly mixing.*

Proof. Consider pairs U_1, U_2 and V_1, V_2 of nonempty open sets of $HS(X)$. Then $q_X^{-1}(U_1), q_X^{-1}(U_2)$ and $q_X^{-1}(V_1), q_X^{-1}(V_2)$ are pairs of nonempty open sets of $C(X)$. The weakly mixing property of the map $\bar{f}$ implies the existence of an $n \in \mathbb{N}$, such that $f^n(q_X^{-1}(U_i)) \cap q_X^{-1}(V_i) \neq \phi$, for $i = 1, 2$. Now arguing as in Theorem 4.2, we have the result. □

Theorem 4.5. *Let (X, f) be a dynamical system such that $(C(X), \bar{f})$ is mixing. Then the dynamical system $(HS(X), HS(f))$ is also mixing.*

Proof. For a pair U, V of nonempty open sets of $HS(X)$, $q_X^{-1}(U)$ and $q_X^{-1}(V)$ constitute a pair of nonempty open sets of $C(X)$. The mixing property of $\bar{f}$ on $C(X)$ provides an $N \in \mathbb{N}$, such that $f^n(q_X^{-1}(U)) \cap q_X^{-1}(V) \neq \phi$, for $n \geq N$. Now proceeding as in the proof of Theorem 4.2, we have the result. □

Theorem 4.6. *Let (X, f) be a dynamical system such that $(C(X), \bar{f})$ is topologically exact. Then the dynamical system $(HS(X), HS(f))$ is also topologically exact.*

Proof. Let U be a nonempty open set of $HS(X)$. Then $q_X^{-1}(U)$ is a nonempty open set of $C(X)$. The topological exactness of $\bar{f}$ on $C(X)$ yields an $n \in \mathbb{N}$, such that $f^n(q_X^{-1}(U)) = C(X)$. Applying q_X and using Lemma 4.1, the result follows. □

Theorem 4.7. *If the dynamical systems (X, f) and (Y, g) are topologically conjugate, then the dynamical systems $(HS(X), HS(f))$ and $(HS(Y), HS(g))$ are also topologically conjugate.*

Proof. Let φ be a conjugacy between the pairs (X, f) and (Y, g). Then by Proposition 2.5, the dynamical systems $(C(X), \bar{f})$ and $(C(Y), \bar{g})$ are topologically conjugate, the induced map $\bar{\varphi}$ being the topological conjugacy. Since $\bar{\varphi}$ maps $F_1(X)$ homeomorphically onto $F_1(Y)$, $HS(\bar{\varphi}) : HS(X) \to HS(Y)$ is the required topological conjugacy between the dynamical systems $(HS(X), HS(f))$ and $(HS(Y), HS(g))$. □

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