Applied General Topology
© Universidad Politécnica de Valencia
Volume 8, No. 2, 2007
pp. 151-159

Products of straight spaces with compact spaces

Kusuo Nishijima and Kohzo Yamada

Dedicated to the memory of Jan Pelant

Abstract. A metric space X is called straight if any continuous real-valued function which is uniformly continuous on each set of a finite cover of X by closed sets, is itself uniformly continuous. Let C be the convergent sequence $\{1/n : n \in \mathbb{N}\}$ with its limit 0 in the real line with the usual metric. In this paper, we show that for a straight space X, $X \times C$ is straight if and only if $X \times K$ is straight for any compact metric space K. Furthermore, we show that for a straight space X, if $X \times C$ is straight, then X is precompact. Note that the notion of straightness depends on the metric on X. Indeed, since the real line $\mathbb{R}$ with the usual metric is not precompact, $\mathbb{R} \times C$ is not straight. On the other hand, we show that the product space of an open interval and C is straight.

2000 AMS Classification: Primary 54C30, 54D05; Secondary 54C20, 54E35

Keywords: Straight spaces, convergent sequence, precompact, uniformly continuous.

1. Introduction

All spaces are metric spaces and one fixed metric on a space X will be denoted by d_X, and $C(X)$ denotes the set of all continuous real-valued functions of a space X. Let (X, d_X) and (Y, d_Y) be metric spaces. For a subspace M of X, we consider the restriction $d_X|_{M\times M}$ to $M \times M$ as a metric on M, which is denoted by d_M. A metric $d_{X\times Y}$ on the product space $X \times Y$ will be defined by

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \sqrt{(d_X(x_1, x_2))^2 + (d_Y(y_1, y_2))^2}.$$

In this paper we study notions that use metrics in their definitions. However, the symbols of metrics will simply be denoted by d or be often omitted except when it is necessary to be clear which metric we consider.

Let X be a metric space and $\{F_i : i = 1, 2, \ldots, n\}$ be a finite closed cover of X. Then it is well-known that every function f on X is continuous if the restriction $f|_{F_i}$ of f is continuous on F_i for each $i = 1, 2, \ldots, n$. However, it is not valid for uniform continuity. Indeed, consider the subspace $X = \{e^{i\theta} : 0 < \theta < 2\pi\}$ of the complex plane with the Euclidean metric and the function $f(e^{i\theta}) = \theta$ defined on X. Then the function f is not uniformly continuous on X, but its restrictions on $\{e^{i\theta} : 0 < \theta \leq \pi\}$ and $\{e^{i\theta} : \pi \leq \theta < 2\pi\}$ are uniformly continuous. The following facts are useful to determine whether a given continuous function on a metric space is uniformly continuous or not.

Lemma 1.1. *Let $f \in C(X)$. Then the following are equivalent:*

(1) *f is uniformly continuous;*
(2) *for every pair of sequences $\{x_n\}$ and $\{y_n\}$ in X if $\lim_{n\to\infty} d(x_n, y_n) = 0$, then $\lim_{n\to\infty} |f(x_n) - f(y_n)| = 0$.*
(3) *for every pair of sequences $\{x_n\}$ and $\{y_n\}$ in X if $\lim_{n\to\infty} d(x_n, y_n) = 0$, then there are subsequences $\{x_{k_n}\}$ of $\{x_n\}$ and $\{y_{k_n}\}$ of $\{y_n\}$ such that $\lim_{n\to\infty} |f(x_{k_n}) - f(y_{k_n})| = 0$.*

Applying Lemma 1.1, it is easy to see that the above function $f(e^{i\theta}) = \theta$ is not u.c., because let $\alpha_n = \dfrac{\pi}{n}$ and $\beta_n = 2\pi - \dfrac{\pi}{n}$ for each $n \in \mathbb{N}$, and if we consider the sequences $\{x_n = e^{i\alpha_n}\}$ and $\{y_n = e^{i\beta_n}\}$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$, but $\lim_{n\to\infty} |f(x_n) - f(y_n)| = 2\pi$.

Recently, Berarducci, Dikranjan and Pelant [3] defined the following notion.

Definition 1.2 ([3])**.** *A metric space X is straight if whenever X is the union of finitely many closed sets, then $f \in C(X)$ is uniformly continuous (briefly, u.c.) iff its restriction to each of the closed sets is u.c.*

Recall that a metric space X is called UC [1, 2] provided every continuous function on X is u.c. and a metric space is called *uniformly locally connected* if for every $\varepsilon > 0$ there is $\delta > 0$ such that any two points at distance $< \delta$ lie in a connected set of diameter $< \varepsilon$. Clearly, all compact spaces are UC and all UC spaces are straight. Berarducci, Dikranjan and Pelant [3] prove that all uniformly locally connected spaces are straight. Hence, since the real line $\mathbb{R}$ and an open interval in $\mathbb{R}$ with the usual metric are clearly uniformly locally connected, they are straight, and of course, they are not UC.

The product space of two compact spaces is compact, and hence UC. However, in general, the product space $X \times Y$ of a UC space X and a compact space Y need not be UC. Indeed, Atsuji's result [2, Theorem 6] yields that if the product space $X \times Y$ of a non-compact and non-uniformly discrete UC space X and a space Y is UC, then Y must be uniformly discrete or finite (recall that a space is *uniformly discrete* if there is $\delta > 0$ such that any two distinct points are at distance at least δ). On the other hand, there are non-compact and non-uniformly discrete straight spaces whose products with compact spaces are straight, for example, $\mathbb{R} \times I$ is uniformly locally connected, and hence straight, where I means that the unit closed interval.

In this paper, we consider properties of a straight space whose product with any compact space is straight. Let X be a straight space and C be the convergent sequence $\{1/n : n \in \mathbb{N}\}$ with its limit 0 in the real line with the usual metric. Recall that a metric space X is *precompact* if for every $\varepsilon > 0$ there are finite points $x_1, x_2, \ldots, x_n$ in X such that $X = \bigcup_{k=1}^{n} B_\varepsilon(x_k)$, where $B_\varepsilon(x) = \{z \in X : d(x,z) < \varepsilon\}$. Then we will show the following:

(1) $X \times C$ is straight if and only if $X \times K$ is straight for any compact space K;

(2) if $X \times C$ is straight, then X is precompact.

We can know, from the result (2), that $\mathbb{R} \times C$ is not straight. On the other hand, we prove that the product space of an open interval and C is straight. However, we cannot decide whether the inverse implication of the result (2) is valid or not (cf. Acknowledgement).

2. Results

We first introduce the terminology that is defined in [3]. Let X be a metric space. A pair E and F of closed sets of X is *u-placed* if $d(E_\varepsilon, F_\varepsilon) > 0$ for every $\varepsilon > 0$, where $E_\varepsilon = \{x \in E : d(x, E \cap F) \geq \varepsilon\}$ and $F_\varepsilon = \{x \in F : d(x, E \cap F) \geq \varepsilon\}$. Note that if $E \cap F = \varnothing$, then $E_\varepsilon = E$ and $F_\varepsilon = F$. Hence, a partition $X = E \cup F$ of X into clopen sets is u-placed iff $d(E, F) > 0$.

Berarducci and Dikranjan and Pelant give the following characterizations of straight spaces in the same paper.

Theorem 2.1 ([3]). *For a metric space X the following are equivalent:*

(1) *X is straight;*

(2) *whenever X is the union of two closed sets, then $f \in C(X)$ is u.c. iff its restriction to each of the closed sets is u.c.;*

(3) *every pair of closed subsets, which form a cover of X, is u-placed.*

According to Theorem 2.1, we can conclude that the space $\mathbb{Q}$ of rational numbers and the space $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers with the usual metric are not straight. Applying Lemma 1.1 and Theorem 2.1, we will show the following, which says that for given straight space X, it suffices to check whether $X \times C$ is straight in order to know whether $X \times K$ is straight for any compact space K.

Theorem 2.2. *For a straight space X $X \times C$ is straight if and only if $X \times K$ is straight for any compact space K.*

Proof. Assume that $X \times C$ is straight and let K be a compact space. From the definition of the straightness we assume that K is an infinite compact space. To show that $X \times K$ is straight, take a closed cover $\{E, F\}$ of $X \times K$ and $f \in C(X \times K)$ on $X \times K$ such that the restrictions $f|_E$ and $f|_F$ are u.c. If we can show that f is u.c., then, from Theorem 2.1, our proof is complete.

Consider sequences $\{x_n\}$ and $\{y_n\}$ in $X \times K$ such that $\lim_{n\to\infty} d_{X\times K}(x_n, y_n) = 0$. We shall find subsequences $\{x_{k_n}\}$ of $\{x_n\}$ and $\{y_{k_n}\}$ of $\{y_n\}$ such that $\lim_{n\to\infty} |f(x_{k_n}) - f(y_{k_n})| = 0$. We denote the projection of $X \times K$ onto K by π_K. We consider the following cases.

Case 1: *$\pi_K(\{x_n : n \in \omega\})$ is a finite set.*

Take a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ and $z \in K$ such that $\pi_K(x_{k_n}) = z$ for each $n \in \omega$.

Case 1.1: *$\pi_K(\{y_{k_n} : n \in \omega\})$ is a finite set.*

In this case, since $d(x_{k_n}, y_{k_n})$ converges to 0, there is an infinite subset $Y \subseteq \{y_{k_n} : n \in \omega\}$ for which $\pi_K(Y) = \{z\}$. So we may assume that $\{y_{k_n} : n \in \omega\}$ is the infinite set. Put $E_z = E \cap (X \times \{z\})$ and $F_z = F \cap (X \times \{z\})$. Then we can know that

(i) $\{E_z, F_z\}$ is a closed cover of $X \times \{z\}$ and
(ii) the restrictions $f|_{E_z}$ and $f|_{F_z}$ are u.c.

Since X is straight and isometric to $X \times \{z\}$, $X \times \{z\}$ is straight. It follows that the restriction $f|_{X\times\{z\}}$ is u.c. Observe that the sequences $\{x_{k_n}\}$ and $\{y_{k_n}\}$ lie in $X \times \{z\}$ and $\lim_{n\to\infty} d_{X\times\{z\}}(x_{k_n}, y_{k_n}) = 0$. Hence, we have that

$$\lim_{n\to\infty} |f(x_{k_n}) - f(y_{k_n})| = \lim_{n\to\infty} |f|_{X\times\{z\}}(x_{k_n}) - f|_{X\times\{z\}}(y_{k_n})| = 0.$$

Case 1.2: $\pi_K(\{y_{k_n} : n \in \omega\})$ is an infinite set.

Since K is compact, $\pi_K(\{y_{k_n} : n \in \omega\})$ contains a non-trivial convergent sequence. We may assume that $\pi_K(\{y_{k_n} : n \in \omega\})$ is the non-trivial convergent sequence and also $\pi_K(y_{k_m}) \neq \pi_K(y_{k_n})$ if $m \neq n$. Note that $d(x_{k_n}, y_{k_n})$ converges to 0. Hence, it follows that z is the convergent point of the sequence $\{\pi_K(y_{k_n})\}$. Put $z_n = \pi_K(y_{k_n})$ for each $n \in \omega$ and $Z = \{z_n : n \in \omega\} \cup \{z\}$. Define a mapping $g : X \times C \to X \times Z$ by $g(x, 1/n) = (x, z_n)$ for each $x \in X$ and $n \in \omega$, and $g(x, 0) = (x, z)$ for each $x \in X$. Clearly, g is a uniformly homeomorphism. Put

$$H = g^{-1}(E \cap (X \times Z)), I = g^{-1}(F \cap (X \times Z)),$$

$$a_{k_n} = g^{-1}(x_{k_n}), b_{k_n} = g^{-1}(y_{k_n}) \text{ for each } n \in \omega, \text{ and}$$

$$h = f \circ g : X \times C \to \mathbb{R}.$$

Then we can show the following:

(i) $\{H, I\}$ is a closed cover of $X \times C$,
(ii) $\lim_{n\to\infty} d_{X\times C}(a_{k_n}, b_{k_n}) = 0$, and
(iii) $h|_H = f|_{E\cap(X\times Z)} \circ g|_H$, and $h|_I = f|_{F\cap(X\times Z)} \circ g|_I$, and hence $h|_H$ and $h|_I$ are u.c.

Since $X \times C$ is straight, h is u.c. Hence $\lim_{n\to\infty} |h(a_{k_n}) - h(b_{k_n})| = 0$. It follows that

$$\begin{aligned}\lim_{n\to\infty} |f(x_{k_n}) - f(y_{k_n})| &= \lim_{n\to\infty} |f(g(a_{k_n})) - f(g(b_{k_n}))| \\ &= \lim_{n\to\infty} |h(a_{k_n}) - h(b_{k_n})| \\ &= 0.\end{aligned}$$

Case 2: $\pi_K(\{x_n : n \in \omega\})$ *is an infinite set.*

Since X is compact, we can pick a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ and $z \in K$ such that $\pi_K(x_{k_m}) \neq \pi_K(x_{k_n})$ if $m \neq n$ and $\{\pi_K(x_{k_n})\}$ converges to z. Note that $\lim_{n\to\infty} d(x_{k_n}, y_{k_n}) = 0$. Hence, this yields that the sequence $\{\pi_K(y_{k_n})\}$ also converges to z. Let $\{z_n : n \in \omega\}$ be an enumeration of $\pi_K(\{x_{k_n} : n \in \omega\} \cup \{y_{k_n} : n \in \omega\})$ such that $z_m \neq z_n$ if $m \neq n$. Then, the sequence $\{z_n\}$ converges to z. Consider the same mapping $g : X \times C \to X \times (\{z_n : n \in \omega\} \cup \{z\})$ as in Case 1.2. Then, with the same argument in Case 1.2, if we put $a_{k_n} = g^{-1}(x_{k_n})$ and $b_{k_n} = g^{-1}(y_{k_n})$ for each $n \in \omega$ and $h = g \circ f$, then we can show that h is u.c., and hence $\lim_{n\to\infty} |h(a_{k_n}) - h(b_{k_n})| = 0$. Consequently,

$$\begin{aligned}\lim_{n\to\infty} |f(x_{k_n}) - f(y_{k_n})| &= \lim_{n\to\infty} |f(g(a_{k_n})) - f(g(b_{k_n}))| \\ &= \lim_{n\to\infty} |h(a_{k_n}) - h(b_{k_n})| \\ &= 0.\end{aligned}$$

Therefore, in any case, we can find subsequences $\{x_{k_n}\}$ of $\{x_n\}$ and $\{y_{k_n}\}$ of $\{y_n\}$ such that $\lim_{n\to\infty} |f(x_{k_n}) - f(y_{k_n})| = 0$. It follows, from Lemma 1.1, that f is u.c. Consequently, $X \times K$ is straight. □

The following result gives a necessary condition of X for which $X \times C$ is straight.

Theorem 2.3. *For a straight space X if $X \times C$ is straight, then X is precompact.*

Proof. Put $Y = X \times C$. Suppose that X is not precompact and pick $\varepsilon > 0$ and an infinite set $\{x_n : n \in \mathbb{N}\}$ such that $\overline{B_\varepsilon(x_m)} \cap \overline{B_\varepsilon(x_n)} = \varnothing$ if $m \neq n$. For each $n \in \mathbb{N}$ let $a_n = (x_n, \frac{1}{n}) \in Y$ and $b_n = (x_n, \frac{1}{n+1}) \in Y$. Clearly, $\lim_{n\to\infty} d_Y(a_n, b_n) = 0$. Hence, we can find $N \in \mathbb{N}$ such that $b_n \in B_{\varepsilon/2}(a_n)$ for every $n \geq N$. Put $M = Y \setminus \bigcup_{n \geq N} B_\varepsilon(a_n)$. Then M is a closed subset of Y. For each $n \geq N$ put

$$\begin{aligned}A_n &= (X \times \{\frac{1}{i} : i \leq n\}) \cap \overline{B_\varepsilon(a_n)}, \\ B_n &= (X \times (\{\frac{1}{i} : i \geq n+1\} \cup \{0\})) \cap \overline{B_\varepsilon(a_n)}.\end{aligned}$$

Note that the collection $\{\overline{B_\varepsilon(a_n)} : n \in \mathbb{N}\}$ is closed discrete in Y, and hence so are $\{A_n : n \in \mathbb{N}\}$ and $\{B_n : n \in \mathbb{N}\}$. If we put $E = M \cup \bigcup_{n \geq N} A_n$ and $F = M \cup \bigcup_{n \geq N} B_n$, then we have that

(a) $\{E, F\}$ is a closed cover of Y,
(b) $E \cap F = M$,
(c) for each $n \geq N$ $d(a_n, E \cap F) = d(a_n, M) \geq d(a_n, Y \setminus B_\varepsilon(a_n)) = \varepsilon$, and
(d) for each $n \geq N$ $d(b_n, E \cap F) = d(b_n, M) \geq d(b_n, Y \setminus B_\varepsilon(a_n)) = \frac{\varepsilon}{2}$, because $b_n \in B_{\varepsilon/2}(a_n)$.

The conditions (c) and (d) imply that $\{a_n : n \geq N\} \subseteq E_{\varepsilon/2}$ and $\{b_n : n \geq N\} \subseteq F_{\varepsilon/2}$. Since $\lim_{n\to\infty} d(a_n, b_n) = 0$, it follows that $d(E_{\varepsilon/2}, F_{\varepsilon/2}) = 0$. That is, the pair E and F is not u-placed. Consequently, by Theorem 2.1 we can prove that $X \times C$ is not straight. □

Remark 2.4. Since $\mathbb{R}$ is a straight space that is not precompact, Theorem 2.3 says that $\mathbb{R} \times C$ is not straight. Indeed, we can construct a pair of closed sets E and F which is not u-placed. For example, let $E = \bigcup_{n\in\mathbb{N}} ([2n, \infty) \times \{\frac{1}{n}\})$ and $F = \bigcup_{n\in\mathbb{N}} ((-\infty, 2n] \times \{\frac{1}{n}\}) \cup \mathbb{R} \times \{0\}$. Then $\{E, F\}$ is a closed cover of $\mathbb{R} \times C$ and $E \cap F = \{(2n, \frac{1}{n}) : n \in \mathbb{N}\}$. Put $x_n = (2n+1, \frac{1}{n})$ and $y_n = (2n+1, \frac{1}{n+1})$ for each $n \in \mathbb{N}$. Then we can see that $\lim_{n\to\infty} d(x_n, y_n) = 0$, $\{x_n : n \in \mathbb{N}\} \subseteq E_{1/2}$ and $\{y_n : n \in \mathbb{N}\} \subseteq F_{1/2}$. Hence $d(E_{1/2}, F_{1/2}) = 0$. This means that the pair E and F is not u-placed.

Corollary 2.5. *For a complete straight space X the following are equivalent:*

(1) *X is precompact;*
(2) *X is compact;*
(3) *$X \times C$ is straight;*
(4) *$X \times K$ is straight for any compact space K.*

We don't know whether the inverse implication of Theorem 2.3 is true or not, however, we can show that the product space of an open interval and C is straight (cf. Acknowledgment). We need the following lemmas.

Lemma 2.6. *A metric space X which is represented as a topological sum of a family $\{X_\alpha : \alpha \in A\}$ of a spaces is straight if* $\inf\{d(X_\alpha, X_\beta) : \alpha \neq \beta\} > 0$.

The following lemma is introduced in [3, Theorem 5.3] and proved in [4, Proposition 2.4].

Lemma 2.7 ([3, 4])**.** *Let X be a metric space and $X = K \cup Y$, where K is a compact subspace of X and Y is a closed subset of X. Then X is straight iff Y is straight.*

Theorem 2.8. *The product space of a half open interval and C is straight.*

Proof. Let $X = (a, b] \times C$, where $a < b$. To show that X is straight, let E and F be closed sets in X with $E \cup F = X$ and take an arbitrary (small) positive number $\varepsilon > 0$. To avoid confusion we use notations such as E^X_ε and $(E \cap Y)^Y_\varepsilon$, and which mean that

$$E^X_\varepsilon = \{x \in E : d_X(x, E \cap F) \geq \varepsilon\} \text{ and}$$
$$(E \cap Y)^Y_\varepsilon = \{x \in E \cap Y : d_Y(x, E \cap F \cap Y) \geq \varepsilon\},$$

where E, F and Y are subsets of a space X. According to Theorem 2.1, we shall show that the pair E and F is u-placed. Assuming that $b = a + 1$ and we can pick $N \in \mathbb{N}$ for which $\dfrac{1}{N+1} < \dfrac{\varepsilon}{\sqrt{2}} \leq \dfrac{1}{N}$, put

$$U = (a,\, a + \frac{\varepsilon}{\sqrt{2}}) \times (\{\frac{1}{n} : n \geq N+1\} \cup \{0\}) \text{ and } Y = X \setminus U.$$

Case 1. $U \cap (E \cap F) \neq \varnothing$.

In this case, since the diameter of U is less than ε, $U \subseteq B^X_\varepsilon(E \cap F \cap U) \subseteq B^X_\varepsilon(E \cap F)$. Thus

$$E^X_\varepsilon \cup F^X_\varepsilon \subseteq X \setminus U = Y. \tag{2.1}$$

It follows that $d_X(E^X_\varepsilon, F^X_\varepsilon) = d_Y(E^X_\varepsilon, F^X_\varepsilon)$. To show that $E^X_\varepsilon \subseteq (E \cap Y)^Y_\varepsilon$, let $x \in E^X_\varepsilon$. Then $x \in E$ and $d_X(x, E \cap F) \geq \varepsilon$. Since $x \in E \cap Y$ by (2.1) and

$$d_Y(x, (E \cap Y) \cap (F \cap Y)) \geq d_X(x, E \cap F) \geq \varepsilon,$$

we can see that $x \in (E \cap Y)^Y_\varepsilon$. Therefore $E^X_\varepsilon \subseteq (E \cap Y)^Y_\varepsilon$. In the same way, we can show that $F^X_\varepsilon \subseteq (F \cap Y)^Y_\varepsilon$. On the other hand, Lemma 2.6 and Lemma 2.7 yield that Y is straight, and hence $d_Y((E \cap Y)^Y_\varepsilon, (F \cap Y)^Y_\varepsilon) > 0$. So, we can get that

$$d_X(E^X_\varepsilon, F^X_\varepsilon) \geq d_X((E \cap Y)^Y_\varepsilon, (F \cap Y)^Y_\varepsilon) = d_Y((E \cap Y)^Y_\varepsilon, (F \cap Y)^Y_\varepsilon) > 0.$$

Case 2. $U \cap (E \cap F) = \varnothing$.

In this case, for every $p \in \{\frac{1}{n} : n \geq N+1\} \cup \{0\}$

$$(a,\, a + \frac{\varepsilon}{\sqrt{2}}) \times \{p\} \subseteq E \cup F \tag{2.2}$$

$$(E \cap F) \cap ((a,\, a + \frac{\varepsilon}{\sqrt{2}}) \times \{p\}) = \varnothing. \tag{2.3}$$

Since every $(a,\, a + \frac{\varepsilon}{\sqrt{2}}) \times \{p\}$ is connected,

$$(a,\, a + \frac{\varepsilon}{\sqrt{2}}) \times \{p\} \subseteq E \text{ or } (a,\, a + \frac{\varepsilon}{\sqrt{2}}) \times \{p\} \subseteq F.$$

Now, we assume that $(a,\, a+\dfrac{\varepsilon}{\sqrt{2}})\times\{0\}\subseteq E$. Then, from the conditions (2.2) and (2.3), we can find $M\geq N+1$ such that

$$(a,\, a+\frac{\varepsilon}{\sqrt{2}})\times(\{\frac{1}{n}:n\geq M\}\cup\{0\})\subseteq E.$$

Put $V=(a,\, a+\dfrac{\varepsilon}{2\sqrt{2}})\times(\{\dfrac{1}{n}:n\geq M+1\}\cup\{0\})$ and $Z=X\setminus V$. Then $E\cap F\subseteq F\subseteq Z$. Lemma 2.6 and Lemma 2.7 claim that Z is straight. So, we can say that

$$d_Z((E\cap Z)^Z_\varepsilon,(F\cap Z)^Z_\varepsilon)>0. \tag{2.4}$$

Here, we shall show that

$$E^X_\varepsilon\cap Z\subseteq(E\cap Z)^Z_\varepsilon \text{ and } F^X_\varepsilon\subseteq(F\cap Z)^Z_\varepsilon. \tag{2.5}$$

Let $x\in E^X_\varepsilon\cap Z$. Then $x\in E\cap Z$ and $d_X(x,E\cap F)\geq\varepsilon$. Since $E\cap F\subseteq Z$, $d_Z(x,(E\cap Z)\cap(F\cap Z))=d_Z(x,E\cap F)=d_X(x,E\cap F)\geq\varepsilon$. It follows that $x\in(E\cap F)^Z_\varepsilon$, and hence $E^X_\varepsilon\cap Z\subseteq(E\cap Z)^Z_\varepsilon$. Next, let $x\in F^X_\varepsilon$. Then $x\in F$ and $d_X(x,E\cap F)\geq\varepsilon$. Since $F\subseteq Z$, $x\in F\cap Z$ and

$$d_Z(x,(E\cap Z)\cap(F\cap Z))=d_Z(x,E\cap F)=d_X(x,E\cap F)\geq\varepsilon.$$

It follows that $x\in(F\cap Z)^Z_\varepsilon$, and hence $F^X_\varepsilon\subseteq(F\cap Z)^Z_\varepsilon$.

The conditions (2.4) and (2.5) yield that

(2.6)

$$d_X(E^X_\varepsilon\cap Z,F^X_\varepsilon)\geq d_X((E\cap Z)^Z_\varepsilon,(F\cap Z)^Z_\varepsilon)=d_Z((E\cap Z)^Z_\varepsilon,(F\cap Z)^Z_\varepsilon)>0.$$

Furthermore, since

$$V=(a,\, a+\frac{\varepsilon}{2\sqrt{2}})\times(\{\frac{1}{n}:n\geq M+1\}\cup\{0\}) \text{ and}$$
$$\left((a,\, a+\frac{\varepsilon}{\sqrt{2}})\times(\frac{1}{n}:n\geq M\}\cup\{0\})\right)\cap F=\varnothing,$$

we can see that $d_X(V,F)>0$, and hence

$$d_X(E^X_\varepsilon\cap V,F^X_\varepsilon)>0. \tag{2.7}$$

The fact $E^X_\varepsilon=(E^X_\varepsilon\cap V)\cup(E^X_\varepsilon\cap Z)$ and the conditions (2.6) and (2.7) yield that $d_X(E^X_\varepsilon,F^X_\varepsilon)>0$.

In any case, we can get $d_X(E^X_\varepsilon,F^X_\varepsilon)>0$. Consequently, we can conclude that $X=(a,\,b]\times C$ is straight. With the same argument we can prove that $[a,b)\times C$ is also straight. □

Corollary 2.9. *The product space of an open interval and C is straight.*

Proof. Let $X=(a,\,b)\times C$, where $a<b$. Take real numbers c and d for which $a<c<d<b$ and put $Y=(a,\,c]\times C$, $Z=[d,\,b)\times C$ and $K=[c,\,d]\times C$. Then Theorem 2.7 and Lemma 2.6 yield that $Y\cup Z$ is straight. Therefore, since $Y\cup Z$ is a straight closed subspace of X, K is compact and $X=(Y\cup Z)\cup K$, applying Lemma 2.8, we can show that X is straight. □

Finally, we obtain the following from Corollary 2.5 and Corollary 2.9.

Corollary 2.10. *The product of an open interval and a compact metric space is straight.*

Acknowledgements. When the second author gave a talk about the results in this paper at the III Japan-Mexico Joint Meeting in Topology and its Applications, he has learned from Professor Dikranjan that A. Berarducci, D. Dikranjan and J. Pelant in [4] proved independently that $\mathbb{R} \times C$ is not straight and the product space of an open interval and C is straight. After that, Professor Dikranjan sent us their paper [5] quite recently, in which they obtain the interesting result such as the inverse implication of Theorem 2.3 is valid. Therefore, we can know now that for a straight space X $X \times K$ is straight for every compact metric space K if and only if X is precompact.

References

[1] M. Atsuji, *Uniform continuity of continuous functions on metric spaces*, Pacific J. Math. **8** (1958), 11–16.
[2] M. Atsuji, *Uniform continuity of continuous functions on metric spaces*, Canad. J. Math. **13** (1961), 657–663.
[3] A. Berarducci, D. Dikranjan and J. Pelant, *An additivity theorem for uniformly continuous functions*, Topology Appl. **146-147** (2005), 339–352.
[4] A. Berarducci, D. Dikranjan and J. Pelant, *Local connectedness and extension of uniformly continuous functions*, preprint.
[5] A. Berarducci, D. Dikranjan and J. Pelant, *Products of straight spaces*, preprint.

Received August 2005

Accepted August 2006

Kusuo Nishijima (lkusuo@ezweb.ne.jp)
Faculty of Education, Shizuoka University, Shizuoka, 422-8529 Japan.

Kohzo Yamada (eckyama@ipc.shizuoka.ac.jp)
Faculty of Education, Shizuoka University, Shizuoka, 422-8529 Japan.