

## On countable star-covering properties

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**ABSTRACT.** We introduce two new notions of topological spaces called a countably starcompact space and a countably absolutely countably compact (= countably acc) space. We clarify the relations between these spaces and other related spaces and investigate topological properties of countably starcompact spaces and countably acc spaces. Some examples showing the limitations of our results are also given.

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### 1. INTRODUCTION

By a space, we mean a topological space. Let us recall that a space  $X$  is *countably compact* if every countable open cover of  $X$  has a finite subcover. Fleischman defined in [4] a space  $X$  to be *starcompact* if for every open cover  $\mathcal{U}$  of  $X$ , there exists a finite subset  $B$  of  $X$  such that  $St(B, \mathcal{U}) = X$ , where

$$St(B, \mathcal{U}) = \bigcup \{U \in \mathcal{U} : U \cap B \neq \emptyset\}.$$

He proved that every countably compact space  $X$  is starcompact. Conversely, van Douwen-Reed-Roscoe-Tree [2] proved that every starcompact  $T_2$ -space is countably compact, but this does not hold for  $T_1$ -spaces (see Example 2.5 below). Strengthening the definition of starcompactness, Matveev defined in [5] a space  $X$  to be *absolutely countably compact* (= *acc*) if for every open cover  $\mathcal{U}$  of  $X$  and every dense subspace  $D$  of  $X$ , there exists a finite subset  $F$  of  $D$  such that  $St(F, \mathcal{U}) = X$ . Every acc  $T_2$ -space is countably compact (see [5]), but an acc  $T_1$ -space need not be countably compact (see Example 2.4 below). Restricting the definitions of starcompactness and absolutely countably compactness to countable open covers, we define the following classes of spaces:

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**Definition 1.1.** A space  $X$  is countably starcompact if for every countable open cover  $\mathcal{U}$  of  $X$ , there exists a finite subset  $B$  of  $X$  such that  $St(B, \mathcal{U}) = X$ .

**Definition 1.2.** A space  $X$  is countably absolutely countably compact (= countably acc) if for every countable open cover  $\mathcal{U}$  of  $X$  and every dense subspace  $D$  of  $X$ , there exists a finite subset  $F$  of  $D$  such that  $St(F, \mathcal{U}) = X$ .

The purpose of this paper is to clarify the relationship among these spaces and to consider topological properties of a countably starcompact space and a countably acc space, respectively. The systematic study of countable star-covering properties is very important to supply the research of star-covering properties. From the definitions and above remarks, we have the following diagram, where  $A \rightarrow B$  means that every  $A$ -space is a  $B$ -space:

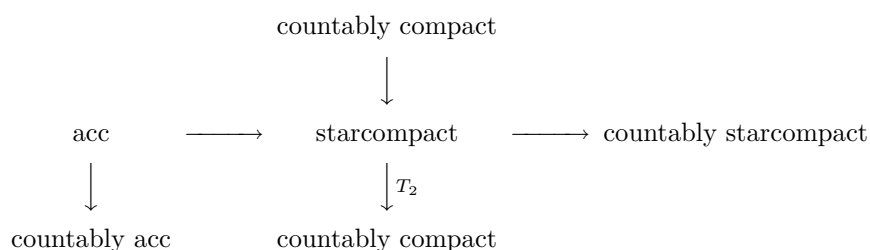


Diagram 1

The cardinality of a set  $A$  is denoted by  $|A|$ . For a cardinal  $\kappa$ ,  $\kappa^+$  denotes the smallest cardinal greater than  $\kappa$ . Let  $\mathfrak{c}$  denote the cardinality of the continuum,  $\omega$  the first infinite cardinal and  $\omega_1 = \omega^+$ . As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. For each ordinals  $\alpha, \beta$  with  $\alpha < \beta$ , we write  $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$ ,  $[\alpha, \beta) = \{\gamma : \alpha \leq \gamma < \beta\}$  and  $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$ . Other terms and symbols will be used as in [3].

## 2. RELATIONS AMONG SPACES

In this section, we consider the relations among countably acc spaces, countably starcompact spaces and other related spaces.

**Proposition 2.1.** Every countably compact space is countably acc and every countably acc space is countably starcompact.

*Proof.* Let  $X$  be a countably compact space. Let  $\mathcal{U}$  be a countable open cover of  $X$  and let  $D$  be a dense subspace of  $X$ . Then, there exists a finite subcover  $\{U_1, U_2, \dots, U_n\}$  of  $\mathcal{U}$ , since  $X$  is countably compact. Pick a point  $x_i \in U_i \cap D$  for  $i = 1, 2, \dots, n$ . Then,  $St(\{x_1, x_2, \dots, x_n\}, \mathcal{U}) = X$ , which shows that  $X$  is countably acc. Hence, every countably compact space is countably acc. It follows immediately from the definitions that every countably acc space is countably starcompact.  $\square$

**Proposition 2.2.** *For a  $T_2$ -space  $X$ , the following conditions are equivalent:*

- (1)  $X$  is countably compact;
- (2)  $X$  is countably acc;
- (3)  $X$  is countably starcompact.

*Proof.* The implications (1)  $\Rightarrow$  (2) and (2)  $\Rightarrow$  (3) are true by Proposition 2.1. It remains to show that (3)  $\Rightarrow$  (1). Suppose that  $X$  is not countably compact. Then, there exists an infinite closed discrete subset  $D = \{x_n : n \in \omega\}$  of  $X$ . For each  $m \in \omega$ , let  $D_m = \{x_n : 2^m \leq n < 2^{m+1}\}$ ; then  $|D_m| = 2^m$ . Since  $X$  is a  $T_2$ -space, there exists a collection  $\mathcal{U}_m = \{U_n : 2^m \leq n < 2^{m+1}\}$  of pairwise disjoint open sets in  $X$  such that  $U_n \cap D = \{x_n\}$  for each  $n \in \omega$ . Take such a collection  $\mathcal{U}_m$  for each  $m \in \omega$  and let

$$\mathcal{U} = \{X \setminus D\} \cup \bigcup_{m \in \omega} \mathcal{U}_m.$$

Then,  $\mathcal{U}$  is a countable open cover of  $X$ . Let  $B$  be any finite subset of  $X$  with  $|B| = k$ . Since  $|B| < 2^k = |\mathcal{U}_k|$  and  $\mathcal{U}_k$  is disjoint, some  $U_n \in \mathcal{U}_k$  does not intersect  $B$ . Then,  $x_n \notin St(B, \mathcal{U})$ , because  $U_n$  is only member of  $\mathcal{U}$  containing  $x_n$ . Hence,  $X$  is not countably starcompact. This proves that (3)  $\Rightarrow$  (1).  $\square$

**Proposition 2.3.** *Every countably starcompact space  $X$  is pseudocompact.*

*Proof.* Let  $f$  be a continuous real-valued function on  $X$ , and let  $U_n = \{x \in X : n - 1 < f(x) < n + 1\}$  for each  $n \in \mathbb{Z}$ . Then,  $\mathcal{U} = \{U_n : n \in \mathbb{Z}\}$  is a countable open cover of  $X$ . Since  $X$  is countably starcompact, there exists a finite subset  $B$  of  $X$  such that  $St(B, \mathcal{U}) = X$ . Since  $\mathcal{U}$  is point-finite, the set  $\{U \in \mathcal{U} : U \cap B \neq \emptyset\}$  is finite, say  $\{U_{n_1}, U_{n_2}, \dots, U_{n_k}\}$ . If we put  $M = \max\{|n_i| + 1 : i = 1, 2, \dots, k\}$ , then  $|f(x)| \leq M$  for each  $x \in X$ . Hence, every continuous real-valued function on  $X$  is bounded, which means that  $X$  is pseudocompact.  $\square$

Summing up the above results, we have the following diagram, where the implications (1)–(6) hold for arbitrary spaces and the inverses of implications (2)–(5) also hold for  $T_2$ -spaces:

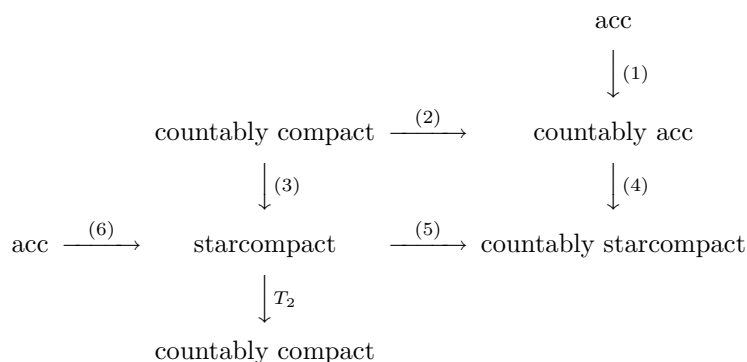


Diagram 2

In the rest of this section, we give examples which show the implications (1)–(6) in Diagram 2 cannot be reversed in the realm of  $T_1$ -spaces. The first one shows that the inverses of the implications (2) and (3) do not hold for  $T_1$ -spaces.

**Example 2.4.** There exists an acc  $T_1$ -space which is not countably compact.

*Proof.* Let  $\kappa$  be an infinite cardinal and  $A$  a set of cardinality  $\kappa$ . Define  $X = \kappa^+ \cup A$ . We topologize  $X$  as follows:  $\kappa^+$  has the usual order topology and is an open subspace of  $X$ , and a basic neighborhood of  $a \in A$  is of the form

$$G_\beta(a) = (\beta, \kappa^+) \cup \{a\}, \quad \text{where } \beta < \kappa^+.$$

Then,  $X$  is a  $T_1$ -space which is not countably compact, because  $A$  is infinite discrete closed in  $X$ . To show that  $X$  is absolutely countably compact, let  $\mathcal{U}$  be an open cover of  $X$ . Let  $D$  be the set of all isolated points of  $\kappa^+$ . Then,  $D$  is dense in  $X$  and every dense subspace of  $X$  includes  $D$ . Thus, it suffices to show that there exists a finite subset  $F \subseteq D$  such that  $St(F, \mathcal{U}) = X$ . Since  $\kappa^+$  is absolutely countably compact, there is a finite subset  $F' \subseteq D$  such that  $\kappa^+ \subseteq St(F', \mathcal{U})$ . For each  $a \in A$ , there is  $\beta(a) < \kappa^+$  such that  $G_{\beta(a)}(a)$  is included in some member of  $\mathcal{U}$ . If we choose  $\beta \in D$  with  $\beta > \sup\{\beta(a) : a \in A\}$ , then  $A \subseteq St(\beta, \mathcal{U})$ . Let  $F = F' \cup \{\beta\}$ . Then,  $St(F, \mathcal{U}) = X$ . Hence,  $X$  is absolutely countably compact, which completes the proof.  $\square$   $\square$

The second example shows that the converses of the implications (3), (4) and (6) in Diagram 2 do not hold for  $T_1$ -spaces.

**Example 2.5.** There exists a starcompact  $T_1$ -space which is not countably acc.

*Proof.* Let  $Y = (\omega + 1) \times \omega_1$ , where both  $\omega + 1$  and  $\omega_1$  have the usual order topologies and  $Y$  has the Tychonoff product topology. Let  $X = \omega \cup Y$ . We topologize  $X$  as follows:  $Y$  is an open subspace of  $X$ ; a basic neighborhood of a point  $n \in \omega$  takes the form

$$O_\alpha(n) = \{n\} \cup ((n, \omega] \times (\alpha, \omega_1)) \quad \text{where } \alpha < \omega_1.$$

Then,  $X$  is a  $T_1$ -space. To show that  $X$  is starcompact, let  $\mathcal{U}$  be an open cover of  $X$ . Then, there exists a finite subset  $F_1$  of  $Y$  such that  $Y \subseteq St(F_1, \mathcal{U})$ , since  $Y$  is countably compact. For each  $n \in \omega$ , there is  $\alpha_n < \omega_1$  such that  $O_{\alpha_n}(n)$  is included in some member of  $\mathcal{U}$ . If we choose  $\alpha_0 < \omega_1$  with  $\alpha_0 > \sup\{\alpha_n : n \in \omega\}$ , then  $\omega \subseteq St(\langle \omega, \alpha_0 \rangle, \mathcal{U})$ . Let  $F_0 = F_1 \cup \{\langle \omega, \alpha_0 \rangle\}$ . Then,  $X = St(F_0, \mathcal{U})$ , which shows that  $X$  is starcompact.

Next, we show that  $X$  is not countably acc. Let  $D = \omega \times \omega_1$ . Then,  $D$  is dense in  $X$ . Therefore, it suffices to show that there exists a countable open cover  $\mathcal{V}$  of  $X$  such that  $St(A, \mathcal{V}) \neq X$  for any finite subset  $A$  of  $D$ . Let us consider the countable open cover

$$\mathcal{V} = \{Y\} \cup \{O_0(n) : n \in \omega\}.$$

Let  $A$  be any finite subset of  $D$ . Then, there exists  $n \in \omega$  such that  $([n, \omega] \times \omega_1) \cap A = \emptyset$ . Hence,  $n \notin St(A, \mathcal{V})$ , since  $O_0(n)$  is only element of  $\mathcal{V}$  such that  $n \in O_0(n)$ . This shows that  $X$  is not countably acc.  $\square$

The third example shows that the converses of the implications (1) and (5) in Diagram 2 do not hold for  $T_1$ -spaces.

**Example 2.6.** There exists a countably acc  $T_1$ -space which is not a starcompact space.

*Proof.* Let  $X = \omega_1 \cup A$ , where  $A = \{a_\alpha : \alpha \in \omega_1\}$  is a set of cardinality  $\omega_1$ . We topologize  $X$  as follows:  $\omega_1$  has the usual order topology and is an open subspace of  $X$ ; a basic neighborhood of a point  $a_\alpha \in A$  takes the form

$$O_\beta(a_\alpha) = \{a_\alpha\} \cup (\beta, \omega_1) \quad \text{where } \beta < \omega_1.$$

Then,  $X$  is a  $T_1$ -space. To show that  $X$  is countably acc, let  $\mathcal{U}$  be a countable open cover of  $X$ . Let  $D$  be the set of all isolated points of  $\omega_1$ . Then,  $D$  is dense in  $X$  and every dense subspace of  $X$  includes  $D$ . Thus, it suffices to show that there exists a finite subset  $F \subseteq D$  such that  $St(F, \mathcal{U}) = X$ . Since  $\omega_1$  is absolutely countably compact, there is a finite subset  $F' \subseteq D$  such that  $\omega_1 \subseteq St(F', \mathcal{U})$ . Let  $\mathcal{V} = \{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ . For each  $U \in \mathcal{V}$ , there exists a  $\beta_U < \omega_1$  such that  $(\beta_U, \omega_1) \subseteq U$ . Since  $\mathcal{V}$  is countable, we can choose  $\beta \in D$  with  $\beta > \sup\{\beta_U : U \in \mathcal{V}\}$ . Thus,  $A \subseteq St(\beta, \mathcal{V}) \subseteq St(\beta, \mathcal{U})$ , since  $\beta \in U$  for each  $U \in \mathcal{V}$ . Let  $F = F' \cup \{\beta\}$ . Then,  $X = St(F, \mathcal{U})$ , which shows that  $X$  is countably acc.

Next, we show that  $X$  is not starcompact. Let us consider the open cover

$$\mathcal{V} = \{\omega_1\} \cup \{O_\alpha(a_\alpha) : \alpha < \omega_1\}.$$

Let  $A$  be any finite subset of  $X$ . Then, there exists  $\alpha < \omega_1$  such that  $A \cap ((\alpha, \omega_1) \cup \{a_\beta : \beta > \alpha\}) = \emptyset$ . Choose  $\beta > \alpha$ . Then  $a_\beta \notin St(A, \mathcal{V})$ , since  $O_\alpha(a_\alpha)$  is only element of  $\mathcal{V}$  containing  $a_\alpha$  for each  $\alpha \in \omega_1$ . This shows that  $X$  is not starcompact.  $\square$

**Remark 2.7.** Pavlov [8] proved that a countably compact space need not be acc even if it is a normal  $T_2$ -space.

### 3. DISCRETE SUM AND SUBSPACES

We begin with a proposition which follows immediately from the definitions of a countably starcompact space and a countably acc space:

**Proposition 3.1.** *The discrete sum of a finite collection of countably starcompact (resp. countably acc) spaces is countably starcompact (resp. countably acc).*

It is well known that a closed subspace of a countably compact space is countably compact. However, a similar result does not hold for starcompactness, countable starcompactness and countable acc properties. In fact, the following example shows that these properties are not preserved by taking regular closed subspaces.

**Example 3.2.** There exists an acc  $T_1$ -space having a regular-closed subspace which is not countably starcompact.

*Proof.* Let  $S_1 = \omega \cup \mathcal{R}$  be the Isbell-Mrówka space [7], where  $\mathcal{R}$  is a maximal almost disjoint family of infinite subsets of  $\omega$  so that  $|\mathcal{R}| = \mathfrak{c}$ . Since  $S_1$  is not countably compact,  $S_1$  is not countably starcompact by Proposition 2.2.

Let  $S_2 = \mathfrak{c}^+ \cup A$ , where  $A$  is a set of cardinality  $\mathfrak{c}$ . We topologize  $S_2$  as follows:  $\mathfrak{c}^+$  has the usual order topology and is an open subspace of  $S_2$ , and a basic neighborhood of  $a \in A$  takes the form

$$G_\beta(a) = (\beta, \mathfrak{c}^+) \cup \{a\}, \quad \text{where } \beta < \mathfrak{c}^+.$$

We assume that  $S_1 \cap S_2 = \emptyset$ . Let  $\varphi : \mathcal{R} \rightarrow A$  be a bijection. Let  $X$  be the quotient space obtained from the discrete sum  $S_1 \oplus S_2$  by identifying  $r$  with  $\varphi(r)$  for each  $r \in \mathcal{R}$ . Let  $\pi : S_1 \oplus S_2 \rightarrow X$  be the quotient map. It is easy to check that  $\pi(S_1)$  is a regular-closed subset of  $X$ , however, it is not countably starcompact, since it is homeomorphic to  $S_1$ .

Next, we show that  $X$  is acc. For this end, let  $\mathcal{U}$  be an open cover of  $X$ . Let  $S$  be the set of all isolated points of  $\mathfrak{c}^+$  and let  $D = \pi(S \cup \omega)$ . Then,  $D$  is dense in  $X$  and every dense subspace of  $X$  includes  $D$ . Thus, it suffices to show that there exists a finite subset  $F$  of  $D$  such that  $X = St(F, \mathcal{U})$ . By the proof of Example 2.4,  $S_2$  is acc. Since  $\pi(S_2)$  is homeomorphic to the space  $S_2$ ,  $\pi(S_2)$  is acc, hence, there exists a finite subset  $F_0$  of  $\pi(S)$  such that  $\pi(S_2) \subseteq St(F_0, \mathcal{U})$ . On the other hand, since  $\pi(S_1)$  is homeomorphic to  $S_1$ , every infinite subset of  $\pi(\omega)$  has an accumulation point in  $\pi(S_1)$ . Hence, there exists a finite subset  $F_1$  of  $\pi(\omega)$  such that  $\pi(\omega) \subseteq St(F_1, \mathcal{U})$ . For, if  $\pi(\omega) \not\subseteq St(B, \mathcal{U})$  for any finite subset  $B \subseteq \pi(\omega)$ , then, by induction, we can define a sequence  $\{x_n : n \in \omega\}$  in  $\pi(\omega)$  such that  $x_n \notin St(\{x_i : i < n\}, \mathcal{U})$  for each  $n \in \omega$ . By the property of  $\pi(S_1)$  mentioned above, the sequence  $\{x_n : n \in \omega\}$  has a limit point  $x_0$  in  $\pi(S_1)$ . Pick  $U \in \mathcal{U}$  such that  $x_0 \in U$ . Choose  $n < m < \omega$  such that  $x_n \in U$  and  $x_m \in U$ . Then,  $x_m \in St(\{x_i : i < m\}, \mathcal{U})$ , which contradicts the definition of the sequence  $\{x_n : n \in \omega\}$ . Let  $F = F_0 \cup F_1$ . Then,  $X = St(F, \mathcal{U})$ . Hence,  $X$  is acc, which completes the proof.  $\square$

#### 4. MAPPINGS

It is well known that a continuous image of a countably compact space is countably compact (see [3]) and a continuous image of a starcompact space is starcompact (see [2]). Similarly, we have the following proposition.

**Proposition 4.1.** *A continuous image of a countably starcompact space is countably starcompact.*

*Proof.* Suppose that  $X$  is a countably starcompact space and  $f : X \rightarrow Y$  a continuous onto map. Let  $\mathcal{U}$  be a countable open cover of  $Y$ . Then,  $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{U}\}$  is a countable open cover of  $X$ . Since  $X$  is countably starcompact, there exists a finite set  $B \subseteq X$  such that  $St(B, \mathcal{V}) = X$ . Let  $F = f(B)$ . Then,  $F$  is a finite set of  $Y$  and  $St(F, \mathcal{U}) = Y$ . Hence,  $Y$  is countably starcompact.  $\square$

Matveev showed in [5, Example 3.1] that a continuous image of an acc space need not be acc. Now, we give an example showing that a continuous image of an acc  $T_1$ -space need not be countably acc.

**Example 4.2.** There exist an acc  $T_1$ -space  $X$  and a continuous map  $f : X \rightarrow Y$  onto a space  $Y$  which is not countably acc.

*Proof.* Let  $X_1 = (\omega + 1) \times \omega_1$  with the Tychonoff product topology, where both  $\omega + 1$  and  $\omega_1$  have the usual order topologies. Then,  $X_1$  is acc by [5, Theorem 2.3], since  $\omega + 1$  is a first countable, compact space and  $\omega_1$  is acc by [5, Theorem 1.8].

Let  $X_2 = \omega_1 \cup \omega$ . We topologize  $X_2$  as follows:  $\omega_1$  has the usual order topology and is an open subspace of  $X_2$ , and a basic neighborhood of  $n \in \omega$  takes the form

$$G_\beta(n) = (\beta, \omega_1) \cup \{n\}, \quad \text{where } \beta < \omega_1.$$

By the proof of Example 2.4,  $X_2$  is acc.

Let  $X = X_1 \oplus X_2$  be the discrete sum of  $X_1$  and  $X_2$ . Then,  $X$  is acc by Proposition 1.3 [5].

Let  $Y = X_1 \cup X_2$ . We topologize  $Y$  as follows:  $X_1$  is an open subspace of  $Y$ ; a basic neighborhood of a point  $\beta < \omega_1 \subseteq X_2$  takes the form

$$O_{\gamma,m}(\beta) = ([m, \omega] \times \omega_1) \cup (\gamma, \beta], \quad \text{where } \gamma < \beta \text{ and } m \in \omega.$$

a basic neighborhood of a point  $n \in \omega$  takes the form

$$O_\alpha(n) = ([n, \omega] \times \omega_1) \cup (\alpha, \omega_1) \cup \{n\}, \quad \text{where } \alpha < \omega_1;$$

To show that  $Y$  is not countably acc. Let  $D = \omega \times \omega_1$ . Then,  $D$  is dense in  $Y$ . Therefore, it is sufficient to show that there exists a countable open cover  $\mathcal{V}$  of  $Y$  such that  $St(A, \mathcal{V}) \neq Y$  for any finite subset  $A$  of  $D$ . Let us consider the countable open cover

$$\mathcal{V} = \{X_1 \cup \omega_1\} \cup \{O_0(n) : n \in \omega\}.$$

Let  $A$  be any finite subset of  $D$ . Then, there exists a  $n \in \omega$  such that  $([n, \omega] \times \omega_1) \cap A = \emptyset$ . Hence,  $n \notin St(A, \mathcal{V})$ , since  $O_0(n)$  is the only element of  $\mathcal{V}$  such that  $n \in O_0(n)$  for each  $n \in \omega$ , which shows that  $Y$  is not countably acc.

Let  $f : X \rightarrow Y$  be the identity map. Then,  $f$  is continuous. This completes the proof.  $\square$

Recall from [5] or [6] that a continuous mapping  $f : X \rightarrow Y$  is *varpseudopen* provided  $\text{int}_Y f(U) \neq \emptyset$  for every nonempty open set  $U$  of  $X$ . In [5], it was proved that a continuous varpseudopen image of an acc space is acc. Similarly, we prove the following proposition.

**Proposition 4.3.** *A continuous varpseudopen image of a countably acc space is countably acc.*

*Proof.* Suppose that  $X$  is a countably acc space and  $f : X \rightarrow Y$  is a continuous varpseudopen onto map. Let  $\mathcal{U}$  be a countable open cover of  $Y$  and  $D$  a dense subspace of  $Y$ . Then,  $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{U}\}$  is a countable open cover

of  $X$ , and  $f^{-1}(D)$  is a dense subspace of  $X$  since  $f$  is a varpseudopen map. Hence, there exists a finite set  $B \subseteq f^{-1}(D)$  such that  $St(B, \mathcal{V}) = X$ . Let  $F = f(B)$ . Then,  $F$  is a finite set of  $D$  and  $St(F, \mathcal{U}) = Y$ , which shows that  $Y$  is a countably acc space.  $\square$

Now, we consider preimages. It is well known that a perfect preimage of a countably compact space is countably compact (see [3, Theorem 3.10.10]) but a perfect preimage of an acc space need not be acc (see [1, Example 3.2]). Now, we give an example showing that

- (1) a perfect preimage of a starcompact space need not be starcompact,
- (2) a perfect preimage of a countably starcompact space need not be countably starcompact, and
- (3) a perfect preimage of a countably acc space need not be countably acc.

Our example uses the Alexandorff duplicate  $A(X)$  of a space  $X$ : The underlying set of  $A(X)$  is  $X \times \{0, 1\}$ ; each point of  $X \times \{1\}$  is isolated and a basic open neighborhood of  $\langle x, 0 \rangle \in X \times \{0\}$  is a set of the form  $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1 \rangle\})$ , where  $U$  is an open neighborhood of  $x$  in  $X$ .

**Example 4.4.** There exists a perfect onto map  $f : X \rightarrow Y$  such that  $Y$  is an acc  $T_1$ -space, but  $X$  is not countably starcompact.

*Proof.* Let  $Y = \omega_1 \cup \omega$ . We topologize  $Y$  as follows:  $\omega_1$  has the usual order topology and is an open subspace of  $Y$ , and a basic neighborhood of  $n \in \omega$  takes the form

$$G_\beta(n) = (\beta, \omega_1) \cup \{n\}, \quad \text{where } \beta < \omega_1.$$

By the proof of Example 2.4,  $Y$  is an acc space.

Let  $X = A(Y)$  be the Alexandorff duplicate of  $Y$ . Then,  $X$  is not countably starcompact, since  $\omega \times \{1\}$  is countable discrete, open and closed in  $X$  and countable starcompactness is preserved by open and closed set.

Let  $f : X \rightarrow Y$  be the projection. Then,  $f$  is a perfect onto map. This completes the proof.  $\square$

## 5. PRODUCTS

It is well known that the product of a countably compact space and a compact space is countably compact. However, the product of an acc Tychonoff space with a compact  $T_2$ -space need not be acc (see [5, Example 2.2]). Also, in [4, Example 3], an example was given showing that the product of a starcompact  $T_1$ -space with a compact metric space need not be starcompact. Now, we show that the same example also shows that the product of a countably starcompact (resp. countably acc)  $T_1$ -space with a compact metric space need not be countably starcompact (resp. countably acc).

**Example 5.1** (Fleischman). There exist an acc  $T_1$ -space  $X$  and a compact metric space  $Y$  such that  $X \times Y$  is not countably starcompact.



*Proof.* Let  $X = \omega_1 \cup A$  and  $\omega_1 \cap A = \emptyset$ , where  $A = \{a_n : n \in \omega\}$  is a countable set. We topologize  $X$  as follows:  $\omega_1$  has the usual order topology and is an open subspace of  $X$ , and a basic neighborhood of each  $a_n \in A$  takes the form

$$G_\beta(a_n) = (\beta, \omega_1) \cup \{a_n\}, \quad \text{where } \beta < \omega_1.$$

Then,  $X$  is an acc  $T_1$ -space. By the proof of Example 2.4. Let  $Y = \omega + 1$  with the usual order topology. Then,  $Y$  is a compact metric space.

Next, we prove that  $X \times Y$  is not countably starcompact. Let  $U_n = [n, \omega_1) \cup \{a_n\}$  and  $V_n = (n, \omega]$  for each  $n \in \omega$ . Let

$$\mathcal{U} = \{U_n \times V_n : n \in \omega\} \cup \{X \times \{n\} : n \in \omega\}.$$

Then,  $\mathcal{U}$  is a countable open cover of  $X \times Y$ . Let  $F$  be a finite subset of  $X \times Y$ . Then, there exists a  $n \in \omega$  such that  $(X \times \{n\}) \cap F = \emptyset$ . Hence,  $\langle a_n, n \rangle \notin St(F, \mathcal{U})$ , since  $X \times \{n\}$  is the only element of  $\mathcal{U}$  such that  $\langle a_n, n \rangle \in X \times \{n\}$  for each  $n \in \omega$ . This completes the proof.  $\square$

**Remark 5.2.** By Example 5.1, we can see that

- (1) an open perfect preimage of a starcompact space need not be starcompact,
- (2) an open perfect preimage of a countably starcompact space need not be countably starcompact, and
- (3) an open perfect preimage of a countably acc space need not be countably acc.

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