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Extension of Compact Operators from DF-spaces to C(K) spaces

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Abstract. It is proved that every compact operator from a DF-space, closed subspace of another DF-space, into the space C(K) of continuous functions on a compact Hausdorff space K can be extended to a compact operator of the total DF-space.

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1. Introduction

Let E and X be topological vector spaces with E a closed subspace of X. We are interested in finding out when a continuous operator $T : E \to C(K)$ has an extension $\tilde{T} : X \to C(K)$, where $C(K)$ is the space of continuous real functions on a compact Hausdorff space K and $C(K)$ has the norm of the supremum. When this is the case we will say that (E, X) has the extension property. Several advances have been made in this direction, a basic resume and bibliography for this problem can be found in [5]. In this work we will focus in the case when the operator T is a compact operator. In [4], p.23 , it is proved that (E, X) has the extension property when E and X are Banach spaces and $T : E \to C(K)$ is a compact operator. In this paper we extend this result to the case when E and X are DF-spaces (to be defined below), for this, we use basic tools from topological vector spaces.

2. Notation and Basic Results in DF-spaces.

We will use basic duality theory of topological vector spaces. For concepts in topological vector spaces see [3] or [2]. All the topological vector spaces in this work are Hausdorff and locally convex.

Let (X, t) be a topological vector space and $E < X$ be a closed vector subspace. Let $X' = (X, t)'$, $E' = (E, t)'$ be the topological duals of X and E respectively.

A topological vector space (X, t) possesses a *a fundamental sequence of bounded sets* if there exists a sequence $B_1 \subset B_2 \subset \cdots$ of bounded sets in (X, t), such that every bounded set B is contained in some B_k.

We take the following definition from [3], p. 396.

Definition 2.1. *A locally convex topological vector space (X,t) is said to be a DF-space if*

(1) *it has a fundamental sequence of bounded sets, and*
(2) *every strongly bounded subset M of X' which is the union of countably many equicontinuous sets is also equicontinuous*

A quasi-barrelled locally convex topological vector space with a fundamental sequence of bounded set is always a DF-space. Thus every normed space is a DF-space. Later we will mention topological vector spaces which are DF-spaces but they are not normed spaces.

First, we state some theorems to be used in the proof of the main result.

If K is a compact Hausdorff topological space, we define, for each $k \in K$ the injective evaluation map $\hat{k} : C(K) \to \mathbb{R}$, $\hat{k}(f) = f(k)$ which is linear and continuous, that is $\hat{k} \in C(K)'$. Let $\hat{K} = \{\hat{k} \mid k \in K\} \subset C(K)'$ and $cch(\hat{K})$ the balanced, closed and convex hull of $\hat{K}$ (which is bounded).

Theorem 2.2. *With the notation above we have*

(1) *$\hat{K}$ is $\sigma(C(K)', C(K))$-compact and K is homeomorphic to $(\hat{K}, \sigma(C(K)', C(K))$. Here $\sigma(C(K)', C(K))$ denotes the weak-* topology on $C(K)'$.*
(2) *If $T : E \to C(K)$ is a compact operator then $A = \overline{T'(cch(\hat{K}))}^{\beta}$ is $\beta(E', E)$-compact. Here $\beta(E', E)$ is the strong topology on E', this topology is generated by the polars sets of all bounded sets of (E, t).*

Proof. See [1], p. 490. □

Theorem 2.3. *If (X, t) is a DF-space then $(X', \beta(X', X))$ is a Frechet space.*

Proof. See [3], p. 397 □

Theorem 2.4. *Let M be paracompact, Z a Banach space, $N \subset Z$ convex and closed, and $\varphi : M \to \mathfrak{F}(N)$ lower semicontinuous (l.s.c.) Then φ has a selection.*

Proof. See [6] □

In the above theorem, $\mathfrak{F}(N) = \{S \subset N : S \neq \varnothing,\ S \text{ closed in } N \text{ and convex}\}$; $\varphi : M \to \mathfrak{F}(N)$ is l.s.c. if $\{m \in M : \varphi(m) \cap V \neq \varnothing\}$ is open in M for every open V in N, and $f : M \to N$ is a *selection* for φ if f is continuous and $f(m) \in \varphi(m)$ for every $m \in M$.

Theorem above remains true if Z is only a complete, metrizable, locally convex topological vector space (see [7]).

3. Main Results

Lemma 3.1. *Let $A \subset E'$. If there is a continuous map*

$$f : (A, \sigma(E', E)) \to (X', \tau'), \qquad \sigma(X', X) \leq \tau' \leq \beta(X', X)$$

such that

(1) $f(a)|_E = a$ *and*

(2) $f(A)$ *is an equicontinuous subset of* X'.

Then every linear and continuous map $T : E \to C(K)$ has a linear and continuous extension $\tilde{T} : X \to C(K)$.

Proof. Let us define $\tilde{T} : X \to C(K)$ in the following way: for each $x \in X$, $\tilde{T}(x) : K \to \mathbb{R}$ is given by $\tilde{T}(x)(k) = f(T'(\hat{k}))(x)$. Here, $\hat{k}$ is the injective evaluation map defined before Theorem 2.2. It is easy to check that $\tilde{T}$ is linear and extends T.

First, let us show that $\tilde{T}(x) \in C(K)$ for each $x \in X$. For this let $O \subset \mathbb{R}$ be an open set. We have that $\tilde{T}(x)^{-1}(O) = T'^{-1}(f^{-1}(x^{-1}(O)))$. Since $x : X'[\sigma(X', X)] \to \mathbb{R}$, f and T' are all continuous maps with the *weak** topology, $\tilde{T}(x)^{-1}(O)$ is open in K. This proves that $\tilde{T}(x) \in C(K)$.

Let us check that $\tilde{T}$ is continuous. Let $\{x_\lambda\}_\Lambda \xrightarrow{t} 0$ in X, we need to show that $\{\tilde{T}(x_\lambda)\} \xrightarrow{||\cdot||_{C(K)}} 0$.

For this, let $\epsilon > 0$. By hypothesis $f(A)$ is a equicontinuous subset of X', so that, $\epsilon f(A)^\circ \subset X$ is a t-neighborhood of 0. Here $f(A)^\circ$ denotes de polar set of $f(A)$. Hence, there is $\lambda_0 \in \Lambda$ such that $x_\lambda \in \epsilon f(A)^\circ$ for all $\lambda \geq \lambda_0$. From part 2 of Theorem 2.2 we have $T'(\hat{K}) \subset A$, hence

$$|\tilde{T}(x_\lambda)(\hat{k})| = |f(T'(\hat{k}))(x_\lambda)| \leq \epsilon \text{ for all } \lambda \geq \lambda_0$$

This implies that

$$||\tilde{T}(x_\lambda)||_{C(K)} = \sup\{\, |f(T'(\hat{k}))(x_\lambda)| \,/\, k \in K\} \leq \epsilon \text{ for all } \lambda \geq \lambda_0$$

This proves that $\{\tilde{T}(x_\lambda)\} \xrightarrow{||\cdot||_{C(K)}} 0$. □

Let $i : E \to X$ be the inclusion map and $i' : X' \to E'$ the dual map of i, that is, if $y \in X'$, $i'(y) = y|_E$.

Let $\mathcal{P}(X') = \{Y \mid Y \neq \varnothing,\ Y \subset X'\}$ and define $\psi : E' \to \mathcal{P}(X')$ by $\psi(e') = \{$ extensions of e' to $X\}$. Notice that $y \in \psi(i'(y))$ for all $y \in X'$ and $\psi(e') \in \mathfrak{F}(X')$.

With this notation, we have

Proposition 3.2. *Let (E, t) and (X, t) be DF-spaces, with $E < X$ a closed subspace. If $\mathcal{O} \subset X'$ is a $\beta(X', X)$-open set then the set $\mathcal{U}_\mathcal{O} = \{z \in E' \mid \psi(z) \cap \mathcal{O} \neq \varnothing\}$ is an open set in $(E', \beta(E', E))$.*

Proof. Notice that $\mathcal{U}_\mathcal{O} = \{z \in E' \mid \psi(z) \cap \mathcal{O} \neq \varnothing\} = i'(\mathcal{O})$. By Theorem 2.3 $(X', \beta(X', X))$ and $(E', \beta(E', E))$ are Frechet spaces. By the Banach-Schauder theorem (see [3], p. 166), the map $i' : (X', \beta(X', X)) \to (E', \beta(E', E))$ is an open map. Since $i'(\mathcal{O})$ is open in E', $\mathcal{U}_\mathcal{O}$ is also open. □

Corollary 3.3. *Let (E, t) and (X, t) be DF-spaces, with $E < X$ a closed subspace. Let $A = \overline{T'(cch(\hat{K}))}^{\beta}$ be as in part 2 of Theorem (2.2) Then $\varphi : (A, \beta(E', E)) \to \mathcal{P}(X')$ given by $\varphi = \psi|_A$ is a lower semicontinuous function, X' provided with the strong topology $\beta(X', X)$.*

Proof. It follows from

$$\{z \in A \mid \varphi(z) \cap \mathcal{O} \neq \varnothing\} = \{z \in E' \mid \psi(z) \cap \mathcal{O} \neq \varnothing\} \cap A$$

and Proposition 3.2. □

With the notation in Corollary 3.3, we have

Proposition 3.4. *If (X,t) is a DF-space then $\varphi : (A, \beta(E', E)) \to \mathcal{P}(X')$ admits a selection, that is, there is a continuous function $f : (A, \beta(E', E)) \to (X', \beta(X', X))$ such that $f(a) \in \varphi(a)$.*

Proof. From Theorem 2.3, (X,t) DF-space implies $(X', \beta(X', X))$ Frechet. From Theorem 2.2, part 2, A is $\beta(E', E)$-compact, hence A is a paracompact set. By Corollary 3.3, φ is a lower semi continuous function, therefore, by Theorem 2.4, φ admits a selection. □

Theorem 3.5. *If (X,t) and the closed subspace E are DF-spaces then every compact operator $T : E \to C(K)$ has a compact extension $\tilde{T} : X \to C(K)$.*

Proof. Let A be as in Proposition 3.4 and $f : (A, \beta(E', E)) \to (X', \beta(X', X))$ a selection function. Since A is $\beta(E', E)$-compact and f is continuous, $f(A)$ is compact, hence $f(A)$ is an equicontinuous set. Let $\tilde{T}$ be the linear extension of T given in Lemma 3.1.

Let us prove that $\tilde{T}$ is a compact operator. For this, we need to show that there is a t-neighborhood V such that $\tilde{T}(V)$ is a relatively compact set.

Since $f(A) \subset X'$ is an equicontinuous set and X is a DF space, [2] (p. 260 and p. 214) tells us that there is $V \subset X$ a balanced, closed and convex t-zero-neighborhood such that $f(A) \subset V^{\circ}$ and the topologies $\beta(X', X)$ and $\rho_{V^{\circ}}$ coincide on $f(A)$. Here $\rho_{V^{\circ}}$ is the Minkowski functional of V°. In this case $\rho_{V^{\circ}}$ is a norm and $(X'_{V^{\circ}}, \rho_{V^{\circ}})$ is a Banach space.

By using the Arzela-Ascoli Theorem, we will show that $\tilde{T}(V) \subset C(K)$ is relatively compact.

First, $\tilde{T}(V)$ is pointwise bounded because, for each $x \in V$ and $k \in K$, $|\tilde{T}(x)(k)| = |f(T'(\hat{k}))(x)| \leq 1$ since $f(A) \subset V^{\circ}$.

Now let us prove that $\tilde{T}(V)$ is equicontinuous in $C(K)$.

Choose and fix $k_0 \in K$ and $\epsilon > 0$. Since the chain of functions

$$K \xrightarrow{\hat{}} \hat{K} \xrightarrow{T'} (A, \beta(X', X)) \xrightarrow{f} (f(A), \beta(X', X))$$

is continuous, given a β-neighborhood W of $f(T'(\hat{k_0}))$ on $f(A)$, there exists $O \subset K$ neighborhood of k_0 such that $k \in O \Rightarrow f(T'(\hat{k})) \in W$. Since $\rho_{V^{\circ}}|_{f(A)} = \beta(X', X)|_{f(A)}$, we can say that

$$k \in O \Rightarrow \rho_{V^{\circ}}\Big(f(T'(\hat{k})) - f(T'(\hat{k_0}))\Big) < \epsilon$$

For each $x \in X$, $x : (X'_{V^\circ}, \rho_{V^\circ}) \to \mathbb{R}$ is linear and continuous, moreover, $|x'(x)| \leq ||x||_{\rho_{V^\circ}} \rho_{V^\circ}(x')$ for all $x' \in X'$, where

$$||x||_{\rho_{V^\circ}} = \sup\{|x'(x)| \mid x' \in V^\circ\}$$

If $x \in V$, $||x||_{\rho_{V^\circ}} \leq 1$. Therefore, for every $k \in O$ and every $x \in V$

$$\left|(f(T'(\hat{k})) - f(T'(\hat{k_0}))(x)\right| \leq ||x||_{\rho_{V^\circ}} \rho_{V^\circ}\Big(f(T'(\hat{k})) - f(T'(\hat{k_0}))\Big) \leq (1)(\epsilon)$$

This proves that $\tilde{T}(V)$ is equicontinuous in $C(K)$ and, by the Arzela-Ascoli Theorem, $\tilde{T}(V)$ is relatively compact which means that $\tilde{T}$ is a compact operator. □

In [3] (p. 402) it is shown that the topological inductive limit of a sequence of DF-spaces is a DF-space. In particular, if (E_n) is a sequence of Banach spaces such that E_n is a proper subspace of E_{n+1}, its inductive limit is DF-space. This inductive limit is not metrizable (see [8] p. 291). For this kind of spaces, Theorem 3.5 can be applied, i.e., given a fixed n, a compact operator $T : E_n \to C(K)$ can be extended to a compact operator of the inductive limit.

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