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Criteria of strong nearest-cross points and strong best approximation pairs

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Abstract. The concept of strong nearest-cross point (strong n.c. point) is introduced, which is the generalization of strong uniqueness of best approximation from a single point. The relation connecting to localization is discussed. Some criteria of strong n.c. points are given. The strong best approximation pairs are also studied.

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1. Introduction

In [6], [5], [9] the first author of the paper studied the nearest cross points (in short, n.c. points) of two subsets of a normed space. More precisely, let G and F be two disjoint subsets of a normed space X. A point $y_0 \in G$ is called a n.c. point of G to F if $\rho(y_0, F) = \rho(G, F)$, where $\rho(G, F) = \inf_{y \in G, x \in F} \rho(y, x)$, $\rho(y, x) = \|x - y\|$ is the norm of $x - y$ in the space X. Moreover, if $x_0 \in F$ satisfies $\rho(x_0, y_0) = \rho(F, G)$, we say that (x_0, y_0) is a best approximation pair of F and G. For details, one can see [4]. Obviously, if (x_0, y_0) is a best approximation pair of F and G, then y_0 is a n.c. point of G to F, and x_0 is the best approximation of y_0 from F. The analogous result for x_0 also holds. However, the inverse is not true. If both n.c. points of F to G and G to F exist, a best approximation pair of F and G may not exist. But if n.c. points of G to F exist and F is a proximal set, then the best approximation pair of F and G exists. In [5], the author discussed the uniqueness of n.c. points (if it exists) and obtained that the n.c. point of G to F is unique if G is strict convex and F is convex. In this paper, we shall discuss a property which is stronger than being a n.c. point, which we will call a strong n.c. point. A strong n.c. point is

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the generalization of strong best approximation in a single best approximation problem. For strong best approximation, one can see [3],[7],[2] in detail.

The organization of the paper is as follows. In Section 2, we will give the definition of strong n.c. point. In Section 3, we shall discuss the criteria of strong nearest cross point. In Section 4, we shall discuss strong n.c. points and strong best approximation pairs by way of the concept of cusp. And we shall give more examples about the relation between strong best approximation pairs and strong n.c. points.

Finally, we declare that we will work in complex norm spaces in this paper and use the following notation. Let X be a normed space. Denote by X^* the dual space of X. For a complex number u, we shall write $Re\, u$ to denote the real part of u. If F denotes a subset of a normed space X, then $\|F\| = \sup_{x\in F} \|x\|$.

2. Definition of strong n.c. point

Definition 2.1. *Let F and G be two disjoint sets, $y_0 \in G$ and a constant r, $0 < r < 1$. If the condition*

$$\rho(y,F) - \rho(y_0,F) \geq r\rho(y,y_0) \tag{1}$$

holds for every $y \in G$, then y_0 is called a strong n.c. point of G to F.

Notice that a strong nearest point is a n.c. point. In fact, since $\rho(y,F) - \rho(y_0,F) \geq 0$ for every $y \in G$, we have $\rho(G,F) = \rho(y_0,F)$. Then, for every $y \in G, y \neq y_0$, $\rho(y,F) > \rho(y_0,F)$, thus the n.c. point is unique. In Definition 2.1, the constant r, $r < 1$ holds automatically because $|\rho(y,F) - \rho(y',F)| \leq \rho(y,y')$ always holds. In fact, to say y_0 is a strong n.c. point, it suffices to remark that it exists a sufficiently small r, such that (1) holds. If F is a singleton x_0, y_0 is a strong n.c. point of G to F, then y_0 is the strong (unique) best approximation of x_0 from G; see [3], [7], [2].

Definition 2.2. *Consider F, G, r, y_0 as in Definition 2.1. If (1) holds only for $y \in V_0 \cap G$, V_0 a neighborhood of y_0, then we say y_0 is a local strong n.c. point of G to F.*

Obviously, if y_0 is a strong n.c. point then y_0 is a local strong n.c. point, but the converse does not hold in general, as the following example shows.

Example 2.3. In the Euclidean space $\mathbb{R}^2$, let $F = \{(\xi,\eta) : (\xi-2)^2+\eta^2 = 1, \xi \geq 2\}$, $G = \{(\xi,\eta) : \xi^2 + \eta^2 = 1, \xi \leq 0\}$. If $y_0 = (0,1)$, then y_0 is a local strong n.c. point of G to$F$, since for every $y \in G$, y be near y_0, $\rho(y,F)$ is equivalent to $\rho(y,x_0) = 2 + \rho(y,y_0)$. $x_0 = (2,1)$. So $\rho(y,F) - \rho(y_0,F)$ is equivalent to $\rho(y,y_0)$. But y_0 is not a strong n.c. point. In fact, choose y' suffice to $(0,-1)$, then $\rho(y_0,y')$ is to 2, $\rho(y_0,F) = 2$, thus $\rho(y',F) - \rho(y_0,F)$ converges to 0, so (1) does not hold. Moreover, (2,-1) is a n.c. point of F to G, (0,1) is a nearest cross point of G to F, and (2,-1), (0,1) are not strong n.c. points.

In the above example, F and G are not convex sets, but under convexity, we shall have different results. To state our results, we need the following lemma, which is well known.

Lemma 2.4. *Let (X, ρ) be a metric space and let $F \subset X$. Then the function $\rho(\cdot, F)$ is uniformly continuous. Moreover if F is a convex set, then $\rho(\cdot, F)$ is a convex function.*

Theorem 2.5. *Let F, G be convex sets, and let $y_0 \in G$. y_0 is a strong nearest cross point of G to F if and only if y_0 is a local strong n.c. point of G to F.*

Proof. As above stated, we only need to show that if y_0 is a local strong n.c. point, then y_0 is a strong n.c. point. Let V be a neighborhood of y_0, where y_0 is a strong n.c. point of $G \cap V$ to F. If y_0 is not a strong n.c. point of G to F, then for every $r_n \to 0$, there exist $y_n \in G, n = 1, 2, \dots$, such that $\rho(y_n, F) - \rho(y_0, F) < r_n \|y_n - y_0\|$. In the segment $[y_n, y_0]$, pick $z_n = \lambda_n y_n + (1 - \lambda_n) y_0$, $0 < \lambda_n < 1, \lambda_n \to 0$, such that for n sufficiently large, $z_n \in G \cap V$. By Lemma 2.4, $\rho(y, F)$ is a convex function. Thus,

$$\begin{aligned}\rho(z_n, F) &\le \lambda_n \rho(y_n, F) + (1 - \lambda_n)\rho(y_0, F) \\ &< \lambda_n r_n \|y_n - y_0\| + \lambda_n \rho(y_0, F) + (1 - \lambda_n)\rho(y_0, F) \\ &= \lambda_n r_n \|y_n - y_0\| + \rho(y_0, F).\end{aligned}$$

So $\rho(z_n, F) - \rho(y_0, F) < \lambda_n r_n \|y_n - y_0\|$. Since $\|y_n - y_0\| = 1/\lambda_n \|z_n - y_0\|$, $\rho(z_n, F) - \rho(y_0, F) \le r_n \|z_n - y_0\|$, $z_n \in G \cap V$. This contradicts the definition of a local strong n.c. point. This completes the proof. $\square$

In Theorem 2.5 we suppose F and G are convex sets. If one of them is not convex, then the result does not hold.

Example 2.6. In the Euclidean space $\mathbb{R}^2$, let $G = \{(\xi, \eta) : \xi^2 + \eta^2 = 1, \xi \le 0\}$. Notice that G is not convex and $F = \{(2, 0)\}$ is a singleton. Then $y_0 = (0, 1)$ is a local strong n.c. point of G to F, but y_0 is not a strong n.c. point of G to F. In fact, pick $y = (0, -1)$. Then $\rho(y, F) - \rho(y_0, F) = \sqrt{5} - \sqrt{5} = 0$, but $\|y - y_0\| = 2$.

Example 2.7. In the Euclidean space $\mathbb{R}^2$, let $G = \{(\xi, \eta) : -3 \le \xi \le 2,\ \eta = 0\}$ be a convex set, indeed, a segment, and $F = \{(\xi, \eta) : \xi^2 + \eta^2 = 25\}$ a non-convex set. Then $y_0 = (2, 0)$ is a local strong n.c. point, but y_0 is not a strong n.c. point, even if it is not a n.c. point. In fact, (-3,0) is a strong n.c. point.

3. Kolmogorov type and differential type criteria of strong n.c. points

In [6], the author gave sufficient and necessary conditions for a point to be a n.c. point by means of linear functions and differentials. Following the same idea we first obtain a sufficient condition for strong n.c. points.

Theorem 3.1. *Let F, G be disjoint sets, $y_0 \in G$, and r a constant, $0 < r < 1$. If y_0 satisfies one of the following conditions, then y_0 is a strong n.c. point of G to F.*
(i) for every $\epsilon > 0$, there exists $f^\epsilon \in X^*$, $\|f^\epsilon\| = 1$, such that $\inf\limits_{x \in F} Re\, f^\epsilon(x - y) = \rho(y_0, F)$ and for every $y \in G$, $Re\, f^\epsilon(y_0 - y) + \epsilon \ge r \|y_0 - y\|$ holds.

(ii) for every $\epsilon > 0$, and $y \in G$, there exists $f^{y,\epsilon} \in X^*$, $\|f^{y,\epsilon}\| = 1$, such that $\inf\limits_{x\in F} Re\, f^{y,\epsilon}(x - y_0) \geq \rho(y_0, F)$, and $Re\, f^{y,\epsilon}(y_0 - y) + \epsilon \geq r\|y_0 - y\|$.

Proof. Obviously, (i) implies (ii). So we only need to show (ii). For this, given $\epsilon > 0$, $y \in G$, and $f^{y,\epsilon}$ as in (ii), we have

$$r\|y_0-y\| \leq Re\, f^{y,\epsilon}(y_0-y)+\epsilon = Re\, f^{y,\epsilon}(y_0-x)+Re\, f^{y,\epsilon}(x-y)+\epsilon \leq \|x-y\|-\rho(y_0, F)+\epsilon.$$

If in the right side of the above inequality, we take the infimum over $x \in F$, we have $r\|y_0 - y\| \leq \rho(y, F) - \rho(y_0, F)$. Thus y_0 is the strong n.c. point of G to F. This completes the proof. □

One may ask immediately if either (i) or (ii) are necessary conditions. To answer this question, we begin with directional derivatives of the distance function $\rho(y, F)$ (for details one can see [6]). From Lemma 2.4, we know $\rho(y, F)$ is a convex function. Furthermore,

$$\rho'_+(y, h, F) = \lim_{t\to 0^+} \frac{\rho(y + th, F) - \rho(y, F)}{t}$$

exists for $h \neq 0 \in X$, and $\rho'_+(y, h, F)$ is a subadditive homogenous function in the variable h. Theorem 2.2 in [6] says that whenever y_0 is a n.c. point of G to F, $\rho'_+(y_0, y - y_0) \geq 0$ for all $y \in G$. In the following, we shall obtain a necessary condition for strong n.c. points.

Theorem 3.2. *Let F, G be two disjoint convex sets of X. If $y_0 \in G$ is a strong n.c. point of G to F, then*

$$(D_s)\ \rho'_+(y_0, y - y_0, F) \geq r\|y - y_0\|$$

for all $y \in G$, $r > 0$.

Proof. From Definition 2.1, $\rho(y, F) - \rho(y_0, F) \geq r\|y - y_0\|$. Put $h = y - y_0$,

$$\frac{\rho(y_0 + th, F) - \rho(y_0, F)}{t} \geq \frac{r\|th\|}{t} = r\|h\|.$$

Set t towards to 0 from right of 0, then $\rho'_+(y_0, h, F) \geq r\|y - y_0\|$. Note that $t > 0$, t is sufficiently small and $y_0 + th \in G$, since G is convex. This completes the proof. □

We shall consider whether condition (D_s) is sufficient. Some lemmas are required. To state them we give first some notation.

$\Gamma = \{\varphi \in X^* :\ \|\varphi\| = 1,\ Re\,\varphi(u) \leq \varphi(y_0),$ for $u \in H\}$, here, $H = \{u : \rho(u, F) \leq \rho(F, G)\}$.

$N_F = \{f \in X^* :\ \|f\| = 1,\ \inf\limits_{x\in F} Re\, f(x) = \inf\limits_{u\in F} \|u\|\}$.

The following lemma is Lemma 2.3 and Lemma 2.4 in [6].

Lemma 3.3. *For every $h \neq 0 \in X$, $\sup_{\varphi\in\Gamma} \frac{Re\,\varphi(h)}{\varphi(y_0)} = \frac{\rho'_+(y_0,h,F)}{\rho(y_0,F)}$, and $-\Gamma = N_{F-y_0}$.*

Theorem 3.4. *Let F be a subspace of X, G a convex subset of X, $F \cap G = \varnothing$ and $0 < r < 1$. If (D_s) holds, then the following condition holds*

(K_s) *for every $\epsilon > 0$, and every $y \in G$, there exists f_0 (depend on ϵ, y) $\in X^*$ with $\|f_0\| = 1$, such that $Re\, f_0(y_0 - y) + \epsilon \geq r\|y_0 - y\|$.*

Proof. By the condition (D_s), Theorem 3.1 and Lemma 3.3, we have

$$\sup_{f \in N_{F-y_0}} \frac{Re\ f(y_0 - y)}{Re\ f(y_0)} \geq \frac{r\|y - y_0\|}{\rho(y_0, F)}.$$

So for every ϵ, $y \in G$, there exists $f_0 \in N_{F-y_0}$, such that $\frac{Re\ f_0(y_0-y)}{Re\ f_0(y_0)} + \frac{\epsilon}{\rho(y_0,F)} \geq \frac{r\|y-y_0\|}{\rho(y_0,F)}$. Since $0 \in F$, the definition of N_{F-y_0}, $Re\ f_0(y_0) \geq \rho(y_0)$. This implies that $Re\ f_0(y_0 - y) + \epsilon \geq r\|y_0 - y\|$, so (K_s) holds. This completes the proof. $\square$

Finally, we give a condition (B_s) which is equivalent to (K_s) for general disjoint sets F, G. Before stating it, we require a notation. Let F be a subset of the space X. Denote

$$Q_F = \{u : Re\,\phi(u) \leq \|F\|, \text{ for all } \phi \in N_F\}.$$

Notice that Q_F is a cone type set including F. For if $z \in Q_F$, $x \in F, z' = x + t(z - x), t > 0$, then $z' \in Q_F$. Because for every $\phi \in N_F$, $Re\,\phi(z) \leq \|F\|$, $Re\,\phi(z') = Re\,\phi(x) + tRe\,\phi(z - x) = (1-t)Re\,\phi(x) + t\phi(z) \leq (1-t)\|F\| + t\|F\| = \|F\|$. Specially, if F is a singleton x_0, then Q_F is a cone including ball $B(0, \|x_0\|)$; see [3].

Theorem 3.5. *Let F, G be two disjoint sets, then (K_s) is equivalent to*

$$(B_s) \quad Q_{F-y_0} \cap cone(y_0 - G) \text{ is bounded,}$$

cone(E) denotes the cone closure of E.

Proof. Suppose (K_s) holds, for every $y \in G$. Then

$$\sup_{y \in N_{F-y_0}} Re\, f(y_0 - y) \geq r\|y_0 - y\|,\ 0 < r < 1.$$

We should conclude that $Q_{F-y_0} \cap cone(y_0 - G) \subset B(0, \|F - y_0\|/r)$. If this is not true, there exists $t > 0$, and some $y \in G$ such that $t(y_0 - y) \in Q_{F-y_0}$, $\|t(y_0 - y)\| > 1/r\|F - y_0\|$. From the definition of Q_{F-y_0}, for every $f \in N_{F-y_0}$, $|tRe\, f(y_0 - y)| \leq \|F - y_0\|$. If we take the supremum over all $f \in N_{F-y_0}$, then $\|F - y_0\| \geq tr\|y_0 - y\| > \|F - y_0\|$, which leads us to a contradiction.

Now if (B_s) holds, from the above statement, there exists a sufficient large number α, such that $Q_{F-y_0} \cap cone(y_0 - G) \subset intB(0, \alpha)$, $\alpha > 0$. So for every $y \in G, y_0 - y \in y_0 - G$, and $\frac{y_0-y}{\|y_0-y\|} \in cone(y_0 - G)$. But since $\alpha\frac{y_0-y}{\|y_0-y\|} \notin intB(0, \alpha)$, then $\alpha\frac{y_0-y}{\|y_0-y\|} \notin Q_{F-y_0}$. There exists $f_0 \in N_{F-y_0}$, such that $Re\, f_0(\frac{\alpha(y_0-y)}{\|y_0-y\|}) > \|F - y_0\|$. It means that $Re\, f_0(y_0 - y) > \frac{1}{\alpha}\|F - y_0\|\|y_0 - y\|$. If we put $r = \frac{\|F-y_0\|}{\alpha}$ and we take α large enough such that $0 < r < 1$, then (K_s) holds. $\square$

4. Cusp and strong best approximation pairs

In this section, we shall discuss the case when either strong nearest cross point or strong best approximation pair involve a cusp. In the end of this section, we shall give three examples of strong best approximation pairs. Let us begin with the definition of a cusp.

Definition 4.1. *Let G be a nonempty subset of X, and let ∂G be the boundary of G. Given $y_0 \in G \cap \partial G$, a point y_0 is called a cusp of G if there exists a hyperplane P supporting G at y_0, and $\frac{\rho(y,P)}{(y,y_0)} > \sigma > 0$ holds for every $y \in G$, where σ is a constant.*

Obviously, every cusp is a strongly exposed point. We say that y_0 is a strongly exposed point of G if there exists a hyperplane P supporting G at y_0, $x \in P, f(x) = c$ and such that if for every arbitrary $\epsilon > 0$, there exists $\delta > 0$, such that $|f(y) - f(y_0)| < \delta$ for $y \in G$, then $\rho(y, y_0) < \epsilon$. In fact, $\rho(y, P) = \frac{|f(y)-c|}{\|f\|}$. Without loss of generality, we assume $\|f\| = 1$, then $\rho(y, P) = |f(y) - f(y_0)|$. Since y_0 is a cusp of G, $\frac{\rho(y,P)}{\rho(y,y_0)} > \sigma$, $\rho(y, y_0) < |f(y) - f(y_0)|/\sigma$ for $y \in G$. For exposed points and strongly exposed points, one can see [1], [8] and the references there in.

Let F, G be two nonempty sets with $\rho(F, G) > 0$. We say that two hyperplanes P, Q regular separate F and G, if P, Q are parallel, F and G are in two outer sides of P and Q, and $\rho(P, Q) = \rho(F, G) = \rho(F, P) = \rho(G, Q)$. Furthermore, if $y_0 \in G$ is such that $\rho(y_0, F) = \rho(F, G)$, and $\frac{\rho(y,P)}{\rho(y,y_0)} > \delta > 0$, we say that y_0 is cusp of G to F. Obviously, if y_0 is a cusp of G to F, then y_0 is a cusp of G.

Theorem 4.2. *Let F, G be two disjoint convex sets and let $y_0 \in G$. If y_0 is a cusp of G to F, then y_0 is a strong n.c. point of G to F.*

Proof. From the definition of cusp of G to F, there exist hyperplanes P, Q separating F, G such that $\frac{\rho(y,P)}{\rho(y,y_0)} > \delta$ for every $y \in G$. Then, $\rho(y, F) - \rho(y_0, F) \geq \rho(y, Q) - \rho(y_0, F)$. Note that since P is parallel to Q, then $\rho(y, Q) = \rho(y, P) + \rho(P, Q)$ and $\rho(P, Q) = \rho(y_0, F)$. Thus $\rho(y, F) - \rho(y_0, F) = \rho(y, P) > \delta\rho(y, y_0)$. This means that y_0 is a strong n.c. point. □

Let h be the Hausdorff metric $h(F_0, F_1) = max\{\triangle(F_0, F_1), \triangle(F_1, F_0)\}$, where $\triangle(F_0, F_1) = \sup_{x\in F_0} \inf_{x'\in F_1} \|x - x'\|$. We have

Theorem 4.3. *(Freud type proposition) Suppose y_0 is a strong n.c. point of G to F. If y_1 is a strong n.c. point of G to F_1, then $\|y - y_0\| < 2/rh(F_0, F_1)$.*

Proof. According to Definition 2.1, there exists $0 < r < 1$, such that $r\|y_1 - y_0\| \leq \triangle(F, G) + \rho(y_1, F) - \rho(y_0, F)$. It is easy to see that $\rho(y, B) - \rho(y, A) \leq (A, B)$ holds for every y. Thus

$$\begin{aligned} r\|y_1 - y_0\| &\leq \triangle(F_1, F_0) + \rho(y_1, F_1) - \rho(y_0, F_0) \\ &\leq \triangle(F_1, F_0) + \rho(y_0, F_1) - \rho(y_0, F_0) \\ &\leq \triangle(F_1, F_0) + \triangle(F_0, F_1) \leq 2h(F_0, F_1), \end{aligned}$$

(to obtain the second inequality, we used that y_1 is a strong n.c. point of G to F). This completes the proof. □

In smooth normed spaces, F is a singleton and G is a normed subspace, then Theorem 4.3 is the result of Wulbert [3, page 95].

Theorem 4.4. *Let F, G be two disjoint sets with F convex and G a linear subspace and $y_0 \in G$. If $\rho(y, F)$ is Gateaux differential at y_0, then y_0 is not a strong n.c. point of G to F.*

Proof. Suppose y_0 is a strong n.c. point. By the Gateaux differentiable of $\rho(y, F)$, we have $\rho'_{-}(y_0, h, F) + \rho'_{+}(y_0, -h, F) = 0$, for $h \neq 0$. By Theorem 3.2, $\rho'_{+}(y_0, y - y_0, F) \geq r\|y - y_0\|$ holds for all $y \neq y_0,\ y \in G$, where $0 < r < 1$. If $y - y_0$ is either h or $-h$, we have $0 \geq 2r\|y - y_0\|$, which is a contradiction. The proof is complete. □

Definition 4.5. *Let F, G be two disjoint sets, $x_0 \in F$, and $y_0 \in G$. We say that (x_0, y_0) is a strong best approximation pair of F and G if there exist positive constants r, r' such that $\rho(y, y) - \rho(x_0, y_0) \geq r\|x - x_0\| + r'\|y - y_0\|$ for all $x \in F,\ y \in G$.*

Obviously, a strong best approximation pair of F and G is a best approximation pair of F and G; for best approximation pairs one can see [5] in detail. In the following, we shall discuss the connection between strong best approximation pairs and strong n.c. points.

Theorem 4.6. *If F, G are two disjoint sets, then (x_0, y_0) is the strong best approximation pair of F and G, if and only if, y_0 is a strong n.c. point of G to F, x_0 is the strong n.c. point of F to G, and (x_0, y_0) is a best approximation pair of F and G. In this case, it is unique.*

Proof. If (x_0, y_0) is the strong best approximation pair of F and G, by Definition 4.5, $\rho(x_0, y_0) = \rho(F, G) = \rho(y_0, G)$ and $\rho(x, y) - \rho(y_0, F) \leq r'\|y - y_0\|$ for all $x \in F$. If we take the infimum over all $x \in F$, we have $\rho(y, F) - \rho(y,F) \geq r'\|y - y_0\|$. Thus y_0 is the strong n.c. point of G to F. Similarly, x_0 is a strong n.c. point of F to G.

Conversely, since (x_0, y_0) is a best approximation pair of F and G, then $\rho(x_0, y_0) = \rho(F, G) = \rho(y_0, F) = \rho(x_0, G)$. Note that $\rho(x, y) \geq \rho(y, F)$ and y_0 is a strong n.c. point of G from F. So, $\rho(x, y) - \rho(x_0, y_0) \geq \rho(y, F) - \rho(y_0, F) \geq r'\|y - y_0\|$. Similarly, $\rho(x, y) - \rho(x_0, y_0) \geq r\|x - x_0\|$. Thus, $\rho(x, y) - \rho(x_0, y_0) \geq r/2\|x - x_0\| + r'/2\|y - y_0\|$. Therefore (x_0, y_0) is a strong best approximation pair. This completes the proof. □

Theorem 4.7. *Let F, G be two disjoint sets, $\rho(F, G) > 0$, and (x_0, y_0) a best approximation pair of F and G. If y_0 is a cusp of G to F, and x_0 is a cusp of F to G, then (x_0, y_0) is a strong best approximation pair.*

Proof. By the definition, y_0 is a cusp of G to F, and there exist parallel hyperplanes separating P, Q, such that F and G are in the outer side of P and Q, $y_0 \in P$, and $\rho(P, Q) = \rho(F, G) = \rho(P, F) = \rho(Q, G) = \rho(y_0, F)$. Obviously,

$\rho(x,y) \geq \rho(y,Q)$ for all $x \in F,\ y \in G$. Since (x_0, y_0) is a best approximation pair of F and G, $\rho(x_0,y_0) = \rho(y_0,F)$. So $\rho(x,y) - \rho(x_0,y_0) \geq \rho(y,Q) - \rho(y_0,F)$. Note that since $\rho(y_0,F) = \rho(P,Q)$, and $\rho(y,Q) = \rho(y,P) + \rho(P,Q)$, then $\rho(x,y) - \rho(x_0,y_0) > \rho(y,P) > \sigma'\|y-y_0\|$. Thus $\rho(x,y) - \rho(x_0,y_0) > \sigma/2\|x - x_0\| + \sigma'/2\|y-y_0\|$. This completes the proof. □

Finally, we shall give three examples.

Example 4.8. Denote C[0,1] be all continuous function f on [0,1] with norm $\|f\| = \max_{t\in[0,1]} |f(t)|$. In C[0,1], let $F = \{\mu t : -\infty < \mu < \infty\}$, $G = \{\lambda t^2 : \sqrt{2}+1 \leq \lambda \leq 5\}$. We consider n.c. points, best approximation pairs, strong n.c. points and strong best approximation pairs between F and G. Denote $x(t) = \mu t,\ y(t) = \lambda t^2$. Then

$$\rho(x,y) = \|\mu t - \lambda t^2\| = \begin{cases} \lambda - \mu, & \text{for} \quad \mu/\lambda \leq \sqrt{8}-2 \\ \mu^2/4\lambda, & \text{for} \quad \mu/lz \geq \sqrt{8}-2. \end{cases}$$

First we compute $\inf\limits_{-\infty<\mu<\infty} \|\mu t - \lambda t^2\|$. For fixed λ, $\|\mu t - \lambda t^2\|$ takes its infimum at $\mu = \mu_\lambda$. By the representation of $\|\mu t - \lambda t^2\|$, μ_λ satisfies $\lambda - \mu_\lambda = \mu_\lambda^2/4\lambda$, so $\mu_\lambda = (\sqrt{8}-2)\lambda$. Thus

$$\begin{aligned} \rho(F,G) &= \inf\nolimits_{\sqrt{2}+1\leq\lambda\leq 5} \|\lambda t^2 - (\sqrt{8}-2)\lambda t\| \\ &= (\sqrt{2}+1)\|t^2 - (\sqrt{8}-2)t\| \\ &= (\sqrt{2}+1)(3-\sqrt{8}) = \sqrt{2}-1. \end{aligned}$$

From above we obtain that $\|x_0 - y_0\| = \rho(F,G)$, when $x_0(t) = \mu_0 t,\ y_0(t) = \lambda_0,\ \lambda_0 = \sqrt{2}+1,\ \mu_0 = 2$. We declare that $y_0(t)$ is a strong n.c. point of G to F, since for every $y(t) = \lambda t^2 \in F$,

$$\rho(y,F) = \inf_{-\infty<\mu<\infty} \|\lambda t^2 - \mu t\| = \|\lambda t^2 - \mu_\lambda t\| = \lambda\|t^2 - (\sqrt{8}-2)t\| = (3-\sqrt{8})\lambda.$$

Therefore, $\rho(y_0,F) = \rho_0 = (3-\sqrt{8})\lambda_0$, $\rho(y,F) - \rho(y_0,F) = (3-\sqrt{8})(\lambda - \lambda_0) = r'\|y-y_0\|$, $r' = 3-\sqrt{8}$. Similarly, we have that x_0 is a strong n.c. point of F to G, since for every $x(t) = \mu t$,

$$\rho(x,G) = \inf_{\sqrt{2}+1\leq\lambda\leq 5} \|\lambda t^2 - \mu t\| = \inf_{\sqrt{2}+1\leq\lambda\leq 5} \begin{cases} \lambda - \mu, & \text{for} \lambda \geq \mu/\sqrt{8}-2 \\ \mu^2/4\lambda, & \text{for} \lambda \leq \mu/\sqrt{8}-2. \end{cases}$$

So, $\rho(x,G)$ takes the infimum at $\lambda_\mu = \sqrt{2}+1/2\mu$, and $\rho(x,G) = \lambda_\mu - \mu = \sqrt{2}/2\mu$. Thus, $\rho(x,G) - \rho)x_0,G) = \sqrt{2}+1/2\mu - \sqrt{2}-1 = r\|x-x_0\|$, where $r = \sqrt{2}-1/2$. By Theorem 3.4, we obtain that (x_0,y_0) is the strong best approximation pair of F and G.

The following example shows that a n.c. point always exists, but strong n.c. points can fail to exist.

Example 4.9. Denote $\ell_1^2 = \{(\xi,\eta) : \xi,\eta \in \mathbb{R}, \|(\xi,\eta)\| = |\xi| + |\eta|\}$. In ℓ_1^2 space, let $F = \{(\xi,\eta) : \xi = \eta\}$, $G = \{(\xi,\eta) : \eta = 0, 2 \leq \xi \leq 3\}$. It is easy to see that $y_0 = (2,0)$ is a n.c. point of G to F. But the best approximation of (2,0) from F is not unique. Thus the nearest cross points of G to F is not unique. Therefore a strong n.c. point of G to F does not exist .

At the end, we shall give an example, which shows that it is possible to find best approximation pairs which are not strong best approximation pairs.

Example 4.10. In the Euclidean $\mathbb{R}^2$ space, set $F = \{(\xi, \eta) : \xi^2 + \eta^2 \leq 1\}$, $G = \{(\xi, \eta) : \xi \geq 2, -\xi + 2 \geq \eta \geq \xi - 2\}$. It is easy to see that $x_0 = (1, 0),\ y_0 = (2, 0)$ is the unique best approximation pair of F and G. Denote an arbitrary point of F as $x = (\cos\theta, \sin\theta),\ 0 \leq \theta < 2\pi$. Put $y = (2 + \delta, \delta) \in G,\ \delta > 0$. Then $\rho_0 = \rho(x_0, y_0) = 1,\ \rho^2 = \rho^2(x, y) = (2 + \delta - \cos\theta)^2 + (\delta - \sin\theta)^2$. Setting $\theta \to 0,\ \delta \to 0$, then $\rho^2 - \rho_0^2$ is asymptotic to $2(\rho - \rho_0)$. $\|x - x_0\| = |\sin\theta|,\ \|y - y_0\| = \sqrt{2}\delta$. Since $\frac{\delta}{\rho^2 - \rho_0^2}$, and $\frac{\theta}{\rho^2 - \rho_0^2}$ are not bounded, then $\frac{\|y - y_0\|}{\rho - \rho_0}$, and $\frac{\|x - x_0\|}{\rho - \rho_0}$ are also not bounded. This means that (x_0, y_0) is not a strong best approximation pair. However, y_0 is a strong n.c. point of G to F.

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