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δ-closure, θ-closure and generalized closed sets

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Abstract. We study some new classes of generalized closed sets (in the sense of N. Levine) in a topological space via the associated δ-closure and θ-closure. The relationships among these new classes and existing classes of generalized closed sets are investigated. In the last section we provide an extensive and more or less complete survey on separation axioms characterized via singletons.

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1. Introduction and Preliminaries

Let (X, τ) be a topological space. Recall that a point $x \in X$ is said to be in the δ-closure (resp. θ-closure) of a subset $A \subseteq X$ (see [15]) if for each open neighbourhood U of x we have $int(cl(U)) \cap A \neq \varnothing$ (resp. $cl(U) \cap A \neq \varnothing$) . We shall denote the δ-closure (resp. θ-closure) of A by $cl_\delta(A)$ (resp. $cl_\theta(A)$). A subset $A \subseteq X$ is called *δ-closed* (resp. *θ-closed*) if $A = cl_\delta(A)$ (resp. $A = cl_\theta(A)$). The complement of a δ-closed (resp. θ-closed) set is called *δ-open* (resp. *θ-open*). It is very well known that the families of all δ-open (resp. θ-open) subsets of (X, τ) are topologies on X which we shall denote by τ_δ (resp. τ_θ). From the definitions it follows immediately that $\tau_\theta \subseteq \tau_\delta \subseteq \tau$. The space (X, τ_δ) is also called the semi-regularization of (X, τ) . A space (X, τ) is said to be semi-regular if $\tau_\delta = \tau$. (X, τ) is regular if and only if $\tau_\theta = \tau$. It should be noted that $cl_\delta(A)$ is the closure of A with respect to (X, τ_δ). In general, $cl_\theta(A)$ will not be the closure of A with respect to (X, τ_θ). It is easily seen that one always has $A \subseteq cl(A) \subseteq cl_\delta(A) \subseteq cl_\theta(A) \subseteq \overline{A}^{\,\theta}$ where $\overline{A}^{\,\theta}$ denotes the closure of A with respect to (X, τ_θ).

Definition 1.1. *A subset A of a space (X, τ) is called*

(i) α-closed *if* $cl(int(cl(A))) \subseteq A$,
(ii) α-open *if* $X \setminus A$ *is α-closed, or equivalently, if* $A \subseteq int(cl(int(A)))$,

(iii) semi-closed *if* $int(cl(A)) \subseteq A$,
(iv) semi-open *if* $X \setminus A$ *is semi-closed, or equivalently, if* $A \subseteq cl(int(A))$,
(v) preclosed *if* $cl(int(A)) \subseteq A$,
(vi) preopen *if* $X \setminus A$ *is preclosed, or equivalently, if* $A \subseteq int(cl(A))$,
(vii) β-closed *if* $int(cl(int(A))) \subseteq A$,
(viii) β-open *if* $X \setminus A$ *is* β*-closed, or equivalently, if* $A \subseteq cl(int(cl(A)))$.

For a subset A of (X, τ) the α-closure (resp. semi-closure, preclosure, β-closure) of A is the smallest α-closed (resp. semi-closed, preclosed, β-closed) set containing A. These closures are denoted by $cl_\alpha(A)$, $cl_s(A), cl_p(A)$ and $cl_\beta(A)$, respectively. It is known that $cl_\alpha(A) = A \cup cl(int(cl(A)))$, $cl_s(A) = A \cup int(cl(A))$, $cl_p(A) = A \cup cl(int(A))$ and $cl_\beta(A) = A \cup int(cl(int(A)))$.

For the sake of completeness, a subset A of (X, τ) is called *regular open* (resp. *regular closed*, *nowhere dense*) if $A = int(cl(A))$ (resp. $A = cl(int(A))$, $int(cl(A)) = \varnothing$). It is well known that the family of regular open subsets of (X, τ) form a base for τ_δ .

In 1970, N. Levine [11] defined a subset A of a space (X, τ) to be *generalized closed* (briefly, *g-closed*) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$. By considering other generalized closures or classes of generalized open sets, numerous additional notions analogous to Levine's g-closed sets have been introduced. We refer the reader to [1] for further details.

In 2001, Cao, Greenwood and Reilly [3] provided a general framework to deal with these notions by introducing the concept of a qr-closed set. For convenience it is useful to denote closed (resp. semi-closed, preclosed) by τ-closed (resp. s-closed, p-closed), and $cl(A)$ by $cl_\tau(A)$ for a subset $A \subseteq X$. Similarly, open (resp. semi-open, preopen) are denoted by τ-open (resp. s-open, p-open). If $\mathcal{P} = \{\tau, \alpha, s, p, \beta\}$ and $q, r \in \mathcal{P}$ then a subset $A \subseteq X$ is called qr-closed if $cl_q(A) \subseteq U$ whenever $A \subseteq U$ and U is r-open. Using this notation, a set A is g-closed if and only if it is $\tau\tau$-closed, and most types of generalized closed sets can be captured within this notation. One basic result (Theorem 2.5 in [3]) says that if $q, r \in \mathcal{P}$ then every qr-closed subset of (X, τ) is q-closed if and only if each singleton of X is either q-open or r-closed.

The aim of this paper is to continue the discussion initiated in [3] by considering the expanded family $\mathcal{P}^* = \{\tau, \alpha, s, p, \beta, \delta, \theta\}$. It is easily observed that Theorem 2.5 in [3] still remains valid.

Remark 1.2. If $q, r \in \mathcal{P}^*$ then every qr-closed subset of (X, τ) is q-closed if and only if each singleton of X is either q-open or r-closed.

So far we are aware of three relevant notions of generalized closed sets that have appeared in the literature for the δ-topology. A subset A of a space (X, τ) is called (i) δg*-closed* [6] if $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$, (ii) $g\delta$*-closed* [5] if $cl_\tau(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_\delta$, (iii) δg^**-closed* [5] if $cl_\delta(A) \subseteq U$

whenever $A \subseteq U$ and $U \in \tau_\delta$. In terms of the qr-closed notation of [3], (i) is equivalent to $\delta\tau$-closed, (ii) is equivalent to $\tau\delta$-closed and (iii) is equivalent to $\delta\delta$-closed.

The most important notion of generalized closed set involving the θ-topology is due to Dontchev and Maki. $A \subseteq X$ is said to be (iv) *θg-closed* [7] if $cl_\theta(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$. Clearly, (iv) is equivalent to $\theta\tau$-closed.

2. $\delta\theta$-closed sets and $\theta\delta$-closed sets

We shall investigate what happens when we mix δ and θ in the context of generalized closed sets.

Definition 2.1. *A subset A of a space (X, τ) is called*

(i) $\delta\theta$-closed, *if $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_\theta$,*
(ii) $\theta\delta$-closed, *if $cl_\theta(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_\delta$.*

Remark 2.2. Obviously every δ-closed (resp. θ-closed) set is $\delta\theta$-closed (resp. $\theta\delta$-closed). Since $\tau_\theta \subseteq \tau_\delta \subseteq \tau$, every $\theta\delta$-closed set is $\delta\theta$-closed. If $x \in U$ and $U \in \tau_\theta$ then there is $V \in \tau$ such that $x \in V \subseteq cl(V) \subseteq U$. Since $cl(V)$ is δ-closed we have $cl_\delta(\{x\}) \subseteq U$, i.e. every singleton in any space is always $\delta\theta$-closed. Now let X be an infinite set and $p \in X$. Let τ be the topology on X consisting of X and all subsets of X not containing p . If $x \neq p$ then $\{x\}$ is δ-open and $cl(\{x\}) = \{x, p\} \subseteq cl_\theta(\{x\})$. Thus $\{x\}$ is $\delta\theta$-closed but fails to be $\theta\delta$-closed.

Clearly every δg-closed subset is δg^*-closed, and every δg^*-closed subset is $\delta\theta$-closed. Moreover, every θg-closed subset is $\theta\delta$-closed. Consider the space (X, τ) in Remark 2.2. If $x \neq p$ then $\{x\}$ is $\delta\theta$-closed but obviously not δg^*-closed. Now let X be an infinite set and τ be the cofinite topology on X. Then $\tau_\theta = \tau_\delta = \{\varnothing, X\}$ hence every subset of X is $\theta\delta$-closed. If A is a proper cofinite subset of X then A is $\theta\delta$-closed but not θg-closed.

Remark 1.2 suggests the consideration of the following properties as candidates for possibly new separation properties.

Definition 2.3. *A space (X, τ) satisfies property*

(i) $\mathcal{A}$ if every $\delta\theta$-closed set is δ-closed, i.e. each singleton is either δ-open or θ-closed,
(ii) $\mathcal{B}$ if every $\theta\delta$-closed set is θ-closed, i.e. each singleton is either θ-open or δ-closed.

We shall see, however, that we do not obtain any new separation axioms. Recall that a space (X, τ) is said to be $T_{1/2}$ [8] if each singleton is either open or closed. (X, τ) is called *weakly Hausdorff* [13] (resp. *almost weakly Hausdorff* [6]) if (X, τ_δ) is T_1 (resp. $T_{1/2}$). We also mention the folklore result that a space (X, τ) is Hausdorff if and only if (X, τ_θ) is T_1 if and only if (X, τ_θ) is $T_{1/2}$ if and only if (X, τ_θ) is T_0.

Theorem 2.4. *For a space (X,τ) the following are equivalent:*

(a) *(X,τ) is Hausdorff,*
(b) *(X,τ) satisfies $\mathcal{A}$,*
(c) *(X,τ) is almost weakly Hausdorff and δ-closed singletons are θ-closed.*

Proof. (a) $\Rightarrow$ (b): If (X,τ) is Hausdorff then (X,τ_θ) is T_1, i.e. singletons are θ-closed. Thus (X,τ) satisfies $\mathcal{A}$.
(b) $\Rightarrow$ (a): If (X,τ) satisfies $\mathcal{A}$ then, by Remark 2.2, each singleton is either δ-clopen or θ-closed. Hence (X,τ_θ) is T_1 and thus (X,τ) is Hausdorff.
(b) $\Rightarrow$ (c): Suppose that (X,τ) satisfies $\mathcal{A}$. Then each singleton is clearly either δ-open or δ-closed, i.e. (X,τ) is almost weakly Hausdorff. If $\{x\}$ is δ-closed then $\{x\}$ is either δ-clopen or θ-closed, hence always θ-closed.
(c) $\Rightarrow$ (b): This is obvious. □

Theorem 2.5 ([14])**.** *For a space (X,τ) the following are equivalent:*

(a) *(X,τ) is weakly Hausdorff,*
(b) *(X,τ) satisfies $\mathcal{B}$.*

Proof. (a) $\Rightarrow$ (b): If (X,τ) is weakly Hausdorff then each singleton is δ-closed. Hence (X,τ) satisfies $\mathcal{B}$.
(b) $\Rightarrow$ (a): This follows from the fact that each θ-open singleton must be clopen. □

3. Separation axioms characterized via singletons

Remark 1.2 suggests to characterize the topological spaces in which each singleton is either q-open or r-closed, where $q,r \in \mathcal{P}^* = \{\tau,\alpha,s,p,\beta,\delta,\theta\}$. We shall first present what is already known and then answer the remaining cases. To do this we need some further preparation.

Observation 3.1. *Let (X,τ) be a space and let $x \in X$.*

(a) *$\{x\}$ is either preopen or nowhere dense* [10]*,*
(b) *$\{x\}$ is either open or preclosed,*
(c) *$\{x\}$ is open $\Leftrightarrow$ $\{x\}$ is α-open $\Leftrightarrow$ $\{x\}$ is semi-open,*
(d) *$\{x\}$ is preopen $\Leftrightarrow$ $\{x\}$ is β-open,*
(e) *$\{x\}$ is nowhere dense $\Rightarrow$ $\{x\}$ is α-closed and thus semi-closed, preclosed and β-closed,*
(f) *$\{x\}$ is semi-closed $\Leftrightarrow$ $\{x\}$ is nowhere dense or regular open.*

Definition 3.2. *A space (X,τ) is said to be*

(i) semi-T_1 *(resp.* pre-T_1*, β-T_1) if each singleton is semi-closed (resp. preclosed, β-closed),*
(ii) a $T_{3/4}$ space [6] *if each singleton is either δ-open or closed,*
(iii) semi-$T_{1/2}$ *if each singleton is either semi-open or semi-closed,*
(iv) feebly T_1 [12] *if each singleton is either nowhere dense or clopen,*
(v) T_{gs} [2] *if each singleton is either preopen or closed.*

The table below exhibits what is already known in the literature and what has been obtained in Section 2. It has to be read in the following way: each column represents a q-open set and each row represents a r-closed set. According to Observation 3.1 we only need to consider $q \in \{p, \tau, \delta, \theta\}$. A separation axiom (P) in a cell means that a space (X, τ) satisfies (P) if and only if each singleton is either q-open or r-closed. The symbol "$\surd$ " in a cell means that in any space each singleton is either q-open or r-closed. Finally, the symbol "?" in a cell means that the property in question has yet to be determined. We also observe that "a.w. T_2" (resp. "w. T_2") means almost weakly Hausdorff (resp. weakly Hausdorff). The present entries in our table can easily be verified by Observation 3.1 and Definition 3.2. The result that (X, τ) is almost weakly Hausdorff if and only if each singleton is either open or δ-closed can be found in [5].

	preopen	open	δ-open	θ-open
β-closed	$\surd$	$\surd$	?	?
preclosed	$\surd$	$\surd$	?	?
semi-closed	$\surd$	semi-$T_{1/2}$	?	?
α-closed	$\surd$	?	?	?
closed	T_{gs}	$T_{1/2}$	$T_{3/4}$	?
δ-closed	?	a.w. T_2	a.w. T_2	w. T_2
θ-closed	?	T_2	T_2	T_2

As an immediate consequence of Observation 3.1 we note that a space (X, τ) is semi-$T_{1/2}$ if and only if each singleton is either α-closed or open.

Proposition 3.3 ([14]). *For a space (X, τ) the following are equivalent:*

(a) (X, τ) *is semi-*T_1 ,
(b) *each singleton is either θ-open or semi-closed,*
(c) *each singleton is either δ-open or semi-closed,*
(d) *each singleton is either δ-open or α-closed.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is obvious. (c) $\Rightarrow$ (d) follows from Observation 3.1 and (d) $\Rightarrow$ (a) is clear. □

By observing that a θ-open singleton must be clopen and Observation 3.1 we have that a space (X, τ) is feebly T_1 if and only if each singleton is either θ-open or α-closed. By a similar argument, (X, τ) is pre-T_1 if and only if each singleton is either θ-open or preclosed. In addition, (X, τ) is T_1 if and only if each singleton is either closed or θ-open.

Proposition 3.4 ([14]). *For a space (X, τ) the following are equivalent:*

(a) (X, τ) *is β-*T_1 ,
(b) *each singleton is either θ-open or β-closed,*
(c) *each singleton is either δ-open or β-closed,*
(d) *each singleton is either δ-open or preclosed.*

Proof. (a) ⇒ (b) ⇒ (c) is obvious. To show that (c) ⇒ (d) let $x \in X$ such that $\{x\}$ is β-closed. If $int(\{x\}) = \varnothing$ then $\{x\}$ is preclosed. Otherwise, $\{x\}$ is open and β-closed and so regular open, i.e. δ-open. (d) ⇒ (a) is clear. □

Definition 3.5. *A space* (X, τ) *is called*

(i) R_1 *if two points* x *and* y *have disjoint neighbourhoods whenever* $cl(\{x\}) \neq cl(\{y\})$,
(ii) subweakly T_2 [4] *if* $cl_\delta(\{x\}) = cl(\{x\})$ *for each* $x \in X$,
(iii) pointwise semi-regular *(briefly* p-semi-regular*)* [5] *if each closed singleton is* δ*-closed,*
(iv) pointwise regular [14] *if each closed singleton is* θ*-closed.*

Proposition 3.6. *Let* (X, τ) *be a space.*

(a) *If* $A \subseteq X$ *is preopen then* $cl(A) = cl_\theta(A)$,
(b) (X, τ) *is* R_1 *if and only if* $cl(\{x\}) = cl_\theta(\{x\})$ *for each* $x \in X$.

Proof. (a) The proof is straightforward, hence it is omitted. The proof of (b) is due to Jankovic [9]. □

Theorem 3.7. *For a space* (X, τ) *the following are equivalent:*

(a) *Each singleton is either* θ*-closed or preopen,*
(b) (X, τ) *is* T_{gs} *and* R_1*,*
(c) (X, τ) *is* T_{gs} *and pointwise regular.*

Proof. (a) ⇒ (b) : Suppose that each singleton is either θ-closed or preopen. Then (X, τ) clearly is T_{gs}. Let $x \in X$. If $\{x\}$ is preopen then $cl(\{x\}) = cl_\theta(\{x\})$ by Proposition 3.6. If $\{x\}$ is θ-closed then $\{x\} = cl_\theta(\{x\}) = cl(\{x\})$. Hence (X, τ) is R_1.
(b) ⇒ (c) : This follows immediately from Proposition 3.6.
(c) ⇒ (a) : Follows straightforward from the definitions. □

Theorem 3.8. *For a space* (X, τ) *the following are equivalent:*

(a) *Each singleton is either* δ*-closed or preopen,*
(b) (X, τ) *is* T_{gs} *and subweakly* T_2*,*
(c) (X, τ) *is* T_{gs} *and p-semi-regular.*

Proof. (a) ⇒ (b) : Suppose that each singleton is either δ-closed or preopen. Then (X, τ) clearly is T_{gs} . Let $x \in X$. If $\{x\}$ is preopen then $cl(\{x\}) = cl(int(cl(\{x\})))$, i.e. $cl(\{x\})$ is regular closed and so $cl(\{x\}) = cl_\delta(\{x\})$. If $\{x\}$ is δ-closed then obviously we have $cl(\{x\}) = cl_\delta(\{x\})$. Thus (X, τ) is subweakly T_2.
(b) ⇒ (c) ⇒ (a) is clear.

□

As our final result we are now able to present the complete table.

	preopen	open	δ-open	θ-open
β-closed	$\surd$	$\surd$	β-T_1	β-T_1
preclosed	$\surd$	$\surd$	β-T_1	pre-T_1
semi-closed	$\surd$	semi-$T_{1/2}$	semi-T_1	semi-T_1
α-closed	$\surd$	semi-$T_{1/2}$	semi-T_1	feebly T_1
closed	T_{gs}	$T_{1/2}$	$T_{3/4}$	T_1
δ-closed	T_{gs} + subweakly T_2	a.w. T_2	a.w. T_2	w. T_2
θ-closed	$T_{gs} + R_1$	T_2	T_2	T_2

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