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A countably compact free Abelian group whose size has countable cofinality

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Abstract. Based on some set-theoretical observations, compactness results are given for general hit-and-miss hyperspaces. Compactness here is sometimes viewed splitting into "κ-Lindelöfness" and "κ-compactness" for cardinals κ. To focus only hit-and-miss structures, could look quite old-fashioned, but some importance, at least for the techniques, is given by a recent result of Som Naimpally, to who this article is hearty dedicated.

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1. Introduction

In 1990, Tkachenko showed that there exists a countably compact group topology on the free Abelian group of size $\mathfrak{c}$, assuming the Continuum Hypothesis (CH). Dikranjan and Shakmatov (see [1]) proved that there is no countably compact group topology on any free group and asked the following question:

For which cardinals κ can the free Abelian group of size κ be endowed with a countably compact group topology?.

In [3], it was shown that it is consistent that $2^{\mathfrak{c}}$ is such a cardinal, and as a consequence, any infinite cardinal $\kappa \leq 2^{\mathfrak{c}}$ with $\kappa = \kappa^{\omega}$. For cardinals that do not satisfy $\kappa = \kappa^{\omega}$, a natural question due to van Douwen [2] needs to be addressed:

1.5 Question. If X is an infinite group (or homogeneous space) which is countably compact, is $|X|^{\omega} = |X|$? Is at least $\mathrm{cf}(|X|) \neq \omega$?

It was shown in [2] that the answer to van Douwen's question is yes under GCH; recently in [6], question 1.5 above was answered in the negative. However, the example contains convergent sequences and all its elements have order 2.

In this note, we obtain the following:

Example 1.1. It is consistent that $2^{\mathfrak{c}}$ is 'arbitrarily large' and that there exists a countably compact group topology on the free Abelian group of size λ for any $\lambda \in [\mathfrak{c}, 2^{\mathfrak{c}}]$. In particular, it is consistent that there are countably compact group topologies on a free Abelian group whose size has countable cofinality.

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2. Preliminaries

We will construct via forcing a free set $X = \{x_\beta : \beta < \kappa\}$ such that for every $\gamma \in [\mathfrak{c}, \kappa)$, the group generated by $\{x_\beta : \beta < \gamma\}$ is countably compact. An element of the group generated is given by $\sum_{\xi \in \operatorname{dom} F} F(\xi) x_\xi$, where F is a function whose domain is a finite subset of κ and the range is $\mathbb{Z}$.

A sequence $\{g_n : n \in \omega\}$ in the group G generated by X can be coded as $\{\mathcal{F}(n) : n \in \omega\}$, where $\operatorname{dom} \mathcal{F}(n) \in [\kappa]^{<\omega}$ and $\operatorname{rng} \mathcal{F}(n) \subseteq \mathbb{Z}$. So $g_n = \sum_{\xi \in dom \mathcal{F}(n)} \mathcal{F}(n)(\xi) x_\xi = x_{\mathcal{F}(n)}$.

For the properties of denseness that will be required later on, it will be useful to work with particular sequences (see [5]):

Definition 2.1. *Let $\mathcal{F} : \omega \longrightarrow \bigcup_{E \in [\kappa]^{<\omega} \setminus \{\varnothing\}} (\mathbb{Z} \setminus \{0\})^E$, where $\mathcal{F}$ is $1-1$.*

We say that $\mathcal{F}$ is a sequence of type I if $\{\operatorname{dom} \mathcal{F}(n) : n \in \omega\}$ is faithfully indexed.

We say that $\mathcal{F}$ is a sequence of type II if there exists E such that $\operatorname{dom} \mathcal{F}(n) = E$ for every $n \in \omega$ and there exists $\mu \in E$ such that $\{\mathcal{F}(n)(\mu) : n \in \omega\}$ is faithfully indexed.

Lemma 2.2. *Let $\{a_n : n \in \omega\}$ be a sequence in $G = \langle \{x_\beta : \beta < \kappa\} \rangle$ that does not contain a constant subsequence. Then there exists a sequence $\mathcal{F}$ of type I or type II such that $\{x_{\mathcal{F}(n)} : n \in \omega\} \subseteq \{a_n : n \in \omega\}$*

Proof. Easy exercise □

Thus, from the lemma above, it suffices to show that every sequence in G coded by a sequence of type I or type II has an accumulation point.

Let $\mathcal{T}$ be the set of opens subarcs in $\mathbb{T}$. We say that $\Phi : D_p \longrightarrow \mathcal{T}$ has finite support if $\{\xi \in D_p : \Phi(\xi) \neq \mathbb{T}\}$ is finite. Given two such functions Φ and Φ^*, we say that $\Phi \leq \Phi^*$ if $\overline{\Phi(\xi)} \subseteq \Phi^*(\xi)$ for each $\xi \in D_p$.

Lemma 2.3. *If $\mathcal{F}$ is of type I or type II and ϕ^* is a function of finite support as above then given an open set W of $\mathbb{T}$ there exists $m \in \operatorname{dom} \mathcal{F}$ and $\Phi \leq \Phi^*$ of finite support such that $\sum_{\xi \in \operatorname{dom} \mathcal{F}(m)} \mathcal{F}(m)(\xi) \Phi(\xi) \subseteq W$.*

Proof. See [5]. □

Definition 2.4. *Let $\{\mathcal{F}_\beta : \beta < \kappa\}$ be an enumeration of all sequences of type I or type II in definition 2.1.*

3. The partial order

All the background information on forcing required in this paper can be found in [4].

Definition 3.1. *Let $\mathbb{P}$ be the family of all element $p = (\alpha_p, \{x_{p,\xi} : \xi \in D_p\}, \{A_{p,\zeta} : \zeta \in E_p)$ satisfying the following conditions:*

i) $\alpha_p \in \omega_1$;
ii) $D_p \in [\kappa]^\omega$;
iii) $x_{p,\xi} \in \mathbb{T}^{\alpha_p}$;
iv) $E_p \in [\kappa]^\omega$;
v) $A_{p,\zeta} \subseteq D_p \cap \mathfrak{c}$;
vi) $\operatorname{dom} \mathcal{F}_\zeta(n) \subseteq D_p$ for each $n \in \omega$, $\zeta \in E_p$;
vii) $x_{p,\xi}$ is an accumulation point of $\{x_{p,\mathcal{F}_\zeta(n)} : n \in \omega\}$ for each $\xi \in A_{p,\zeta}$ and each $\zeta \in E_p$;
viii) $\{x_{p,\xi} : \xi \in D_p\}$ is a faithfully indexed free set.

Given $p, q \in \mathbb{P}$, p extends q if

a) $\alpha_p \geq \alpha_q$;

b) $D_p \supseteq D_q$*;*
c) $x_{p,\xi}|_{\alpha_q} = x_{q,\xi}$ *for all* $\xi \in D_q$*;*
d) $E_p \supseteq E_q$*;*
e) $A_{p,\zeta} \supseteq A_{q,\zeta}$ *for all* $\zeta \in E_q$*.*

The following results will be proven in the next section:

Lemma 3.2. (CH) *The partial order* $\mathbb{P}$ *is countably closed and* ω_2-cc*.*

Lemma 3.3. *The set* $\mathcal{D}_{\alpha,\xi,\zeta,\mu} = \{p \in \mathbb{P} : \alpha_p \geq \alpha, \xi \in D_p, \zeta \in E_p \wedge A_{p,\zeta} \setminus \mu \neq \varnothing\}$ *is dense in* $\mathbb{P}$*, for each* $\alpha, \mu < \omega_1$ *and* $\xi, \zeta < \kappa$*.*

We are ready to prove the main result:

Proof. (of Example 1.1) Start with a model of GCH and let $\mathbb{G}$ be a generic set for the partial order $\mathbb{P}$. The forcing is cardinal preserving by Lemma 3.2, no new countable subsets of the ground model are added, CH holds and $\kappa = 2^{\mathfrak{c}}$.

For each $\xi, \zeta \in \kappa$, let $x_\xi = \bigcup_{p\in\mathbb{G}\wedge\xi\in D_p} x_{p,\xi}$ and $A_\zeta = \bigcup_{p\in\mathbb{G}\wedge\zeta\in E_p} A_{p,\zeta}$. By denseness of the sets in Lemma 3.3, each x_ξ is defined and is a function in $2^{\mathfrak{c}}$ and the set $\{x_\beta : \beta < \kappa\}$ is free. Also, each A_ζ is defined and has size $\mathfrak{c}$. Clearly, x_μ is an accumulation point of $\{x_{\mathcal{F}_\zeta(n)} : n \in \omega\}$ for each $\mu \in A_\zeta$.

Fix $\lambda \in [\mathfrak{c}, \kappa]$. Clearly the group generated by $\{x_\beta : \beta < \lambda\}$ is free Abelian and from Lemma 2.2 it will be countably compact as well. □

4. Some proofs

We start by proving an auxiliary lemma which will be used in the proofs of Lemmas 3.2 and 3.3.

Lemma 4.1. *Let* $r = (\alpha_r, \{x_{r,\xi} : \xi \in D_r\}, \{A_{r,\zeta} : \zeta \in E_r\})$ *satisfying all conditions in Definition 3.1 with the exception of condition viii). Then there exists a condition* $p \in \mathbb{P}$ *such that p 'extends' r, that is, conditions a) − e) are satisfied.*

Proof. We shall define p such that $\alpha_p = \alpha_r + \omega$, $D_p = D_r$, $E_p = E_r$, $A_{p,\zeta} = A_{r,\zeta}$ for all $\zeta \in E_p$ and $x_{p,\xi}|_{\alpha_r} = x_{r,\xi}$ for all $\xi \in D_p$.

List $\bigcup_{E\in[D_p]^{<\omega}} (\mathbb{Z} \setminus \{0\})^E$ in length ω as $\{F_n : n \in \omega\}$. We shall define by induction $x_{p,\xi}(\alpha + n)$ for each $\xi \in D_p$ and each $n \in \omega$ so that conditions *vii*) is satisfied (all other conditions are trivially satisfied) and $x_{p,F_n}(\alpha + n) \neq 0 \in \mathbb{T}$. Suppose that condition *vii*) is satisfied for $\{x_{p,\xi}|_{\alpha_r+n} : \xi \in D_p\}$. For each $\zeta \in E_p$ and $\mu \in A_{p,\zeta}$ let $B_{\zeta,\mu}$ be an infinite subset of ω such that $\{x_{p,\mathcal{F}_\zeta(m)}|_{\alpha_r+n} : m \in B_{\zeta,\mu}\}$ converges to $x_{p,\mu}|_{\alpha_r+n}$. Partition each $B_{\zeta,\mu}$ into $\{B_{\zeta,\mu,k} : k \in \omega\}$ each of infinite size and let $\{W_l : l \in \omega\}$ be a basis for $\mathbb{T}$. Enumerate all possible pairs $(B_{\zeta,\mu,k}, W_l)$ as $\{(B_{\zeta_t,\mu_t,k_t}, W_{l_t}) : k \in \omega\}$. We will define by induction a decreasing family $\Phi_t : \omega \longrightarrow \mathcal{T}$ for $t \in \omega$ of finite support as defined prior to Lemma 2.3.

Start with Φ_0 such that the support of Φ_0 contains $\operatorname{dom} F_n$ and $0 \notin \sum_{\xi\in\operatorname{dom} F_n} F_n(\xi)\Phi_0(\xi)$. Since $\mathcal{F}_\zeta|_{B_{\zeta_0,\mu_0,k_0}}$ is of type *I* or *II*, from Lemma 2.3, there exists $m_0 \in B_{\zeta_0,\mu_0,k_0}$ such that $\Phi_1 \leq \Phi_0$ of finite support such that $\sum_{\xi\in\operatorname{dom}\mathcal{F}_{\zeta_0}(m_0)} \mathcal{F}_\zeta(m_0)(\xi)\Phi_1(\xi) \subseteq W_0$.

By induction, using Lemma 2.3, define $\{\Phi_t : t \in \omega\}$ and $\{m_t : t \in \omega\}$ such that $\Phi_{t+1} \leq \Phi_t$ and $\sum_{\xi\in\operatorname{dom}\mathcal{F}_{\zeta_t}(m_t)} \mathcal{F}_{\zeta_t}(m_t)(\xi)\Phi_{t+1}(\xi) \subseteq W_t$. For each $\xi \in D_p$ define $x_{p,\xi}(\alpha+n) \in \bigcap_{t\in\omega} \overline{\Phi_t(\xi)}$. Then condition *vii*) is satisfied by $\{x_{p,\xi}|_{\alpha+n+1} : \xi \in D_p\}$ and $\sum_{\xi\in\operatorname{dom} F_n} F_n(\xi)x_{p,\xi}(\alpha + n) \neq 0$. □

Proof. (of Lemma 3.2) It is straightforward to see that the partial order is countably closed. Indeed, if $p_{n+1} \leq p_n$ for each $n \in \omega$, define $p_\omega = (\alpha_{p_\omega}, \{x_{p_\omega,\xi} : \xi \in D_{p_\omega}\}, \{A_{p_\omega,\zeta} : \zeta \in E_{p_\omega}\})$, where $\alpha_\omega = \sup\{\alpha_{p_n} : n \in \omega\}$, $D_{p_\omega} = \bigcup_{n\in\omega} D_{p_n}$, $x_{p_\omega,\xi} = \bigcup_{\xi\in D_{p_n}} x_{p_n,\xi}$, $E_{p_\omega} = \bigcup_{n\in\omega} E_{p_n}$, $A_{p_\omega,\zeta} = \bigcup A_{p_n,\zeta}$. Then $p_\omega \leq p_n$ for each $n \in \omega$.

We will now check that $\mathbb{P}$ is ω_2-cc. Let $\{p_\beta : \beta < \omega_2\}$ be a subset of $\mathbb{P}$. From CH and the Δ-system lemma, we conclude that there exists $I \in [\omega_2]^{\omega_2}$ and $D \in [\kappa]^{\leq\omega}$ such that $D_{p_\beta} \cap D_{p_\gamma} = D$ for any pair $\{\beta, \gamma\} \in [I]^2$. Without loss of generality, we can assume that there exists $\alpha < \omega_1$ such that $\alpha_{p_\beta} = \alpha$ for every $\beta \in I$. Also we can assume that $|\{x_{p_\beta,\xi} : \beta \in I\}| = 1$ for each $\xi \in D$.

Fix $\beta, \gamma \in I$. Let $D_r = D_{p_\beta} \cup D_{p_\gamma}$. Define $x_{r,\xi}$ as $x_{p_\beta,\xi}$ if $\xi \in D_{p_\beta}$ or $x_{p_\gamma,\xi}$ if $\xi \in D_{p_\gamma} \setminus D$. Set $E_r = E_{p_\beta} \cup E_{p_\gamma}$ and define $A_{r,\zeta}$ as $A_{p_\beta,\zeta}$ if $\zeta \in E_{p_\beta} \setminus E_{p_\gamma}$; $A_{p_\gamma,\zeta}$ if $\zeta \in E_{p_\gamma} \setminus E_{p_\beta}$ or $A_{p_\beta,\zeta} \cup A_{p_\gamma,\zeta}$ if $\zeta \in E_{p_\beta} \cap E_{p_\gamma}$. Note that $r = (\alpha, \{x_{r,\xi} : \xi \in D_r\}, \{A_{r,\zeta} : \zeta \in E_r\})$ may not be a condition in $\mathbb{P}$ but it can be extended to a condition p by applying Lemma 4.1. The condition p extends p_γ and p_β. Therefore, there are no antichains of size ω_2. □

Proof. (of Lemma 3.3) Let q be an arbitrary element of $\mathbb{P}$.

If $\zeta \in E_q$, define $E_r = E_q$. Let $\theta \in (\mu, \mathfrak{c})$ such that $\theta \notin D_q \cup \{\xi\}$, set $D_r = D_q \cup \{\theta\} \cup \{\xi\}$ and define $x_{q,\theta}$ as an accumulation point of the sequence $\{x_{q,\mathcal{F}_\zeta(n)} : n \in \omega\}$. If $\xi \in D_r \setminus D_q$ define $x_{q,\xi} = 0 \in \mathbb{T}^{\alpha_q}$.

If $\zeta \notin E_q$, choose $\theta \in (\mu, \mathfrak{c}) \setminus (D_q \cup \bigcup_{n\in\omega} \operatorname{dom} \mathcal{F}_\zeta(n) \cup \{\xi\})$ and set $D_r = D_q \cup (\bigcup_{n\in\omega} \operatorname{dom} \mathcal{F}_\zeta(n)) \cup \{\theta, \xi\}$. If $\psi \in D_r \setminus (D_q \cup \{\theta\})$, define $x_{q,\psi} = 0 \in \mathbb{T}^{\alpha_q}$. Define $x_{q,\theta}$ as an accumulation point of $\{x_{q,\mathcal{F}_\zeta(n)} : n \in \omega\}$.

In either case, define $A_{r,\rho} = A_{q,\rho}$ for each $\rho \in E_r \setminus \{\zeta\}$ and $A_{r,\zeta} = A_{q,\zeta} \cup \{\theta\}$. If $\alpha_q \geq \alpha$, let $x_{r,\eta} = x_{q,\eta}$ for each $\eta \in D_r$; otherwise, let $x_{r,\eta} = x_{q,\eta} \cup \{(\beta, 0) : \alpha_q \leq \beta < \alpha\}$ for each $\eta \in D_r$. Set $\alpha_r = max\{\alpha, \alpha_q\}$.

The set $r = (\alpha_r, \{x_{r,\eta} : \eta \in D_r\}, \{A_{r,\rho} : \rho \in E_r\})$ satisfies all the conditions to be an element of $\mathbb{P}$ with the possible exception of condition *viii*). Applying the Lemma 4.1, there exists a condition p 'below' r. Such condition p will be an element of $D_{\alpha,\xi,\zeta,\mu}$ and below q. □

Note: Independently from this note, D. Dikranjan and D. Shakmatov produced a model of $\mathrm{ZFC} + \mathrm{CH}$ with $2^{\mathfrak{c}}$ "arbitrarily large" and, in this model they obtained a characterization of Abelian groups of size κ with $\kappa \leq 2^{\mathfrak{c}}$ wich admit a countably compact group topology without non–trivial convergent sequences. Using forcing they constructed a group monomorphism $\pi : \mathbb{Q}^{(\kappa)} \oplus (\mathbb{Q}/\mathbb{Z})^{(\kappa)} \longrightarrow \mathbb{T}^{\omega_1}$ such that for every *almost n– torsion* subset E from $\mathbb{Q}^{(\kappa)} \oplus (\mathbb{Q}/\mathbb{Z})^{(\kappa)}$, $\pi(E)$ is HFD (hereditarily finally dense) in $\mathbb{T}[n]^{\omega_1}$ and has a cluster point in $\pi(\mathbb{Q}^{(\kappa)} \oplus (\mathbb{Q}/\mathbb{Z})^{(\kappa)})$.

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