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On classes of T_0 spaces admitting completions

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Abstract. For a given class **X** of T_0 spaces the existence of a subclass **C**, having the same properties that the class of complete metric spaces has in the class of all metric spaces and non-expansive maps, is investigated. A positive example is the class of all T_0 spaces, with **C** the class of sober T_0 spaces, and a negative example is the class of Tychonoff spaces. We prove that **X** has the previous property (i.e., admits completions) whenever it is the class of T_0 spaces of an hereditary coreflective subcategory of a suitable supercategory of the category **Top** of topological spaces. Two classes of examples are provided.

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1. Introduction

Let **Met** be the category of metric spaces and non-expansive maps and let **CMet** be the full subcategory of complete metric spaces. It is well known (see e.g. [14]) that

(1) for every metric space X there exist a complete metric space X^* and a dense embedding $\gamma : X \to X^*$ such that, for every complete metric space Y and non-expansive map $f : X \to Y$ there exists a (unique) non-expansive map $f^* : X^* \to Y$ for which $\gamma \circ f^* = f$;

(2) if $f : X \to Z$ is a dense embedding into a complete metric space Z then Z coincides, up to isometries, with X^*.

Since in **Met** dense non-expansive maps are epimorphisms, (1) says that **CMet** is epi-embedding reflective in **Met**. It is also well known that the category of compact Hausdorff spaces is embedding-epireflective in the category of Tychonov spaces, being here X^* the Stone-Čech compactification of X. However no property (2) is fulfilled by the latter construction. In this sense we can say

*Dedicated to Professors Miroslav Hušek and Gerhard Preuss on their sixtieth birthday that property (2) distinguishes completions from compactifications (see [6] and [5] for a general treatment of categorical completions).

It is also known (see e.g. [6]) that the category $\mathbf{Sob_o}$ of sober T_0 spaces (every irreducible closed set is the closure of a unique point) is epi-embedding reflective in the category $\mathbf{Top_o}$ of all T_0 spaces and that property (2) is fulfilled (i.e., $\mathbf{Sob_o}$ is *firmly epireflective* in $\mathbf{Top_o}$ in the terminology of [6]).

On the other hand no non trivial epireflective subcategory of **Top** consisting of Hausdorff spaces admits a firm epireflection (cf. [6] Example 1.8(2)).

The aim of this paper is to give sufficient conditions for a class of T_0 objects of a large enough topological category, **SSet**, whose objects are called *affine sets*, to admit a firm epireflection or, as we shall say, to *admit completions*.

For that we introduce and we study in Section 2 the category **SSet** of affine sets and affine maps which properly contains the category **Top** of topological spaces and continuous maps.

In Section 3, we analyze the extension from **Top** to **SSet** of the ordinary closure, the b-closure and the Zariski closure (the last two do not coincide in **SSet**, while they are the same in **Top**).

In Section 4 we restrict our considerations to affine sets satisfying a mild condition (the corresponding full subcategory is denoted by **SSET**) and there we introduce and study the so called separated ($= T_0$) affine sets.

Section 5 contains the main result: a class of separated affine sets admits completions whenever it is the class of all separated affine sets of a subcategory of **SSET** which is stable under affine subsets, disjoint unions and quotients (i.e., of a hereditary coreflective subcategory of **SSET**).

Two methods to produce hereditary coreflective subcategories of **SSET** and of **Top** are considered. The first goes back to a paper of Diers [13] (from which some terminology is derived, e.g., affine set, Zariski closure, etc.) and the second goes back to a paper of Hušek and the author [16].

Finally two proper classes of examples of classes of T_0 spaces admitting completions are provided:

(a) For each infinite regular cardinal α, the class of all T_0 spaces for which every point in the closure of a subset is also in the closure of a smaller subset of cardinality less than α. If α is a successor cardinal we obtain the T_0 spaces of tightness less than α.
(b) For each infinite cardinal α, those T_0 spaces for which the intersection of less than α open sets is open.

For categorical terminology see [1] and [20]. For General Topology we refer to [14].

All the subcategories considered in the paper are full and isomorphism closed. These are frequently identified with classes of objects defined by a given property.

2. The category **SSet** of affine sets

An *affine set* over the two point set $S = \{0,1\}$ is a pair $(X,\mathcal{U})$, where X is a set and $\mathcal{U}$ is a subset of the power set $\mathcal{P}(X)$. An *affine map* from $(X,\mathcal{U})$ to $(Y,\mathcal{V})$ is a function $f : X \to Y$ such that $f^{-1}(V) \in \mathcal{U}$ for every $V \in \mathcal{V}$. **SSet** will denote the *category of affine sets* (over S) and affine maps. The functional isomorphic description of **SSet** is as follows: objects are pairs $(X,\mathcal{A})$ where X is a set and $\mathcal{A}$ is a subset of the power set S^X and the morphisms from $(X,\mathcal{A})$ to $(Y,\mathcal{B})$ are functions $f : X \to Y$ such that $\beta \circ f \in \mathcal{A}$ whenever $\beta \in \mathcal{B}$.

Both descriptions of **SSet** will be utilized throughout the paper. We will denote by $F : \mathbf{SSet} \to \mathbf{Set}$ the obvious forgetful functor.

Proposition 2.1. $(\mathbf{SSet}, F)$ *is a topological category.*

Proof. To show that every F-structured source admits a unique initial lift, let X be a set, $\{(Y_i,\mathcal{U}_i) : i \in I\}$ a family of affine sets and $\{f_i : X \to Y_i : i \in I\}$ a family of functions. Then the subset $\mathcal{U} = \{(f_i)^{-1}(V) : V \in \mathcal{U}_i,\ i \in I\}$ is the unique initial structure in X for the given data. □

Thus an affine map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is *initial* (with respect to F) if and only if every $U \in \mathcal{U}$ is of the form $f^{-1}V$, $V \in \mathcal{V}$. As usual an initial monomorphism will be called an *embedding*. It is clear that every subset M of the underlying set X of an affine set $(X,\mathcal{U})$ carries as initial structure the family $\mathcal{V} = \{U \cap M : U \in \mathcal{U}\}$. In this case we say that $(M,\mathcal{V})$ is an *affine subset* of $(X,\mathcal{U})$.

Some consequences (cf. [1, 17]) of Proposition 2.1 are collected in the

Corollary 2.2.

(i) *Every F-structured sink $\{g_i : F(X_i,\mathcal{U}_i) \to Y,\ i \in I\}$ admits a unique final lift (i.e., there is a largest affine structure in Y for which all the g_i are affine maps);*

(ii) *In* **SSet** *the epimorphisms are the surjective affine maps and the monomorphisms are the injective affine maps;*

(iii) *In* **SSet** *the embeddings coincide with the regular monomorphisms (= equalizers of two affine maps);*

(iv) *Every affine map f admits an essentially unique (surjective, embedding)-factorization. That is: $f = m_f \circ e_f$ for some surjective affine map e_f and embedding m_f and, for every commutative square $f \circ e = m \circ g$ with e surjective and m embedding, there exists a (unique) affine map d such that $d \circ e = g$ and $m \circ d = f$;*

(v) **SSet** *is complete. Every limit is obtained as an initial lift of the corresponding limit in* **Set**.

In particular, the product of a family $\{(X_i,\mathcal{U}_i) : i \in I)\}$ of affine sets is the cartesian product X of the family $\{(X_i) : i \in I\}$ endowed with the affine structure $\mathcal{U} = \{\pi_i^{-1}(U) : U \in \mathcal{U}_i, i \in I\}$ where $\pi_i : X \to X_i$ are the projections;

(vi) **SSet** *is co-complete. Every colimit is obtained as a final lift of the corresponding colimit in* **Set***.*

Example 2.3. (*a*) Every closure (hence every topological) space (cf. [11, 9]), is an affine set and a function between closure (resp. topological) spaces is continuous if and only if it is affine. Thus both the categories **Top** of topological spaces and **Cl** of closure spaces are fully embedded in **SSet**;

(*b*) Affine spaces coincide with so called normal (Boolean) Chu spaces recently introduced by William Pratt as a generalization of Nielsen, Plotkin and Winskel's notion of event structure for modelling concurrent computation [19]. Moreover continuous maps between normal (Boolean) Chu spaces coincide with the affine maps. Thus **SSet** is a full subcategory of the category $\mathbf{Chu_S}$ of (Boolean) Chu spaces and continuous maps.

(*c*) Following L. M. Brown and M. Diker [3] a texture space is a pair $(X, \mathcal{U})$ where $\mathcal{U}$ is a subset of the power set $\mathcal{P}(X)$ which is a complete, completely distributive lattice with respect to the inclusion, which contains X and $\varnothing$, separates the points of X, and for which meet coincides with intersection and finite join with union. Thus texture spaces are particular affine spaces. In our context the natural morphisms between texture spaces are the affine maps. For suitable morphisms in the class of texture spaces see [4].

Remark 2.4. (1) Following the same lines as the functional description of **SSet**, in [15] the category **ASet**, for every set A, was considered. The above category coincides with the full subcategory of the category $\mathbf{Chu_A}$ of Chu spaces with respect to the set A, consisting of normal Chu spaces [19] and, for particular instances of A it contains the category **Fuz** of fuzzy topological spaces ($A = [0, 1]$) and the category **AP** of approach spaces ($A = [0, \infty]$) [15]. The categories of the form **ASet** fulfil both the proposition and the corollary above.

(2) The category **SSet** is not well-fibred even though every set admits a set of affine structures. In fact the empty set admits two affine structures and every one-point set admits four structures. This defect can be removed by assuming that every affine structure contains the empty and the whole set. We shall consider this full subcategory of **SSet**, denoted by **SSET**, in Section 4.

3. Kuratowski and Zariski closure

In this section we extend from **Top** to **SSet** the usual (Kuratowski) closure k, the b-closure in the form introduced by Baron [2] (to characterize the epimorphisms in the category $\mathbf{Top_0}$ of T_0-spaces), and the b-closure in the form considered by Skula [21]. The latter, called Zariski closure and denoted by z, in contrast with the situation in **Top**, does not coincide with Baron closure in **SSet**.

We recall that a *closure operator* c of a topological category (over **Set**) **X** is an assignment, to each subset M of (the underlying set of) any object X of **X**, of a subset $c_X M$ of X such that

(c1) $M \subset c_X M$;
(c2) $c_X M \subset c_X N$ whenever $M \subset N$;
(c3) *c-continuity.* For every $f : X \to Y$ in $\mathbf{X}$ and M subset of X, $f(c_X M) \subset c_Y(fM)$.

Note that by property $(c2)$

(c4) $c_X M \cup c_X N \subset c_X(M \cup N)$.

We will drop X in $c_X M$ when no confusion is possible.

The closure operator c is called

(c5) *idempotent* if $c_X(c_X M) = c_X M$;
(c6) *grounded* if $c_X \varnothing = \varnothing$;
(c7) *additive* if $c_X(M \cup N) = c_X M \cup c_X N$.
(c8) *hereditary* if, for every $M \subset Y \subset X$, $c_Y M = (c_X M) \cap Y$, where Y is endowed with the initial structure induced by the inclusion $Y \subset X$.

A subset $M \subset X$ is called *c-closed* (respectively *c-dense*) in X if $c_X M = M$ (respectively $c_X M = X$). A morphism $f : X \to Y$ is called *c-dense* if $f(X)$ is c-dense in Y and it is called *c-closed* if it sends c-closed subsets into c-closed subsets.

An object X is called

(1) *c-separated* if the diagonal $\Delta_X = \{(x, x) : x \in X\}$ is c-closed in the square X^2;
(2) *absolutely c-closed* if it is c-separated and it is c-closed in every c-separated object in which it can be embedded;
(3) *c-compact* if, for every object Y, the projection $p : X \times Y \to Y$ is c-closed;
(4) *c-connected* if the diagonal Δ_X is c-dense in X^2.

It is well known (e.g. see [14]) that in **Top**, if c is the ordinary closure, then c-separated means Hausdorff, absolutely c-closed means H-closed and c-compact means compact in the usual sense (no Hausdorff condition is included). The c-connected spaces coincide with the irreducible spaces, that is: any disjoint open sets U,V must satisfy $U = \varnothing$ or $V = \varnothing$ (cf. [8]).

For the general theory of closure operators we refer to [12, 10] and [7].

If $(X,\mathcal{U})$ is an affine set and $x \in X$ we will denote by $\mathcal{U}_x$ the family of all $U \in \mathcal{U}$ such that $x \in U$.

Let $(X,\mathcal{U})$ be an affine set and let $M \subset X$:

(a) the *Kuratowski closure* of M in $(X,\mathcal{U})$ is defined by

$$k_{(X,\mathcal{U})} M = \{x \in X : (\forall U \in \mathcal{U}_x)(\exists m \in M)(m \in U)\}$$

(b) the *Baron closure* of M in $(X,\mathcal{U})$ (cf. [2]) is defined by

$$\begin{aligned} b_{(X,\mathcal{U})} M &= \{x \in X : x \in k_{(X,\mathcal{U})}(M \cap k_{(X,\mathcal{U})}\{x\})\} \\ &= \{x \in X : (\forall U \in \mathcal{U}_x)(\exists m \in (U \cap M))(\forall V \in \mathcal{U}_m)(x \in V)\}; \end{aligned}$$

(c) the *Zariski closure* of M in $(X,\mathcal{U})$ (cf. [21]) is defined by

$$z_{(X,\mathcal{U})}M = \{x \in X : (\nexists\, U, V \in \mathcal{U})(U \cap M = V \cap M, x \in (U \smallsetminus V)\}.$$

Proposition 3.1.

(i) *Always* $b_{(X,\mathcal{U})}M \subset z_{(X,\mathcal{U})}M$ *and* $b_{(X,\mathcal{U})}M \subset k_{(X,\mathcal{U})}M$.

(ii) *If* $\varnothing \in \mathcal{U}$ *then* $z_{(X,\mathcal{U})}M \subset k_{(X,\mathcal{U})}M$.

(iii) *If* $\mathcal{U}$ *is a topology then* $z_{(X,\mathcal{U})}M \subset b_{(X,\mathcal{U})}M$.

Proof. (i). If x is not in $z_{(X,\mathcal{U})}M$ and $U, V \in \mathcal{U}$ are such that $U \cap M = V \cap M$ and $x \in (U \smallsetminus V)$, then $U \in \mathcal{U}_x$ and for every $m \in U \cap M$, by $m \in V$, we have $V \in \mathcal{U}_m$ while x is not in V. Consequently x does not belong to $b_{(X,\mathcal{U})}M$.

The inclusion $b_{(X,\mathcal{U})}M \subset k_{(X,\mathcal{U})}M$ is obvious.

(ii). Assume $\varnothing \in \mathcal{U}$ and x not in $k_{(X,\mathcal{U})}M$. Then there exists $U \in \mathcal{U}_x$ such that $U \cap M = \varnothing$, $\varnothing = V \in \mathcal{U}$, $U \cap M = V \cap M$ and $x \in (U \smallsetminus V)$; consequently x is not in $z_{(X,\mathcal{U})}M$.

(iii). Assume x not in $b_{(X,\mathcal{U})}M$. Then there exists $U \in \mathcal{U}_x$ such that, for each $m \in U \cap M$ there is a $V_m \in \mathcal{U}_m$ with x not in V_m. If $\mathcal{U}$ is a topology then the set $V = \bigcup\{V_m \cap U : m \in (M \cap U)\}$ belongs to $\mathcal{U}$ and both conditions $U \cap M = V \cap M$ and $x \in (U \smallsetminus V)$ are fulfilled; consequently x is not in $z_{(X,\mathcal{U})}M$. □

Theorem 3.2.

(i) *The Kuratowski, Baron and Zariski closure are idempotent and hereditary closure operators of* **SSet**.

(ii) *The closure operators k and b are grounded in $(X,\mathcal{U})$ if and only if $\mathcal{U}$ is a cover of X. If, in addition, the empty set belongs to $\mathcal{U}$, then z is grounded.*

(iii) *The closure operators k, b and z are additive in $(X,\mathcal{U})$ whenever $\mathcal{U}$ is stable under intersection of pairs.*

Proof. (i). Properties $(c1)$ and $(c2)$ are trivial and k-continuity of every affine map (i.e., property $(c3)$) directly follows from the defining property of affine map.

Property $(c3)$ for b: if $x \in b_{(X,\mathcal{U})}M$ then $x \in k_{(X,\mathcal{U})}(M \cap k_{(X,\mathcal{U})}\{x\})$ so that $fx \in k_{(Y,\mathcal{V})}(f(M \cap k_{(X,\mathcal{U})}\{x\}))$. Now, by k-continuity of f applied twice, $k_{(Y,\mathcal{V})}(f(M \cap k_{(X,\mathcal{U})}\{x\})) \subset k_{(Y,\mathcal{V})}(fM \cap f(k_{(X,\mathcal{U})}\{x\})) \subset k_{(Y,\mathcal{V})}(fM \cap k_{(Y,\mathcal{V})}\{fx\})$ consequently $fx \in b_{(Y,\mathcal{V})}(fM)$.

Property $(c3)$ for z: assume that y is not $z_{(Y,\mathcal{V})}(fM)$ and let $U, V \in \mathcal{V}$ such that $U \cap M = V \cap M$ and $y \in (U \smallsetminus V)$. Then for every $x \in f^{-1}y$ we have $x \in f^{-1}U$ while x is not in $f^{-1}V$ and $f^{-1}U \cap M = f^{-1}V \cap M$ which means that x is not in $z_{(X,\mathcal{U})}M$, consequently y is not in $f(z_{(X,\mathcal{U})}M)$.

The idempotency of k is clear. Let $x \in b_{(X,\mathcal{U})}(b_{(X,\mathcal{U})}M)$ which means $x \in k_{(X,\mathcal{U})}(b_{(X,\mathcal{U})}M \cap k_{(X,\mathcal{U})}\{x\})$. Then for every $U \in \mathcal{U}_x$, $U \cap b_{(X,\mathcal{U})}M \cap k_{(X,\mathcal{U})}\{x\} \neq \varnothing$. Let, for a fixed U, $y \in U \cap b_{(X,\mathcal{U})}M \cap k_{(X,\mathcal{U})}\{x\}$. For every $V \in \mathcal{U}_y$, by $y \in b_{(X,\mathcal{U})}M$, $V \cap M \cap k_{(X,\mathcal{U})}\{y\} \neq \varnothing$ and by $y \in k_{(X,\mathcal{U})}\{x\}$, $k_{(X,\mathcal{U})}\{y\} \subset k_{(X,\mathcal{U})}\{x\}$

consequently $V \cap M \cap k_{(X,\mathcal{U})}\{x\} \neq \varnothing$. Now $U \in \mathcal{U}_y$ consequently $x \in b_{(X,\mathcal{U})}M$. This shows that b is idempotent. For the idempotency of z note that if $U, V \in \mathcal{U}$ satisfy $U \cap M = V \cap M$ then also $U \cap z_{(X,\mathcal{U})}M = V \cap z_{(X,\mathcal{U})}M$, so that if x is not in $z_{(X,\mathcal{U})}M$ then there exist $U, V \in \mathcal{U}$ such that ($U \cap M = V \cap M$, hence) $U \cap z_{(X,\mathcal{U})}M = V \cap z_{(X,\mathcal{U})}M$ and $x \in (U \setminus V)$, consequently x is not in $z_{(X,\mathcal{U})}(z_{(X,\mathcal{U})}M)$.

The hereditariness directly follows from the fact that the affine structure in an affine subset Y of an affine set $(X, \mathcal{U})$ is $\{U \cap Y : U \in \mathcal{U}\}$.

(ii). $k_{(X,\mathcal{U})}\varnothing = \varnothing$ if and only if $\mathcal{U}$ is a cover is clear. Then the remaining part of (ii) follows from $k_{(X,\mathcal{U})}\varnothing = b_{(X,\mathcal{U})}\varnothing \subset z_{(X,\mathcal{U})}\varnothing$.

(iii). This simple example shows that b, z and k are not additive. Let $X = \{1, 2, 3\}$ and let $\mathcal{U} = \{\{0, 1\}, \{0, 2\}, \varnothing\}$. For each $M \subset X$, $z_{(X,\mathcal{U})}M \subset k_{(X,\mathcal{U})}M$, by $\varnothing \in \mathcal{U}$, $b_{(X,\mathcal{U})}\{i\} = k_{(X,\mathcal{U})}\{i\}$, for $i = 1, 2$, and $b_{(X,\mathcal{U})}\{1, 2\} = k_{(X,\mathcal{U})}\{1, 2\} = X$. Hence $z_{(X,\mathcal{U})}\{1, 2\} = X$.

For additivity, one inclusion is $(c4)$, and the other directly follows from the stability under intersection of pairs of $\mathcal{U}$. □

The next result follows from the idempotency and hereditariness of our closure operators (see [10, 12]).

Corollary 3.3. *Every affine map admits an essentially unique* (c*-dense,* c*-closed embedding*) *factorization for* $c = k, b, z$.

Proof. For a given $f : X \to Y$ denote by $e_f : X \to c_Y(fX)$ the codomain restriction of f to $c_Y(fX)$ and by $m_f : c_Y(fX) \to Y$ the inclusion map. By c idempotent, $c_Y(fX)$ is c-closed and, by c hereditary, e_f is c-dense. Thus $m_f \circ e_f$ is a (c-dense map, c-closed embedding)-factorization of f.

Let now $f \circ e = m \circ g$ ($e : X \to Y, m : Z \to T$) be a commutative square in **SSet**, with e a c-dense affine map and m a c-closed embedding, and let $y \in Y$. Since y is in the c-closure of eX in Y by assumption, then, by c-continuity of f, fy is in the c-closure of $f(eX) = (f \circ e)X$. By commutativity, fy is then in the c-closure of ($m(gX)$, hence) mZ. Now mZ is c-closed in T by assumption, so that $fy \in mZ$, consequently, by injectivity of m, there is unique $z \in Z$ such that $mz = fy$. Set $dy = z$. Then d is an affine map since $m \circ d = f$ and m is initial by assumption. Moreover d is the unique affine map satisfying both $m \circ d = f$ and $d \circ e = g$ since m is injective by assumption. □

Remark 3.4. It should be noted that in the previous proof we have not used the full power of hereditariness of our closures. Indeed what we need is that every subset is c-dense in its closure. That property, called *weak hereditariness*, is, in the presence of idempotency, weaker than hereditariness (see e.g., [12]).

4. Separated ($= T_0$) affine sets

In this section we will refer to the functional description of **SSet**

The extension of the ordinary closure k of **Top** to **SSet** has no interest for the development of such basic topological notions as separation, compactness, absolute closedness and connectedness. Indeed, by the form of products in **SSet** as explained in Section 2, an affine set is k-separated if and only if it is absolutely closed if and only if it has at most one point. Moreover every affine set is k-compact since every projection is k-closed and every affine set is k-connected since the diagonal of every affine set is k-dense in the square of the affine set.

The aim of the last two sections is to show that the above topological notions are not trivial in **SSet** if we refer to the Zariski (or Baron) closure.

For that we restrict our considerations to those affine sets whose affine structure contains the two constant functions **0** and **1**. The corresponding full subcategory, denoted by **SSET**, which is topological too, has many pleasant properties not shared by **SSet**. Among others,

1. it is well-fibred, i.e. every constant function is affine or, equivalently, the affine sets with at most one point have unique affine structure;
2. our closures are grounded there;
3. a nonempty affine set is *indiscrete* if and only if its structure consists of constant functions.

In what follows **S** will denote the two-point set $\{0,1\}$ endowed with the affine structure $\mathcal{A} = \{\mathbf{0}, \mathbf{1}, id_S\}$ ($\mathcal{U} = \{\varnothing, S, \{1\}\}$ in the subset description). **S** will be called the *Sierpinski affine set*.

An affine set $(X, \mathcal{A})$ is called *separated* if $\mathcal{A}$ separates the points of X.

$\mathbf{SSET_0}$ will denote the full subcategory of separated affine sets.

Clearly the Sierpinski affine set **S** is separated.

The following result has a trivial proof but plays an important role:

Lemma 4.1. *A function $f : X \to S$ is an affine map between $(X, \mathcal{A})$ and* **S** *if and only if $f \in \mathcal{A}$.*

Proposition 4.2. *For an affine set $(X, \mathcal{A})$ these are equivalent:*

(i) *$(X, \mathcal{A})$ is separated;*
(ii) *every affine map $f : \mathbf{I_2} \to (X, \mathcal{A})$ is constant, where $\mathbf{I_2}$ is a two-point indiscrete affine set;*
(iii) *Δ_X is z-closed in $(X, \mathcal{A}) \times (X, \mathcal{A})$;*
(iv) *Δ_X is b-closed in $(X, \mathcal{A}) \times (X, \mathcal{A})$;*
(v) *$(X, \mathcal{A})$ is an affine subset of a product of copies of the Sierpinski affine set* **S**.

Proof. (i) $\Leftrightarrow$ (ii). The existence of a non constant affine map from $\mathbf{I_2}$ to $(X, \mathcal{A})$ is equivalent to saying that there are distinct points x and y in X such that $\alpha(x) = \alpha(y)$ for every $\alpha \in \mathcal{A}$, which is equivalent to saying that $(X, \mathcal{A})$ is not separated.

(i) $\Rightarrow$ (iii). If $(x, y) \in X^2 \setminus \Delta_X$ then, by assumption, there is $\alpha \in \mathcal{A}$ such that $\alpha(x) \neq \alpha(y)$. Then the two affine maps $\alpha \circ p_1, \alpha \circ p_2 : X \times X \to X$ coincide in

Δ_X and do not coincide on (x, y) which says that (x, y) is not in the z-closure of Δ_X. Consequently Δ_X is z-closed.

(iii) $\Rightarrow$ (iv). It follows from Proposition 3.1 (i).

(iv) $\Rightarrow$ (ii). Assume $(X, \mathcal{A})$ is not separated and let $x \neq y$ in X such that, for each $\alpha \in \mathcal{A}$, $\alpha(x) = \alpha(y)$. Taking into account that the affine structure of $X \times X$ consists of functions of the form $\alpha \circ p_i$, $i = 1, 2$, it is easy to verify that the point (x, y) is in the b-closure of Δ.

(i) $\Rightarrow$ (v). Let $\mathbf{S}^{\mathcal{A}}$ be the product of $\mathcal{A}$ copies of $\mathbf{S}$ and let

$$\phi : (X, \mathcal{A}) \to \mathbf{S}^{\mathcal{A}}$$

be the map whose components are the elements of $\mathcal{A}$. By virtue of Lemma 4.1, ϕ is affine and initial, and it is (injective, hence) an embedding if (and only if) $(X, \mathcal{A})$ is separated.

(v) $\Rightarrow$ (i). Separation is clearly a productive and hereditary property. □

Remark 4.3. (1) The equivalence (i) $\Leftrightarrow$ (ii) in the above proposition says that the separated affine sets coincide with the so called T_0-objects of the well fibred topological category **SSET**, see [18]. In particular $\mathbf{SSET_o}$ is the largest epireflective non bireflective subcategory of **SSET**. Moreover (i) $\Leftrightarrow$ (v) says that **SSET** is simply cogenerated by the Sierpinski affine set **S**.

(2) The $\mathbf{SSET_o}$-reflection r of an affine set $(X, \mathcal{A})$ is the restriction to the image of the affine map ϕ. Indeed let $(Y, \mathcal{B})$ be a separated affine set, let

$$\psi : (Y, \mathcal{B}) \to \mathbf{S}^{\mathcal{B}}$$

be the canonical map, let $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be any affine map and let $f' : \mathbf{S}^{\mathcal{A}} \to \mathbf{S}^{\mathcal{B}}$ be the affine map associated to f. Then, denoting by $k : \phi(X) \to \mathbf{S}^{\mathcal{A}}$ the inclusion, we obtain the commutative square $(f' \circ k) \circ r = \psi \circ f$. Now r is surjective and ψ is an embedding since $(Y, \mathcal{B})$ is separated, so that, by Corollary 2.2 (iv) there exists a (unique) affine map $d : \phi(X) \to Y$ such that $d \circ r = f$.

(3) Since the $\mathbf{SSET_o}$-reflections are initial (see the proof of (i) $\Rightarrow$ (v) in Proposition 4.2) then the category **SSET** is *universal* in the sense of Marny [18].

Corollary 4.4.

(i) *In* $\mathbf{SSET_o}$ *the epimorphisms are precisely the* z*-dense affine maps and the regular monomorphisms are precisely the* z*-closed embeddings.*

(ii) *In* $\mathbf{SSET_o}$ *every affine map admits an essentially unique factorization by a* z*-dense* (*respectively* b*-dense*) *affine map followed by a* z*-closed* (*respectively,* b*-closed*) *embedding.*

Proof. (i). By Lemma 4.1 the morphisms from an affine set $(X, \mathcal{A})$ to the Sierpinski affine set are precisely the elements of the structure of $\mathcal{A}$. On the other hand our Zariski closure of a subset M is obtained by intersecting all the equalizers, of pairs in $\mathcal{A}$, containing M, so, equivalently intersecting equalizers of pairs of affine maps into **S**. Then, z being the regular closure operator induced by $\mathbf{S}$ hence, in virtue of (i) $\Leftrightarrow$ (v) for $\mathbf{SSET_o}$, it gives the epimorphisms and the regular monomorphisms (for that we use weak hereditariness) in $\mathbf{SSET_o}$ as in (i) (see [10]).

(ii). Every affine subset of a separated affine set is separated, so the statement follows from Corollary 3.3. □

5. Completions

An affine set $(X, \mathcal{A})$ is called *z-injective* if it is injective with respect to z-dense embeddings. That is: for every z-dense embedding $m : (M, \mathcal{C}) \to (Y, \mathcal{B})$ and affine map $f : (M, \mathcal{C}) \to (X, \mathcal{A})$ there exists an affine map $f' : (Y, \mathcal{B}) \to (X, \mathcal{A})$ such that $f' \circ m = f$.

By a standard argument z-injectivity is preserved by products.

Since embeddings are initial affine maps, by Lemma 4.1 the Sierpinski affine set is z-injective.

Proposition 5.1. *For a separated affine set* $(X, \mathcal{A})$ *these are equivalent:*

(i) $(X, \mathcal{A})$ *is* z*-injective;*
(ii) $(X, \mathcal{A})$ *is absolutely* z*-closed;*
(iii) *The canonical map* $\phi : (X, \mathcal{A}) \to \mathbf{S}^{\mathcal{A}}$ *is a* z*-closed embedding.*

Proof. (i) $\Rightarrow$ (ii). Let $(X, \mathcal{A})$ be separated and z-injective, let $k : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be an embedding with $(Y, \mathcal{B})$ separated and $m \circ e$ the (z-dense, z-closed embedding)-factorization of k (see Corollary 4.4 (ii)). Then, since $(X, \mathcal{A})$ is z-injective, there is an affine map g from the codomain of e into $(X, \mathcal{A})$ such that $g \circ e = 1_X$, which says that e is a section, hence an isomorphism since it is an epimorphism in $\mathbf{SSET_o}$.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Assume that the canonical map $\phi : (X, \mathcal{A}) \to \mathbf{S}^{\mathcal{A}}$ is a z-closed embedding, let $k : (M, \mathcal{C}) \to (Y, \mathcal{B})$ be a z-dense embedding and $f : (M, \mathcal{C}) \to (X, \mathcal{A})$ any affine map. Since the elements of $\mathcal{C}$ are the restrictions to M of those of $\mathcal{B}$, there is an affine map $f'' : \mathbf{S}^{\mathcal{B}} \to \mathbf{S}^{\mathcal{A}}$ such that $(f'' \circ \psi) \circ k = \phi \circ f$ consequently, by Corollary 4.4 (ii), there is an affine map $f' : (Y, \mathcal{B}) \to (X, \mathcal{A})$ satisfying, in particular, $f' \circ k = f$. □

We are now ready to prove that $\mathbf{SSET_o}$ admits completions.

Theorem 5.2. *The full subcategory* $\mathbf{CSSET_o}$ *of* $\mathbf{SSET_o}$ *consisting of all the absolutely* z*-closed affine sets is firmly epireflective in* $\mathbf{SSET_o}$.

Proof. The $\mathbf{CSSET_o}$-reflection s of a separated affine set $(X, \mathcal{A})$ is the restriction to the z-closure in $\mathbf{S}^{\mathcal{A}}$ of the image of the affine map ϕ. The proof uses the same argument as Remark 4.3 (2).

The property ($c2$) directly follows from the fact that the Sierpinski affine set is a z-injective cogenerator of $\mathbf{SSET_o}$. □

Recall that a subcategory $\mathbf{X}$ of a category $\mathbf{Y}$ is called *coreflective* in $\mathbf{Y}$ if for every object Y of $\mathbf{Y}$ there exist an object $\hat{Y}$ in $\mathbf{X}$ and a morphism $s : \hat{Y} \to Y$ such that, for every $X \in \mathbf{X}$ and morphism $f : X \to Y$ there exists a unique $f' : X \to \hat{Y}$ satisfying $s \circ f' = f$.

If $\mathbf{Y}$ is topological over **Set** (as is our **SSET**), then the coreflection maps s are bimorphisms, so we may assume, in our context, that $\widehat{(X, \mathcal{A})}$ is X endowed with an affine structure finer than or equal to $\mathcal{A}$ and that the coreflection maps are identities. If a coreflective subcategory of **SSET** is stable under affine subsets we shall say that it is *hereditarily coreflective.*

Let $\mathbf{X}$ be a hereditary coreflective subcategory of **SSET** and let us denote by $\mathbf{T_0X}$ the subcategory of its separated affine sets. We shall show that $\mathbf{T_0X}$ admits completions.

Lemma 5.3. *If* $\mathbf{X}$ *is a hereditarily coreflective subcategory of* **SSET** *then:*

(i) $\hat{\mathbf{S}}$ *is separated and* z*-injective in* $\mathbf{X}$.
(ii) $\mathbf{T_0X}$ *is cogenerated by* $\hat{\mathbf{S}}$ *in* $\mathbf{X}$.

Proof. (i). Since $\hat{\mathbf{S}}$ is a finer modification of $\mathbf{S}$ it is separated. Let $k : X \to Y$ be a z-dense embedding with $X, Y \in \mathbf{X}$ and let $f : Y \to \hat{\mathbf{S}}$ any affine map. Since $\mathbf{S}$ is z-injective there exists $f'' : Y \to S$ with $f'' \circ k = s \circ f$ $(s : \hat{\mathbf{S}} \to S)$. Consequently, by the universal property of coreflections, there is $f' : Y \to S$ such that $s \circ f' = f''$ so that $f' \circ k = f$, by injectivity of s.

(ii). It is clear that a function from an affine set in $\mathbf{X}$ to $\mathbf{S}$ is affine if and only if it is so into $\hat{\mathbf{S}}$. Consequently the canonical map $\phi : (X, \mathcal{A}) \to \hat{\mathbf{S}}^{\mathcal{A}}$ remains an embedding whenever $(X, \mathcal{A})$ is in $\mathbf{X}$ and the product $\hat{\mathbf{S}}^{\mathcal{A}}$ is taken in $\mathbf{X}$. □

Theorem 5.4. *If* $\mathbf{X}$ *is a hereditary coreflective subcategory of* **SSET** *then the affine sets which are affine subsets of products* (*taken in* $\mathbf{X}$) *of copies of* $\hat{\mathbf{S}}$ *form a firm epireflective subcategory of* $\mathbf{T_0X}$.

Proof. The proof follows the lines of proof of Theorem 5.2 (and Remark 4.3 (2)). □

There is a general method to produce hereditary coreflective subcategories of **SSET**:

Recall that an *algebra structure* in a set A is a family of functions

$$\Omega = \{\omega_T : A^T \to A\}$$

where T runs in a given class of sets. Then for every set X, by point-wise extension, the powerset A^X carries an algebra structure. If Ω is an algebra structure in the two-point set S we denote by $\mathbf{SSET(\Omega)}$ the subcategory of **SSET** consisting of those affine sets $(X, \mathcal{A})$ for which $\mathcal{A}$ is a Ω-subalgebra of the function algebra S^X.

It is easy to show that, for every algebra structure Ω in S the corresponding subcategory $\mathbf{SSET(\Omega)}$ of *affine sets over the algebra* (S, Ω) is hereditarily coreflective in **SSET**.

In this way, for suitable Ω, we obtain among others, the topological categories **CS** of closure spaces (cf. [9]), **Top** of topological spaces and **PrOS** of preordered sets.

For **CS** an internal characterization of the complete (= absolutely closed) T_0 spaces is given in [9]; in $\mathbf{Top_0}$ they are the sober T_0 spaces (see the introduction) and in **PrOS** it is easy to see that they are the partially ordered sets.

We do not know any example of a hereditary coreflective subcategory of **SSET** which is not of the form $\mathbf{SSET}(\Omega)$.

There is a second general method to produce hereditary coreflective subcategories, e.g. in **Top**, which is described in [16]. In that paper two proper classes of hereditary coreflective subcategories of **Top** are produced: for every regular cardinal α,

$$Alex(\alpha) = \{X \in \mathbf{Top} : \text{intersection of less than } \alpha \text{ open sets in } X \text{ is open}\},$$

$$Tight(\alpha) = \{X \in \mathbf{Top} : \text{if } x \in \bar{B} \subset X \text{ then } x \in \bar{A} \text{ for some } A \subset B,\ |A| < \alpha\}.$$

While it is easy to show that every category in the first class is of the form $\mathbf{SSET}(\Omega)$ we do not know if (at least) one of the categories of the second class is of the form $\mathbf{SSET}(\Omega)$.

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