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Hausdorff compactifications and zero-one measures II

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Abstract. The notion of *PBS-sublattice* is introduced and, using it, a simplification of the results of [6] and of some results of [5] is obtained. Two propositions concerning Wallman-type compactifications are presented as well.

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1. Introduction

In 1977, V. M. Ul′janov ([15]) obtained a negative answer to the famous Frink's question, posed in [8], whether each Hausdorff compactification of a Tychonoff space X is a Wallman-type compactification (we shall use from now on the term "Wallman compactification" instead of "Wallman-type compactification"). O. Frink introduced the Wallman compactifications of a space X as spaces of all $\mathcal{C}$-ultrafilters, where $\mathcal{C}$ is a ring of subsets of X and a special closed base of X (called *normal base*) (we will denote such compactifications by $\omega(X,\mathcal{C})$). Passing to the complements in X of all elements of a normal base $\mathcal{C}$, one obtains a special open base $\mathcal{B} = \mathcal{C}'$ of X (which is again a ring of sets), called *normal Wallman base*. This leads to a dual description of the Wallman compactifications of X as spaces of the type $\max(\mathcal{B})$ (= maximal spectrum of $\mathcal{B}$), where $\mathcal{B}$ is a normal Wallman base of X (see, e.g., [9]). Hence, in general, not every Hausdorff compactification of a Tychonoff space X can be obtained as a maximal spectrum of a normal Wallman base of X. In our paper [6], using

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the notion of *PB-sublattice* introduced in [5], we answered affirmatively two natural questions. The first one was:

Problem 1.1. *Is it possible to correlate (in a canonical way) to each Tychonoff space X a Boolean algebra B_X and a set $\mathcal{L}_X$ of sublattices of B_X in order to obtain that the set of all, up to equivalence, Hausdorff compactifications of X is represented by the set $\{\max(L) : L \in \mathcal{L}_X\}$?*

This question was motivated also by some measure-theoretic constructions of Hausdorff compactifications. It was well known (see [1, 3, 4, 14]) that, when $\mathcal{C}$ is a normal base of X, then the space $I_R(\mathcal{C})$ (of all regular zero-one measures on the Boolean subalgebra $b(\mathcal{C})$ of the Boolean algebra $\exp(X)$ (of all subsets of X, with the natural operations), generated by the sublattice $\mathcal{C}$ of $\exp(X)$) is a Hausdorff compactification of X equivalent to $\omega(X, \mathcal{C})$ and $\max(\mathcal{C}')$. The second problem was:

Problem 1.2. *Is it possible to construct in a similar way (by means of zero-one measures) every Hausdorff compactification of X?*

In this paper we introduce the notion of *PBS-sublattice* and, using it, we obtain a simplification of the results of [6] and of some results of [5]. We also present the notion of PB-sublattice in a simpler but equivalent form. Finally, a necessary and sufficient condition and a sufficient condition, as well, which a lattice $L \in \mathcal{L}_X$ has to satisfy in order to obtain that $\max(L)$ is a Wallman compactification of X, are stated and proved.

2. Preliminaries

We first fix some notations.

Note 2.1. We denote by ω the set of all *positive* natural numbers. All lattices will be with top (= unit) and bottom (= zero) elements, denoted respectively by 1 and 0 and all sublattices of a lattice L are assumed to contain the top and the bottom elements of L. We don't require the elements 0 and 1 to be distinct.

Let A be a distributive lattice. The set of all ideals of A will be denoted by $\mathrm{Idl}(A)$ and the set of all maximal ideals of A (which will be, as usual, always proper) — by $\max(A)$. Put $\mathcal{T}_A = \{O_I = \{J \in \max(A) : I \not\subseteq J\} : I \in \mathrm{Idl}(A)\}$. The space $(\max(A), \mathcal{T}_A)$ is called *maximal spectrum of* A and the topology $\mathcal{T}_A$ is called *spectral topology* on the set $\max(A)$. $(\max(A), \mathcal{T}_A)$ is always a compact T_1-space (see, e.g., [9]). If the lattice A is *normal* (i.e., for each pair $a, b \in A$ with $a \vee b = 1$, there exist $u, v \in A$ such that $a \vee u = 1 = b \vee v$ and $u \wedge v = 0$) then $(\max(A), \mathcal{T}_A)$ is a compact T_2-space.

If L is a sublattice of a Boolean algebra B then we will denote by $b(L)$ the Boolean subalgebra of B generated by L. By $\exp(X)$ we denote the set of all subsets of the set X.

The ordered set of all, up to equivalence, Hausdorff compactifications of a Tychonoff space X will be denoted by $(K(X), \leq)$.

If $(X,\mathcal{T})$ is a topological space then we write $\mathrm{Coz}(X,\mathcal{T})$ or, simply, $\mathrm{Coz}(X)$ for the set of all cozero-subsets of X; the closure of a subset M of $(X,\mathcal{T})$ will be denoted by $\mathrm{cl}_X M$; a *dense embedding* will mean an embedding with dense image.

By a *proximity* we shall always mean an Efremovič proximity. If δ is a proximity on a set X, then $\overline{\delta}$ will be the complement of the relation δ. If $(X,\mathcal{T})$ is a topological space and δ is a proximity on the set X, we say that δ is a *proximity on the space* $(X,\mathcal{T})$ if the topology $\mathcal{T}_\delta$, generated by δ on the set X, coincides with $\mathcal{T}$. The ordered set of all proximities δ on a topological space $(X,\mathcal{T})$ will be denoted by $(P_{\mathcal{T}}(X),\leq)$.

For all undefined terms and notations see [7], [9] and [11].

We shall recall the Smirnoff Compactification Theorem:

Theorem 2.2 ([13]). *Let $(X,\mathcal{T})$ be a Tychonoff space. If (cX,c) is a Hausdorff compactification of X, then putting, for every $A,B\subseteq X$, $A\overline{\delta_c}B$ iff $\mathrm{cl}_{cX}\, c(A)\cap \mathrm{cl}_{cX}\, c(B)=\varnothing$, we obtain a proximity δ_c on $(X,\mathcal{T})$. The correspondence*

$$s\colon (K(X),\leq)\longrightarrow (P_{\mathcal{T}}(X),\leq),$$

defined by $s(cX,c)=\delta_c$, is an isomorphism.

If $\delta\in P_{\mathcal{T}}(X)$ then the compactification $s^{-1}(\delta)$ of X, which will be denoted by $(c_\delta X, c_\delta)$, is called Smirnoff compactification *of $(X,\mathcal{T})$.*

Definition 2.3 ([9, 8]). Let $(X,\mathcal{T})$ be a topological space. A sublattice $\mathcal{B}$ of $\mathcal{T}$ is called a *Wallman base for X* if $\mathcal{B}$ is a base of $\mathcal{T}$ and satisfies the following condition:

(W) Whenever $U\in\mathcal{B}$ and $x\in U$, there exists $V\in\mathcal{B}$ with $U\cup V=X$ and $x\notin V$.

If $\mathcal{B}$ is a Wallman base for a T_0-space X, then the map

$$\eta_{\mathcal{B}}\colon X\longrightarrow max(\mathcal{B}),\qquad x\mapsto \eta_{\mathcal{B}}(x)=\{U\in\mathcal{B} : x\notin U\},$$

is a dense embedding. Hence, for every T_1-space X, $(\max(\mathcal{B}),\eta_{\mathcal{B}})$ is a T_1-compactification of X. If $\mathcal{B}$ is a normal Wallman base, then $(\max(\mathcal{B}),\eta_{\mathcal{B}})$ is a T_2-compactification of X, called *Wallman compactification.*

A family $\mathcal{C}$ of closed subsets of X, such that the family $\mathcal{B}=\mathcal{C}'=\{X\setminus F : F\in\mathcal{C}\}$ is a normal Wallman base of X, is called a *normal base of X*. Let $\omega(X,\mathcal{C})$ denote the set of all $\mathcal{C}$-ultrafilters. Topologize this set by using as a base for the closed sets all sets of the form $A^-=\{\mathcal{F}\in\omega(X,\mathcal{C}) : A\in\mathcal{F}\}$, where $A\in\mathcal{C}$. Then the map $\omega_{\mathcal{C}}\colon X\longrightarrow\omega(X,\mathcal{C})$, defined by the formula $\omega_{\mathcal{C}}(x)=\{F\in\mathcal{C} : x\in F\}$, where $x\in X$, is a dense embedding of X in $\omega(X,\mathcal{C})$ and $(\omega(X,\mathcal{C}),\omega_{\mathcal{C}})$ is a compactification of X equivalent to $(\max(\mathcal{B}),\eta_{\mathcal{B}})$.

We will need the following theorem of O. Njåstad:

Theorem 2.4 ([12]). *Let $(X,\mathcal{T})$ be a Tychonoff space. A compactification (cX,c) of X is a Wallman compactification if and only if there exists a subfamily $\mathcal{B}$ of $\mathcal{T}$ which is closed under finite unions and satisfies the following two conditions:*

(B1) *If* $U, V \in \mathcal{B}$ *and* $U \cup V = X$ *then* $(X \setminus U)\overline{\delta_c}(X \setminus V)$*;*
(B2) *If* $A, B \subseteq X$ *and* $A\overline{\delta_c}B$ *then there exist* $U, V \in \mathcal{B}$ *such that* $A \subseteq X \setminus U$, $B \subseteq X \setminus V$ *and* $U \cup V = X$*.*

Recall that (see, e.g., [1, 3, 4, 14]) a *measure* on a Boolean algebra A is a non-negative real-valued function μ on A such that $\mu(a \vee b) = \mu(a) + \mu(b)$ for all $a, b \in A$ with $a \wedge b = 0$; in the case when $\mu(A) = \{0, 1\}$, μ is called a *zero-one measure*.

Let B be a Boolean algebra and L be a sublattice of B. A measure μ, defined on the Boolean algebra $b(L)$, is called *L-regular measure* (or, simply, *regular measure*) if $\mu(x) = \sup\{\mu(a) : a \in L, x \geq a\}$ for any $x \in b(L)$. The set of all L-regular zero-one measures on the Boolean algebra $b(L)$ will be denoted by $I_R(L)$. The topology $\mathcal{D}_w$ on $I_R(L)$ is defined as follows: a base for the closed sets of $\mathcal{D}_w$ consists of all sets of the form $W(a) = \{\mu \in I_R(L) : \mu(a) = 1\}$, where $a \in L$. The space $(I_R(L), \mathcal{D}_w)$ is a compact T_1-space.

If X is a Tychonoff space and $\mathcal{C}$ is a normal base of X then $(I_R(\mathcal{C}), \mathcal{D}_w)$ is a compact Hausdorff space. The map $M_{\mathcal{C}} \colon X \longrightarrow (I_R(\mathcal{C}), \mathcal{D}_w)$, defined by the formula $M_{\mathcal{C}}(x) = \mu^x$, where $x \in X$ and, for every element F of the Boolean subalgebra $b(\mathcal{C})$ of $\exp(X)$,

$$\mu^x(F) = 1 \text{ if } x \in F, \text{ and } mu^x(F) = 0 \text{ if } x \notin F,$$

is a dense embedding. $((I_R(\mathcal{C}), \mathcal{D}_w), M_{\mathcal{C}})$ is a compactification of X equivalent to $(\omega(X, \mathcal{C}), \omega_{\mathcal{C}})$ and $(max(\mathcal{C}'), \eta_{\mathcal{C}'})$.

We will recall a theorem of J. Kerstan.

Definition 2.5 ([10, 2])**.** A family $\mathcal{U}$ of open subsets of a topological space space X is called *completely regular* if for every $U \in \mathcal{U}$ there exist two sequences $(U^i)_{i\in\omega}$ and $(V^i)_{i\in\omega}$ in $\mathcal{U}$ such that $U = \bigcup\{U^i : i \in \omega\}$ and $U^i \subseteq X \setminus V^i \subseteq U$ for each $i \in \omega$.

Theorem 2.6 ([10, 2])**.** *A subset of a topological space is a cozero-set if and only if it belongs to a completely regular family.*

3. The Results

Definition 3.1. Let $(X, \mathcal{T})$ be a space and U be an open subset of X. If there is a sequence $(U^i, U^{ci})_{i\in\omega}$ in $\mathcal{T} \times \mathcal{T}$ with $U = \bigcup_{i\in\omega} U^i$, $U^i \subseteq X \setminus U^{ci} \subseteq U^{i+1}$, for every $i \in \omega$, then such a sequence $(U^i, U^{ci})_{i\in\omega}$ will be called $Ur-$*representation* of U. We put $\mathcal{T}_{Ur} = \{U \in \mathcal{T} : U \text{ has an } Ur\text{-representation}\}$.

Definition 3.2. Let $(X, \mathcal{T})$ be a space. Denote by $L(X)$ the set of all Ur-representations of the elements of $\mathcal{T}_{Ur}$. The elements of $L(X)$ will be written in the following way:

$$\bar{U} = (U^i, U^{ci})_{i\in\omega},$$

where $(U^i, U^{ci})_{i\in\omega}$ is a Ur-representation of $U^0 = \bigcup\{U^i : i \in \omega\}$; two elements $\bar{U} = (U^i, U^{ci})_{i\in\omega}$ and $\bar{V} = (V^i, V^{ci})_{i\in\omega}$ of $L(X)$ are equal if $U^i = V^i$, $U^{ci} = V^{ci}$, for every $i \in \omega$. Define two operations $\wedge$ and $\vee$ in $L(X)$ by

$$\bar{U} \vee \bar{V} = (U^i \cup V^i, U^{ci} \cap V^{ci})_{i\in\omega}$$

and

$$\bar{U} \wedge \bar{V} = (U^i \cap V^i, U^{ci} \cup V^{ci})_{i\in\omega},$$

where $\bar{U} = (U^i, U^{ci})_{i\in\omega}$ and $\bar{V} = (V^i, V^{ci})_{i\in\omega}$, and let $\bar{0} = (0^i, 0^{ci})_{i\in\omega}$, $\bar{1} = (1^i, 1^{ci})_{i\in\omega}$, where $\varnothing = 0^i = 1^{ci}$, $X = 1^i = 0^{ci}$, $i \in \omega$.

Fact 3.3. *$(L(X), \vee, \wedge)$ is a distributive lattice and $\bar{0}, \bar{1}$ are its zero and one.*

Definition 3.4 (see also [5])**.** Let X be a Tychonoff space. A sublattice L of $L(X)$ is said to be a PB-*sublattice* if

(L1) The set $L^0 = \{U^0 = \bigcup\{U^i : i \in \omega\} : (U^i, U^{ci})_{i\in\omega} \in L\}$ is an open base of the space X;

(L2) For every $\bar{U} = (U^i, U^{ci})_{i\in\omega} \in L$ and every $j \in \omega$, there exist $k \in \omega$ and $\bar{V} = (V^i, V^{ci})_{i\in\omega}, \bar{W} = (W^i, W^{ci})_{i\in\omega} \in L$ (which depend on the choice of $\bar{U}$ and j) such that $U^{c(j+1)} \subseteq W^k \subseteq W^0 = U^{cj}$, $U^{j-1} \subseteq V^k \subseteq V^0 = U^j$ (for $j > 1$), and $V^0 = U^j$ (for $j = 1$).

Proposition 3.5. *Let L be a PB-sublattice of $L(X)$. Then, for every element $\bar{U} = (U^i, U^{ci})_{i\in\omega}$ of L and for every $i \in \omega$, we have that $U^i, U^{ci} \in \mathrm{Coz}(X)$. Hence, $L^0 \subseteq \mathrm{Coz}(X)$.*

Proof. For every $\bar{U} = (U^i, U^{ci})_{i\in\omega} \in L$ and every $j \in \omega$, we have, by (L2), that there exist $\bar{V} = (V^i, V^{ci})_{i\in\omega} \in L$ and $\bar{W} = (W^i, W^{ci})_{i\in\omega} \in L$ such that $U^j = V^0$ and $U^{cj} = W^0$. Hence, in order to prove our proposition, we need only to show, according to Kerstan Theorem (see 2.6), that L^0 is a completely regular family (see 2.5). So, let $\bar{U} = (U^i, U^{ci})_{i\in\omega} \in L$. Then $\{U^i : i \in \omega\} \subseteq L^0$ and $U^0 = \bigcup\{U^i : i \in \omega\}$. We let $(U^i)_{i\in\omega}$ to be the first required sequence. As it follows from 3.1, $(U^{ci})_{i\in\omega}$ can serve as the second required sequence. Therefore, L^0 is a completely regular family. □

Definition 3.6 ([5])**.** Let (X, τ) be a space. Denote by $L(\mathrm{Coz}(X))$ the set of all Ur-representations of all elements of $\mathrm{Coz}(X)$ by elements of $\mathrm{Coz}(X)$. We will regard $L(\mathrm{Coz}(X))$ as a sublattice of the lattice $L(X)$.

Remark 3.7. Let us remark that in [5] the notion of "PB-sublattice" was introduced with the redundant (as Proposition 3.5 shows now) requirement that a PB-sublattice is (by definition) a sublattice of $L(\mathrm{Coz}(X))$.

Proposition 3.8 ([5])**.** *$(L(\mathrm{Coz}(X)), \vee, \wedge)$ is the greatest (with respect to the inclusion) PB-sublattice of $(L(X), \vee, \wedge)$.*

Note 3.9. Let X be a set. We will denote by $S(X)$ the complete Boolean algebra $(\exp(X))^{\aleph_0}$.

Definition 3.10. Let $(X, \mathcal{T})$ be a topological space. We put

$$\mathrm{OIS}(X, \mathcal{T}) = \{\bar{U} = (U^i)_{i\in\omega} : U^i \in \mathcal{T}, U^i \subseteq U^{i+1}, \forall i \in \omega\}.$$

Instead of $\mathrm{OIS}(X, \mathcal{T})$, we shall often write simply $\mathrm{OIS}(X)$. For every $(U^i)_{i\in\omega} \in \mathrm{OIS}(X)$, we put $U^0 = \bigcup\{U^i : i \in \omega\}$. We will regard $\mathrm{OIS}(X)$ as a sublattice of $S(X)$.

Definition 3.11. Define a relation $\sim$ in $S(X)$ putting: for every $\bar{U} = (U^i)_{i\in\omega}$, $\bar{V} = (V^i)_{i\in\omega} \in \mathrm{OIS}(X)$, $\bar{U} \sim \bar{V}$ if and only if there exists an $i_0 \in \omega$ such that $U^i = V^i$, for every $i \geq i_0$. Then $\sim$ is a congruence relation on the Boolean algebra $S(X)$. So, a quotient Boolean algebra $S(X)/\!\sim$, which will be denoted by $[S(X)]$, is defined. The natural mapping between $S(X)$ and $[S(X)]$ will be denoted by π. We put, for every $\bar{U} \in S(X)$, $\pi(\bar{U}) = [\bar{U}]$.

Definition 3.12. Let $(X, \mathcal{T})$ be a Tychonoff space. A sublattice L of the lattice $\mathrm{OIS}(X)$ is said to be a PBS-*sublattice in* X, if

(LS1) The set $L^0 = \{U^0 : (U^i)_{i\in\omega} \in L\}$ is a base of the space X;

(LS2) For every $\bar{U} = (U^i)_{i\in\omega} \in L$ and for every $j \in \omega$, there exist $\bar{V} = (V^i)_{i\in\omega}, \bar{W} = (W^i)_{i\in\omega} \in L$ and $k \in \omega$ (which depend on the choice of $\bar{U}$ and j) such that $X \setminus U^{j+1} \subseteq V^k \subseteq V^0 \subseteq X \setminus U^j$, and $U^{j-1} \subseteq W^k \subseteq W^0 = U^j$ (for $j > 1$), $U^j = W^0$ (for $j = 1$).

Fact 3.13. *The restriction of the relation $\sim$ (defined in 3.11) to any PBS-sublattice L in X is a congruence relation in L. So, a quotient lattice $[L] = L/\!\sim$ is defined.*

Lemma 3.14. *Let L' be a PB-sublattice of $L(X)$. Then*

$$L = \{\bar{U} = (U^i)_{i\in\omega} : \textit{there exists an } \bar{U}' \in L' \textit{ such that } \bar{U}' = (U^i, U^{ci})_{i\in\omega}\}$$

is a PBS-sublattice in X. For every $\bar{U}' = (U^i, U^{ci})_{i\in\omega} \in L'$ put $p(\bar{U}') = (U^i)_{i\in\omega}$. Then $p\colon L' \longrightarrow L$ is a lattice homomorphism, $L = p(L')$ and the correspondence

$$[p]\colon [L'] \longrightarrow [L], \qquad [\bar{U}'] \longrightarrow [p(\bar{U}')]$$

is a lattice isomorphism.

Proof. For proving that L is a PBS-sublattice in X, we need only to check that the first part in the condition (LS2) (see 3.12) is satisfied.

Let $\bar{U} = (U^i)_{i\in\omega} \in L$ and $j \in \omega$. There exists an $\bar{U}' \in L'$ such that $\bar{U}' = (U^i, U^{ci})_{i\in\omega}$. By (L2) of 3.4, there exist $\bar{W}' = (W^i, W^{ci})_{i\in\omega} \in L'$ and $l \in \omega$ such that $U^j \subseteq W^l \subseteq W^0 = U^{j+1}$. Then $U^j \subseteq W^l \subseteq W^{l+2} \subseteq U^{j+1}$ and hence $X \setminus U^j \supseteq X \setminus W^l \supseteq X \setminus W^{l+2} \supseteq X \setminus U^{j+1}$. We have that $X \setminus W^{l+2} \subseteq W^{c(l+1)} \subseteq X \setminus W^{l+1} \subseteq W^{cl} \subseteq X \setminus W^l$. Using again (L2) of 3.4, we obtain that there exist $\bar{V}' = (V^i, V^{ci})_{i\in\omega} \in L'$ and $k \in \omega$ such that $W^{c(l+1)} \subseteq V^k \subseteq V^0 = W^{cl}$. Therefore $\bar{V} = (V^i)_{i\in\omega} \in L$ and $X \setminus U^{j+1} \subseteq V^k \subseteq V^0 \subseteq X \setminus U^j$.

It is easy to see that $[p]$ is a lattice isomorphism. □

Lemma 3.15. *For every PBS-sublattice L in X there exists a PB-sublattice L' of $L(X)$ such that $p(L') = L$ and $[p]\colon [L'] \longrightarrow [L]$ is a lattice isomorphism (see 3.14 for the notations).*

Proof. Let $\bar{U} = (U^i)_{i\in\omega} \in L$. Then, by (LS2) (see 3.12), for every $j \in \omega$ there exist $\bar{V}_j = (V_j^i)_{i\in\omega} \in L$ and $k_j \in \omega$ such that $U^j \subseteq X \setminus V_j^0 \subseteq X \setminus V_j^{k_j} \subseteq U^{j+1}$. Put $U^{cj} = V_j^0$, for every $j \in \omega$. Then $U^j \subseteq X \setminus U^{cj} \subseteq U^{j+1}$, for every $j \in \omega$, and hence $\bar{U}' = (U^i, U^{ci})_{i\in\omega} \in L(X)$. Put $L'' = \{\bar{U}' : \bar{U} \in L\}$. Then

$L'' \subseteq L(X)$. Let L' be the sublattice of $L(X)$ generated by L''. In order to show that L' is a PB-sublattice of $L(X)$, we need only to check that the first part of the condition (L2) (see 3.4) is satisfied. Let $\bar{U} = (U^i)_{i\in\omega} \in L$. Then $\bar{U}' = (U^i, U^{ci})_{i\in\omega} \in L''$. Let $j \in \omega$. By the construction of $\bar{U}'$, we have that $U^{c(j+1)} \subseteq X \setminus U^{j+1} \subseteq V_j^{k_j} \subseteq V_j^0 = U^{cj}$. Since $(\bar{V}_j)' \in L'' \subseteq L'$ and $k_j \in \omega$, we obtain that (L2) is satisfied by the elements of L''. From the facts that L is a lattice and L'' generates L', we obtain that (L2) is satisfied also by all elements of L'. So, L' is a PB-sublattice of $L(X)$. The construction of L' shows that $p(L') = L$. The rest follows from 3.14. □

Corollary 3.16. *Let L be a PBS-sublattice in $(X, \mathcal{T})$. Then, for every element $\bar{U} = (U^i)_{i\in\omega}$ of L and for every $i \in \omega$, we have that $U^i \in \mathrm{Coz}(X)$. Hence, $L^0 \subseteq \mathrm{Coz}(X)$.*

Proof. It follows from 3.15 and 3.5. □

Theorem 3.17. *Let $(X, \mathcal{T})$ be a Tychonoff space and L be a PBS-sublattice in X. Define for $A, B \subseteq X$: $A\overline{\delta_L}B$ iff there exist $\bar{U} = (U^i)_{i\in\omega} \in L$ and $k \in \omega$ such that $A \subseteq U^k \subseteq U^0 \subseteq X \setminus B$. Then δ_L is an Efremovič proximity on the topological space $(X, \mathcal{T})$. (We will say that the proximity δ_L is* generated by the PBS-sublattice *L in X.) Moreover, for any proximity δ on $(X, \mathcal{T})$ there exists a PBS-sublattice L in X such that $\delta = \delta_L$. The set of all PBS-sublattices in X generating a proximity δ on $(X, \mathcal{T})$ has a greatest element (with respect to the inclusion), which will be denoted by L_δ.*

Proof. By Lemma 3.15, there exists a PB-sublattice L' of $L(X)$ such that $p(L') = L$. In Proposition 2.12 of [5] we show that the relation $\delta_{L'}$ generated by L', defined in the same way as we define here the relation δ_L, is a proximity on the space $(X, \mathcal{T})$. Hence, δ_L is such one, as well. This fact can be also obtained directly, modifying the proof of Proposition 2.12 of [5].

If (cX, c) is a compactification of X then the family $\mathcal{F} = \{f : X \longrightarrow [0,1] : f$ has a continuous extension to $cX\}$ generates (cX, c). The PB-sublattice $L_\mathcal{F}$ of $L(X)$, constructed in Example 2.4 of [5], has the property that $\delta_{L_\mathcal{F}} = \delta_c$ (see Theorem 3.1(a) of [5]). By Lemma 3.14, the lattice $L = p(L_\mathcal{F})$ is a PBS-sublattice in X. Obviously, $\delta_{L_\mathcal{F}} = \delta_L$. Hence, by Theorem 2.2, for any proximity δ on $(X, \mathcal{T})$ there exists a PBS-sublattice L in X such that $\delta = \delta_L$.

Finally, one easily infer from Proposition 2.11 of [5] and our lemmas 3.14 and 3.15 that the set of all PBS-sublattices in X generating a proximity δ on $(X, \mathcal{T})$ has a greatest element (with respect to the inclusion). □

Theorem 3.18. *Let $(X, \mathcal{T})$ be a Tychonoff space and L be a PBS-sublattice in X. Put, for every $x \in X$, $I_x = \{\bar{U} \in L : x \notin U^0\}$. Then:*

(a) *$\pi(I_x) = \{[\bar{U}] : \bar{U} \in I_x\} \in \max([L])$ and the map $e_L : (X, \mathcal{T}) \longrightarrow \max([L])$, defined by the formula $e_L(x) = \pi(I_x)$, is a dense embedding;*

(b) *$(\max([L]), e_L)$ is a Hausdorff compactification of X, equivalent to the Smirnoff compactification $(c_{\delta_L}X, c_{\delta_L})$ (see 3.17 for δ_L and 2.2 for $(c_{\delta_L}X, c_{\delta_L})$).*

Hence, the set $K(X)$ of all, up to equivalence, Hausdorff compactifications of X is represented by the set $\{(\max([L_\delta]), e_{L_\delta}) : \delta \in P_{\mathcal{T}}(X)\}$. Moreover, the following is true: $(c_{\delta_1}X, c_{\delta_1}) \leq (c_{\delta_2}X, c_{\delta_2})$ iff $L_{\delta_1} \subseteq L_{\delta_2}$ (see 3.17 for L_δ).

Therefore, putting $B_X = [S(X)]$ and $\mathcal{L}_X = \{[L_\delta] : \delta \in P_{\mathcal{T}}(X)\}$, we obtain a new (simpler) solution of our Problem 1.

Proof. In [6] the PB-sublattice version of this theorem (i.e., the version obtained by substituting everywhere in the theorem "PBS-" with "PB-") was proved (see Theorem 3.8 there). Now our result follows from it, from lemmas 3.14, 3.15 and Theorem 3.17 proved above, and from 2.17, 2.13 of [5]. □

Definition 3.19. Let B be a Boolean algebra and L be a sublattice of B. A measure μ, defined on the Boolean algebra $b(L)$, is called *u-regular measure* (or *u-L-regular measure*) if $\mu(x) = \inf\{\mu(a) : a \in L, x \leq a\}$ for any $x \in b(L)$. The set of all u-L-regular zero-one measures on the Boolean algebra $b(L)$ will be denoted by $I_{ur}(L)$.

The following lemma is essentialy known (see [1], Theorem 2.1):

Lemma 3.20. *Let B be a Boolean algebra and L be a sublattice of B. Then there exists a bijection between the sets $\max(L)$ and $I_{ur}(L)$.*

Lemma 3.21. *Let $(X, \mathcal{T})$ be a Tychonoff space and L be a PBS-sublattice in X. Then $[L]$ is a sublattice of $[S(X)]$ (see 3.11 and 3.9 for the notations). For every $[\bar{U}] \in [L]$ put*

$$[\bar{U}]^* = \{\mu \in I_{ur}([L]) : \mu([\bar{U}]) = 1\}.$$

Then the family $\mathcal{B}^* = \{[\bar{U}]^* : [\bar{U}] \in [L]\}$ is a base of a topology $\mathcal{T}^*$ on the set $I_{ur}([L])$. If δ is the proximity on $(X, \mathcal{T})$ generated by L (see 3.17), then for every $x \in X$ and every $[\bar{U}] = [(U^i)_{i\in\omega}] \in b([L])$ put:

$$\mu_x([\bar{U}]) = \begin{cases} 0 & \text{if there exists an } i_0 \in \omega \text{ such that } x\overline{\delta}U^i \text{ for every } i \geq i_0, \text{ and} \\ 1 & \text{if there exists an } j_0 \in \omega \text{ such that } x\overline{\delta}(X \setminus U^j) \text{ for every } j \geq j_0. \end{cases}$$

Then, for every $x \in X$, μ_x is a well-defined zero-one u-[L]-regular measure on the Boolean subalgebra $b([L])$ of the complete Boolean algebra $[S(X)]$ and the mapping $m_L\colon (X, \mathcal{T}) \longrightarrow (I_{ur}([L]), \mathcal{T}^*)$, defined by the formula $m_L(x) = \mu_x$, is a dense embedding. $((I_{ur}([L]), \mathcal{T}^*), m_L)$ is a Hausdorff compactification of $(X, \mathcal{T})$ equivalent to the compactification $(max([L]), e_L)$ of $(X, \mathcal{T})$ (and, hence, to the Smirnoff compactification $(c_\delta X, c_\delta)$). The map $\Phi\colon (I_{ur}([L]), \mathcal{T}^*) \longrightarrow \max([L])$, defined by the formula $\Phi(\mu) = \mu^{-1}(0) \cap [L]$, carries out this equivalence.

Proof. In [6] the PB-sublattice version of this lemma was proved (see Lemma 3.16 there). Our result follows from it and from Lemma 3.15 proved above. □

Theorem 3.22 (The Main Theorem). *Let $(X, \mathcal{T})$ be a Tychonoff space. Then for every Hausdorff compactification (cX, c) of X there exists a sublattice $[L]$ of the complete Boolean algebra $[S(X)]$ (where L is a PBS-sublattice in X) such that $(\max([L]), e_L)$ (see 3.18 for the definition of the map e_L) and*

$((I_{ur}([L]), \mathcal{T}^*), m_L)$ *(see 3.19 and 3.21 for the notations) are Hausdorff compactification of* X *equivalent to the compactification* (cX, c) *of* X*.*

Proof. Let (cX, c) be a Hausdorff compactification of $(X, \mathcal{T})$. Then, by Theorem 3.17, there exists a PBS-sublattice L in X such that $\delta_L = \delta_c$ (see 2.2 and 3.17 for the notations). Now, Theorem 3.18, Lemma 3.21 and Theorem 2.2 imply that $(\max([L]), e_L)$ and $((I_{ur}([L]), \mathcal{T}^*), m_L)$ are Hausdorff compactification of X equivalent to the compactification (cX, c) of X. □

Corollary 3.23. *Let* $(X, \mathcal{T})$ *be a Tychonoff space. Put*

$$MA(X) = \{(\max([L_\delta]), e_{L_\delta}) : \delta \in P_{\mathcal{T}}(X)\}$$

and

$$ME(X) = \{((I_{ur}([L_\delta]), \mathcal{T}^*), m_{L_\delta}) : \delta \in P_{\mathcal{T}}(X)\}.$$

Order these sets putting for every $\delta_1, \delta_2 \in P_{\mathcal{T}}(X)$*,*

$$\max([L_{\delta_1}]) \leq max([L_{\delta_2}]) \text{ iff } \delta_1 \leq \delta_2, \text{ and } I_{ur}([L_{\delta_1}]) \leq I_{ur}([L_{\delta_2}]) \text{ iff } \delta_1 \leq \delta_2.$$

Then the ordered sets $(MA(X), \leq)$ *and* $(ME(X), \leq)$ *are isomorphic to the ordered set* $(K(X), \leq)$ *of all, up to equivalence, Hausdorff compactifications of* X*.*

In the next proposition, the O. Njåstad's characterization of Wallman compactifications by means of proximities (see 2.4) is restated in the language of PBS-sublattices.

Proposition 3.24. *Let* $(X, \mathcal{T})$ *be a Tychonoff space and* L *be a PBS-sublattice in* X*. Then* $(max([L]), e_L)$ *is a Wallman compactification of* X *iff there exists a family* $\mathcal{B}$*, consisting of open subsets of* X*, such that*

- (i) $\mathcal{B}$ *is closed under finite unions;*
- (ii) *If* $A, B \in \mathcal{B}$ *and* $A \cup B = X$ *then there exist* $\bar{U} = (U^i)_{i \in \omega} \in L$ *and* $j \in \omega$ *with* $X \setminus A \subseteq U^j \subseteq U^0 \subseteq B$*;*
- (iii) *If* $\bar{U} = (U^i)_{i \in \omega} \in L$ *and* $j \in \omega$ *then there exist* $A, B \in \mathcal{B}$ *such that* $U^j \subseteq X \setminus A \subseteq B \subseteq U^0$*.*

Proof. The proximity generated by the compactification $(\max([L]), e_L)$ is exactly the proximity δ_L (see Theorem 3.18(b)). Hence, by Theorem 2.4, $(\max([L]), e_L)$ is a Wallman compactification of X if and only if there exists a subfamily $\mathcal{B}$ of $\mathcal{T}$ which is closed under finite unions and satisfies the conditions $(B1)$ and $(B2)$. Since our proximity δ_L is generated by L, these conditions can be rewritten now as follows:

- $(B1_L)$ If $U, V \in \mathcal{B}$ and $U \cup V = X$ then there exist $\bar{U} = (U^i)_{i \in \omega} \in L$ and $j \in \omega$ such that $X \setminus U \subseteq U^j \subseteq U^0 \subseteq V$;
- $(B2_L)$ If $A, B \subseteq X$ and there exist $\bar{U} = (U^i)_{i \in \omega} \in L$ and $j \in \omega$ such that $A \subseteq U^j \subseteq U^0 \subseteq X \setminus B$ then there are $V, W \in \mathcal{B}$ with $A \subseteq X \setminus V$, $B \subseteq X \setminus W$ and $V \cup W = X$.

Obviously, condition $(B1_L)$ coincides with condition (ii) of our Proposition and condition (i) is also satisfied. Since for every $\bar{U} = (U^i)_{i\in\omega} \in L$ and $j \in \omega$ we have that $U^j \overline{\delta_L}(X \setminus U^0)$, condition $(B2_L)$ is equivalent to the condition (iii). This completes the proof. □

Now we will give a sufficient condition for $(max([L]), e_L)$ to be a Wallman compactification:

Proposition 3.25. *Let $(X, \mathcal{T})$ be a Tychonoff space and L be a PBS-sublattice in X. If L satisfies the following condition:*

(Wa) *If $\bar{U}, \bar{V} \in L$ and $U^0 \cup V^0 = X$ then there exist $\bar{W} = (W^i)_{i\in\omega} \in L$ and $j \in \omega$ such that $X \setminus U^0 \subseteq W^j \subseteq W^0 \subseteq V^0$,*

then $(\max([L]), e_L)$ is a Wallman compactification of X. In fact, we have that $(\max([L]), e_L)$ is equivalent to the Wallman compactification $(\max(L^0), \eta_{L^0})$ (see 2.3 and 3.12 for the notations).

Proof. Let us recall that O. Njåstad ([12]) proved that if (X, δ) is a proximity space then a subfamily $\mathcal{B}$ of the topology $\mathcal{T}_\delta$, generated by the proximity δ, is a normal Wallman base of $(X, \mathcal{T}_\delta)$ if it is a ring of sets and satisfies the conditions $(B1)$ and $(B2)$ from 2.4; moreover, he showed that $(\max(\mathcal{B}), \eta_{\mathcal{B}})$ and $(c_\delta X, c_\delta)$ are equivalent compactifications of $(X, \mathcal{T}_\delta)$ (see 2.2 and 2.3 for the notations).

By Theorem 3.18(b), we have that $(max([L]), e_L)$ and $(c_{\delta_L} X, c_{\delta_L})$ are equivalent compactifications of $(X, \mathcal{T})$. Obviously, $L^0 (= \{U^0 = \bigcup\{U^i : i \in \omega\} : (U^i)_{i\in\omega} \in L\})$ is a ring of open sets in $(X, \mathcal{T})$ and $\mathcal{T} = \mathcal{T}_{\delta_L}$ (see Theorem 3.17). So, in order to prove our proposition, it is enough to show that the family L^0 satisfies the conditions $(B1)$ and $(B2)$ from 2.4. The condition (Wa) says that if $U^0, V^0 \in L^0$ and $U^0 \cup V^0 = X$ then $(X \setminus U^0)\overline{\delta_L}(X \setminus V^0)$. Hence $(B1)$ is satisfied. For proving $(B2)$, let $A, B \subseteq X$ and $A\overline{\delta_L}B$. Then, by the definition of δ_L, there exist $\bar{U} = (U^i)_{i\in\omega} \in L$ and $j \in \omega$ such that $A \subseteq U^j \subseteq U^0 \subseteq X \setminus B$. Since L is a PBS-sublattice in X, we obtain (by the condition $(LS2)$ from Definition 3.12) that there exist $\bar{V} = (V^i)_{i\in\omega} \in L$ and $k \in \omega$ with $U^j \subseteq X \setminus V^0 \subseteq X \setminus V^k \subseteq U^{j+1} \subseteq U^0$. Hence $A \subseteq X \setminus V^0$, $B \subseteq X \setminus U^0$, $U^0 \cup V^0 = X$ and $U^0, V^0 \in L^0$. Therefore, L^0 satisfies $(B2)$. The proof of our proposition is completed. □

References

[1] G. Bachman and R. Cohen, *Regular Lattice Measures and Repleteness*, Comm. on Pure and Applied Math. **26** (1973), 587–599.

[2] H. Brandenburg and A. Mysior, *Short proof of an internal characterization of complete regularity*, Canad. Math. Bull. **27** (4) (1984), 461–462.

[3] G. Bachman and P. Stratigos, *On the Relation of Lattice Repleteness and C-Real Compactness*, Indian J. Pure and Applied Math. **20** (11) (1989), 1043–1066.

[4] R. Cohen, *Lattice Measures and Topologies*, Ann. Mat. Pura Appl. **109** (1976), 147–164.

[5] G. Dimov and G. Tironi, *Compactifications, A-Compactifications and Proximities*, Ann. Mat. Pura Appl. **169** (1995), 87–108.

[6] G. Dimov and G. Tironi, *Hausdorff Compactifications and Zero-One Measures*, Math. Proc. Camb. Phil. Soc. (2001) (to appear).

[7] R. Engelking, *General Topology*, (PWN, Warszawa, 1977).

[8] O. Frink, *Compactifications and semi-normal spaces*, Amer. J. Math. **86** (1964), 602–607.
[9] P. T. Johnstone, *Stone Spaces*, Cambridge Studies in Advanced Math. no. 3 (Cambridge University Press, 1982).
[10] J. Kerstan, *Eine Charakterisierung der vollständig regulären Räume*, Math. Nachrichten **17** (1958), 27–46.
[11] S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, (Cambridge Univ. Press, 1970).
[12] O. Njåstad, *On Wallman-type Compactifications*, Math. Zeitschr **91** (1966), 267-276.
[13] Y. M. Smirnoff, *On proximity spaces*, Mat. Sb. (Russian) **31** (1952), 543-574; Am. Math. Soc. Transl. (English translation) **38** Ser. 2, 5–35.
[14] A. Sultan, *Measure, compactification and representation*, Can. J. Math. **30** (1) (1978), 54–65.
[15] V. M. Ul′janov, *Solution of a basic problem on compactifications of Wallman type*, Soviet Math. Dokl. **18** (1977), 567–571.
[16] H. Wallman, *Lattices and topological spaces*, Annals of Math. **39** (1938), 112–126.

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