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Fixed point theorems for simulation functions in b-metric spaces via the wt-distance

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Abstract

The purpose of this article is to prove some fixed point theorems for simulation functions in complete $b-$metric spaces with partially ordered by using wt-distance which introduced by Hussain et al. [12]. *Also, we give some examples to illustrate our main results.*

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1. Introduction

Since Banach's fixed point theorem (or Banach's contraction principle) proved by Banach [4] in 1922, many authors have extended, improved and generalized in several ways.

In 2015, Khojasteh et al. [15] introduced the notion of a simulation function to generalize Banach's contraction principle. Recently, Roldán-López-de-Hierroet et al. [18] modified the notion of a simulation function and showed the existence and uniqueness of coincidence points of two nonlinear mappings using the concept of a simulation function.

On the other hand, in 1989, Bakhtin [3] (see also Czerwik [8]) introduced the concept of a b-metric space (or a space of metric type) and proved some fixed point theorems for some contractive mappings in b-metric spaces which are generalizations of Banach's contraction principle in metric spaces.

In 1996, Kada et al. [14] introduced some generalized metric, which is called the w-distance and gave some examples of w-distance and, using the w-distance, they also improved Caristi's fixed point theorem, Ekeland's variational principle and the nonconvex minimization theorem of Takahashi [20]. Later, Shioji et al. [19] studied the relationship between weakly contractive mappings and weakly Kannan mappings under the conditions, the w-distance and the symmetric w-distance. In 2012, Imdad and Rouzkard [13] proved some fixed point theorems in a complete metric space equipped with a partial ordering via the w-distance.

Recently, Hussain et al. [12] introduced the concept of the wt-distance in generalized b-metric spaces, which is a generalization of the w-distance, and also proved some fixed point theorems in a partially ordered b-metric space by using the wt-distance. Also, Abdou et al. [1] proved some common fixed point theorems in Menger probabilistic metric type spaces by using the wt-distance.

In this paper, we consider some simulation functions to show the existence of fixed points of some nonlinear mappings in complete b-metric spaces via the wt-distance. Furthermore, we also give some examples to illustrate the main results. Our result improve, extend and generalize several results given by some authors in literatures.

2. Preliminaries and generalized distances

Now, we give some definitions and their examples

Definition 2.1. Let $(X, \leq)$ be a partially ordered set.The elements $x, y \in X$ are said to be *comparable* with respect to the order $\leq$ if either $x \leq y$ or $y \leq x$.

Let us denote $X_{\leq}$ by the subset of $X \times X$ defined by

$$X_{\leq} = \{(x, y) \in X \times X : x \leq y \text{ or } y \leq x\}.$$

Definition 2.2. Let $(X, \leq)$ be a partially ordered set and $f : X \to X$ be a self-mapping of X. We say that

(1) f is *inverse increasing* if, for all $x, y \in X$, $f(x) \leq f(y)$ implies $x \leq y$;
(2) f is *nondecreasing* if, for all $x, y \in X$, $x \leq y$ implies $f(x) \leq f(y)$.

Definition 2.3. Let $(X, \leq)$ be a partially ordered set and $T : X \to X$ be a self-mapping of X. Then

(1) $F(T) = \{x \in X : T(x) = x\}$, i.e., $F(T)$ denotes the set of all fixed points of T;
(2) T is called a *Picard operator* (briefly, PO) if there exists $x^* \in X$ such that $F(T) = \{x^*\}$ and $\{T^n(x)\}$ converges to x^* for all $x \in X$;
(3) T is said to be *orbitally $\mathcal{U}$-continuous* for any $\mathcal{U} \subset X \times X$ if, for any $x \in X$, $T^{n_i}(x) \to a \in X$ as $i \to \infty$ and $(T^{n_i}(x), a) \in \mathcal{U}$ for any $i \in \mathbb{N}$ imply that $T^{n_i+1}(x) \to Ta \in X$ as $i \to \infty$;
(4) T is said to be *orbitally continuous* on X if $x \in X$ and $T^{n_i}(x) \to a \in X$ as $i \to \infty$ imply that $T^{n_i+1}(x) \to T(a) \in X$ as $i \to \infty$.

Definition 2.4. Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is said to be the *w-distance* on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;
(2) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semi-continuous (i.e., if $x \in X$ and $y_n \to y \in X$, then $p(x, y) \leq \liminf_{n\to\infty} p(x, y_n)$);
(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Let X be a metric space with a metric d. A w-distance p on X is said to be *symmetric* if $p(x, y) = p(y, x)$ for all $x, y \in X$. Obviously, every metric is the w-distance, but not conversely.

Next, we recall some examples in [21] to show that the w-distance is a generalized metric.

Example 2.5. Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = c$ for all $x, y \in X$ is a w-distance on X, where c is a positive real number. But p is not a metric since $p(x, x) = c \neq 0$ for any $x \in X$.

Example 2.6. Let $(X, \|\cdot\|)$ be a normed linear space. A function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = \|x\| + \|y\|$ for all $x, y \in X$ is a w-distance on X.

Example 2.7. Let F be a bounded and closed subset of a metric spaces X. Assume that F contain at least two points and c is a constant with $c \geq \delta(F)$, where $\delta(F)$ is the diameter of F. Then a function $p : X \times X \to [0, \infty)$ defined by

$$p(x, y) = \begin{cases} d(x, y), & \text{if } \ x, y \in F, \\ c, & \text{if } \ x \notin F \ \text{ or } \ y \notin F, \end{cases}$$

is a w-distance on X.

Definition 2.8. Let X be a nonempty set and $s \geq 1$ be a given real number. A functional $D : X \times X \to [0, \infty)$ is called a *b-metric* if, for all $x, y, z \in X$, the following conditions are satisfied:

(1) $D(x, y) = 0$ if and only if $x = y$;
(2) $D(x, y) = D(y, x)$;
(3) $D(x, z) \leq s[D(x, y) + D(y, z)]$.

A pair (X, D) is called a *b-metric space* with coefficient s.

In Definition 2.8, every metric space is a b-metric space with $s = 1$ and hence the class of b-metric spaces is larger than the class of metric spaces.

Some examples of b-metric spaces are given by Berinde [5], Czerwik [9], Heinonen [11] and, further, some examples to show that every b-metric space is a real generalization of metric spaces are as follows:

Example 2.9. The set $\mathbb{R}$ of real numbers together with the functional $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$D(x, y) := |x - y|^2$$

for all $x, y \in \mathbb{R}$ is a b-metric space with coefficient $s = 2$. However, we know that D is not a metric on X since the ordinary triangle inequality is not satisfied. Indeed,

$$D(3, 5) > D(3, 4) + D(4, 5).$$

In 2014, Hussain et al. [12] introduced the concept of the wt-distance as follow:

Definition 2.10. Let (X, D) be a b-metric space with constant $K \geq 1$. A function $P : X \times X \to [0, \infty)$ is called the *wt-distance* on X if the following are satisfied:

(1) $P(x, z) \leq K(P(x, y) + P(y, z))$ for all $x, y, z \in X$;
(2) for any $x \in X$, $P(x, \cdot) : X \to [0, \infty)$ is K-lower semi-continuous (i.e., if $x \in X$ and $y_n \to y \in X$, then $P(x, y) \leq \liminf_{n\to\infty} KP(x, y_n)$;
(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $P(z, x) \leq \delta$ and $P(z, y) \leq \delta$ imply $D(x, y) \leq \varepsilon$.

Example 2.11 ([12]). Let (X, D) be a b-metric space. Then the metric D is a wt-distance on X.

Example 2.12 ([12]). Let $X = \mathbb{R}$ and $D_1 = (x-y)^2$. A function $P : X \times X \to [0, \infty)$ defined by $P(x, y) = \|x\|^2 + \|y\|^2$ for all $x, y \in X$ is a wt-distance on X.

Example 2.13 ([12]). Let $X = \mathbb{R}$ and $D_1 = (x-y)^2$. A function $P : X \times X \to [0, \infty)$ defined by $P(x, y) = \|y\|^2$ for all $x, y \in X$ is a wt-distance on X.

The following two lemmas are crucial for our resuts.

Lemma 2.14 ([12]). *Let (X, D) be a b-metric space with constant $K \geq 1$ and P be a wt-distance on X. Let $\{x_n\}$, $\{y_n\}$ be two sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ two sequences in $[0, \infty)$ converging to zero. Then the following conditions hold: for all $x, y, z \in X$,*

(1) *if $P(x_n, y) \leq \alpha_n$ and $P(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $P(x, y) = 0$ and $P(x, z) = 0$, then $y = z$;*
(2) *if $P(x_n, y_n) \leq \alpha_n$ and $P(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;*

(3) *if $P(x_n, x_m) \leq \alpha_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
(4) *$P(y, x_n) \leq \alpha_n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

3. The classes of simulation functions

In 2015, Khojasteh et al. [15] introduced the notion of a simulation function which generalizes the Banach contraction as follow:

Definition 3.1 ([15])**.** A *simulation function* is a mapping $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$ for all $s, t > 0$;
(ζ_3) if $\{t_n\}$ and $\{s_n\}$ are two sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Now, we recall some examples of the simulation function given by Khojasteh et al. [15].

Example 3.2. Let $\zeta_i : [0,\infty) \times [0,\infty) \to \mathbb{R}$ for $i = 1,2,3$ be defined by

(1) $\zeta_1(t,s) = \psi(s) - \phi(t)$ for all $t, s \in [0,\infty)$, where $\phi, \psi : [0,\infty) \to [0,\infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$ for all $t > 0$;
(2) $\zeta_2(t,s) = s - \dfrac{f(t,s)}{g(t,s)}t$ for all $t, s \in [0,\infty)$, where $f, g : [0,\infty) \times [0,\infty) \to (0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t, s > 0$.
(3) $\zeta_3(t,s) = s - \varphi(s) - t$ for all $t, s \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$

Then ζ_i for $i = 1,2,3$ are a simulation function.

Recently, Roldán-López-de-Hierro et al. [18] modified the notion of a simulation function as follow:

Definition 3.3 ([18])**.** A *simulation function* is a mapping â $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$ for all $s, t > 0$;
(ζ_3) if $\{t_n\}$ and $\{s_n\}$ are two sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Note that the classes of all simulation functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ denote by $\mathcal{Z}$ and every simulation function in the original sense of Khojasteh et al. [15] is also a simulation function in the sense of Roldán-López-de-Hierroet et al. [18], but the converse is not true as in the following example.

Example 3.4 ([18])**.** Let $k \in \mathbb{R}$ be such that $k < 1$ and let $\zeta \in \mathcal{Z}$ be the function defined by

$$\zeta(t,s) = \begin{cases} 2s - 2t, & \text{if } s < t, \\ ks - t, & \text{otherwise.} \end{cases}$$

Then ζ is a simulation function in the sense of Definition 3.3, but ζ does not satisfy the condition (ζ_3) of Definition 3.1.

Definition 3.5. Let (X,d) is a complete metric space. A mapping $T : X \to X$ is called *$\mathcal{Z}$-contraction* if there exists $\zeta \in \mathcal{Z}$ such that

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0 \tag{3.1}$$

for all $x, y \in X$.

Remark 3.6. If we take $\zeta(t,s) = \lambda s - t$ for all $s, t \geq 0$, where $\lambda \in [0,1)$ in Definition 3.5, then the $\mathcal{Z}$-contraction become to the Banach contraction.

4. **Fixed point theorems for simulation functions**

In this section, we consider the concept of a simulation function and show the existence of a fixed point for such mapping in complete b-metric spaces via the wt-distance. First, we improve the notion of a simulation function for our considerations as follow:

Definition 4.1. Let K be a given real number such that $K \geq 1$. A *simulation function* is a mapping $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ satisfying the following conditions:

- (ζ_1) $\zeta(0,0) = 0$;
- (ζ_2) $\zeta(Kt,s) < s - Kt$ for all $s,t > 0$;
- (ζ_3) if $\{t_n\}$ and $\{s_n\}$ are two sequences in $(0,\infty)$ such that $\limsup_{n\to\infty} Kt_n = \limsup_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \zeta(Kt_n, s_n) < 0.$$

Example 4.2. Let $\lambda, K \in \mathbb{R}$ be such that $\lambda < 1$ and $K \geq 1$. Define the mapping â $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\zeta(Kt,s) = \begin{cases} s - Kt, & \text{if } s < t, \\ \dfrac{\lambda s - Kt}{Ks + 1}, & \text{otherwise.} \end{cases}$$

Clearly, ζ verifies (ζ_1), and ζ satisfies (ζ_2). Indeed,

$$s,t > 0, \begin{cases} 0 < s < t & \Rightarrow \zeta(Kt,s) = s - Kt, \\ 0 < t < s, & \Rightarrow \zeta(Kt,s) = \dfrac{\lambda s - Kt}{Ks+1} < \dfrac{s - Kt}{Ks+1} < s - Kt. \end{cases}$$

Next, we will show that ζ satisfies (ζ_3). If $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $\limsup_{n\to\infty} Kt_n = \limsup_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$.

then

$$\begin{aligned}\limsup_{n\to\infty}\zeta(Kt_n, s_n) &= \limsup_{n\to\infty}\left(\frac{\lambda s_n - Kt_n}{Ks_n+1}\right)\\ &< \limsup_{n\to\infty}\left(\frac{s_n - Kt_n}{Kt_n+1}\right)\\ &< \limsup_{n\to\infty}\left(\frac{s_n - Kt_n}{Kt_n}\right)\\ &< \limsup_{n\to\infty}\left(\frac{s_n}{Kt_n} - \frac{Kt_n}{Kt_n}\right)\\ &\leq \limsup_{n\to\infty}\left(\frac{s_n}{Kt_n}\right) - \liminf_{n\to\infty}(1)\\ &\leq 1-1\\ &= 0.\end{aligned}$$

Then ζ is a simulation function in the sense of Definition 4.1, but ζ does not satisfy the condition (ζ_3) of Definition 3.1. Indeed, if we take $K=1$, $t_n = 2\sqrt{2}$ and $s_n = 2\sqrt{2} - \frac{1}{n}$, for all $n \in \mathbb{N}$. Then, $s_n < t_n$

$$\limsup_{n\to\infty}\zeta(t_n, s_n) = \limsup_{n\to\infty}\left(2\sqrt{2} - \frac{1}{n} - 2\sqrt{2}\right) = \limsup_{n\to\infty}\left(-\frac{1}{n}\right) = 0.$$

Theorem 4.3. *Let $(X,\leq)$ be a partially ordered set, (X,D) be a complete $b-$metric space with constant $K \geq 1$ and P be a wt-distance on X. Suppose that $T : X \to X$ is a nondecreasing mapping satisfying the following conditions:*

(i) *there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(KP(Tx, T^2x), P(x, Tx)) \geq 0 \tag{4.1}$$

for all $(x, Tx) \in X_{\leq}$;

(ii) *for all $x \in X$ with $(x, Tx) \in X_{\leq}$,*

$$\inf\{P(x,y) + P(x,Tx)\} > 0$$

for all $y \in X$ with $y \neq Ty$;

(iii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_{\leq}$.*

Then T has a fixed point in X. Moreover, if $Tx = x$, then $P(x,x) = 0$.

Proof. If $Tx_0 = x_0$, then we are done. Suppose that the conclusion is not true. Then there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_{\leq}$. Since T is nondecreasing, we have $(Tx_0, T^2x_0) \in X_{\leq}$. Continuing this process, we obtain $(T^nx_0, T^mx_0) \in X_{\leq}$ for all $n, m \in \mathbb{N}$. Now, we claim that

$$\lim_{n\to\infty} P(T^nx_0, T^{n+1}x_0) = 0. \tag{4.2}$$

By the assumption (i) and the property of ζ, we observe that

$$\begin{aligned} 0 &\leq \zeta(KP(T^nx_0, T^{n+1}x_0), P(T^{n-1}x_0, T^nx_0))\\ &\leq P(T^{n-1}x_0, T^nx_0) - KP(T^nx_0, T^{n+1}x_0)\end{aligned} \tag{4.3}$$

for all $n \in \mathbb{N}$. Since $K \geq 1$ and using (4.3), we get

$$P(T^nx_0, T^{n+1}x_0) \leq KP(T^nx_0, T^{n+1}x_0) \leq P(T^{n-1}x_0, T^nx_0). \tag{4.4}$$

This mean that the sequence $\{P(T^n x_0, T^{n+1}x_0)\}$ is a decreasing sequence of nonnegative real numbers and so it is convergent to some $r \geq 0$. Suppose that $r > 0$.

Case I. If $K > 1$, letting $n \to \infty$ in (4.4), we get $r \leq Kr \leq r$ which is a contradiction.

Case II. If $K = 1$, putting $t_n = P(T^{n+1}x_0, T^{n+2}x_0)$ and $s_n = P(T^n x_0, T^{n+1}x_0)$, the sequences $\{Kt_n\}$ and $\{s_n\}$ have the same positive limit. Also, the sequences $\{Kt_n\}$ and $\{s_n\}$ have the same positive limit superior and verify that $t_n < s_n$ for all $n \in \mathbb{N}$. By the condition (ζ_3) of definition 4.1 we have

$$\limsup_{n\to\infty} \zeta(KP(T^{n+1}x_0, T^{n+2}x_0), P(T^n x_0, T^{n+1}x_0)) = \limsup_{n\to\infty} \zeta(Kt_n, s_n) < 0,$$

which is a contradiction. Therefore $r = 0$, that is, the claim (4.3) holds. Next, we show that

$$\lim_{m,n\to\infty} P(T^n x_0, T^m x_0) = 0. \tag{4.5}$$

Suppose that this is not true. Then we can find $\varepsilon_0 > 0$ with the sequences $\{m_k\}$, $\{n_k\}$ such that, for any $m_k > n_k$ such that

$$P(T^{n_k}x_0, T^{m_k}x_0) > \varepsilon_0 \tag{4.6}$$

for all $k \in \{1, 2, 3, \cdots\}$. We can assume that m_k is a minimum index such that (4.6) holds. Then we also have

$$P(T^{n_k}x_0, T^{m_k-1}x_0) \leq \varepsilon_0. \tag{4.7}$$

Hence we have

$$\begin{aligned} \varepsilon_0 &< P(T^{n_k}x_0, T^{m_k}x_0) \\ &\leq K[P(T^{n_k}x_0, T^{m_k-1}x_0) + P(T^{m_k-1}x_0, T^{m_k}x_0)] \\ &< K\varepsilon_0 + KP(T^{m_k-1}x_0, T^{m_k}x_0). \end{aligned}$$

Taking limit superior as $k \to \infty$ in the above inequality and using (4.2), we have

$$\varepsilon_0 < \limsup_{k\to\infty} P(T^{n_k}x_0, T^{m_k}x_0) \leq K\varepsilon_0. \tag{4.8}$$

Now, we claim that $\limsup_{n\to\infty} P(T^{n_k+1}x_0, T^{m_k+1}x_0) < \varepsilon_0$. If

$$\limsup_{k\to\infty} P(T^{n_k+1}x_0, T^{m_k+1}x_0) \geq \varepsilon_0,$$

then there exists $\{k_r\}$ and $\delta > 0$ such that

$$\limsup_{r\to\infty} P(T^{n_{k_r}+1}x_0, T^{m_{k_r}+1}x_0) = \delta \geq \varepsilon_0. \tag{4.9}$$

By the assumption (i) and the property of ζ, we have

$$\begin{aligned} 0 &\leq \zeta(KP(T^{n_{k_r}+1}x_0, T^{m_{k_r}+1}x_0), P(T^{n_{k_r}}x_0, T^{m_{k_r}}x_0)) \\ &\leq P(T^{n_{k_r}}x_0, T^{m_{k_r}}x_0) - KP(T^{n_{k_r}+1}x_0, T^{m_{k_r}+1}x_0). \end{aligned} \tag{4.10}$$

Hence,

$$KP(T^{n_{k_r}+1}x_0, T^{m_{k_r}+1}x_0) \leq P(T^{n_{k_r}}x_0, T^{m_{k_r}}x_0), \tag{4.11}$$

it follows from (4.8), (4.9) and (4.11), we get that

$$K\delta = \limsup_{r\to\infty} KP(T^{n_{k_r}+1}x_0, T^{m_{k_r}+1}x_0) \leq \limsup_{r\to\infty} P(T^{n_{k_r}}x_0, T^{m_{k_r}}x_0) \leq K\varepsilon_0 \leq K\delta.$$

Therefore the sequence $\{Kt_{k_r} := KP(T^{n_{k_r}+1}x_0, T^{m_{k_r}+1}x_0)\}$ and $\{s_{k_r} := P(T^{n_{k_r}}x_0, T^{m_{k_r}}x_0)\}$ have the same positive limit superior and verify that $t_{k_r} < s_{k_r}$ for all $r \in \mathbb{N}$. By the property (ζ_3), we conclude that

$$\begin{aligned} 0 &\leq \limsup_{r\to\infty} \zeta(KP(T^{n_{k_r}+1}x_0, T^{m_{k_r}+1}x_0), P(T^{n_{k_r}}x_0, T^{m_{k_r}}x_0)) \\ &= \limsup_{r\to\infty} \zeta(Kt_{k_r}, s_{k_r}) < 0, \end{aligned}$$

which is a contradiction and hence (4.5) hold. It follows from Lemma 2.14 (iii) that $\{T^n x_0\}$ is a Cauchy sequence. Since X is a complete $b-$metric space, the sequence $\{T^n x_0\}$ converges to some element $z \in X$. From the fact that $\lim_{m,n\to\infty} P(T^n x_0, T^m x_0) = 0$, for each $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $n > N_\varepsilon$ implies

$$P(T^{N_\varepsilon}x_0, T^n x_0) < \varepsilon.$$

Since $P(x,\cdot)$ is K-lower semi-continuous and the sequence $\{T^n x_0\}$ converges to z, we have

$$P(T^{N_\varepsilon}x_0, z) \leq \liminf_{n\to\infty} KP(T^{N_\varepsilon}x_0, T^n x_0) \leq K\varepsilon. \tag{4.12}$$

Setting $\varepsilon = \frac{1}{k^2}$ and $N_\varepsilon = n_k$, by (4.12), we have

$$\lim_{k\to\infty} P(T^{n_k}x_0, z) = 0. \tag{4.13}$$

Now, we prove that z is a fixed point of T. Suppose that $Tz \neq z$. Since

$$(T^{n_k}x_0, T^{n_k+1}x_0) \in X_{\leq}$$

for each $n \in \mathbb{N}$, using the assumption (ii), (4.2) and (4.13), we have

$$0 < \inf\{P(T^{n_k}x_0, z) + P(T^{n_k}x_0, T^{n_k+1}x_0)\} \to 0$$

as $n \to \infty$, which is a contradiction. Therefore, $Tz = z$.
If $Tx = x$, we distinguish two cases.
case I If $K = 1$, then

$$0 \leq \zeta(P(Tx, T^2x), P(x, Tx)) = \zeta(P(x,x), P(x,x)) \leq P(x,x) - P(x,x) = 0.$$

Hence $\zeta(P(Tx, T^2x), P(x, Tx)) = 0$ and so, by (ζ_1), we obtain $P(x,x) = 0$.
case II If $K > 1$, then

$$\begin{aligned} 0 &\leq \zeta(KP(Tx, T^2x), P(x, Tx)) \\ &= \zeta(KP(x,x), P(x,x)) \\ &\leq P(x,x) - KP(x,x) \\ &= (1-K)P(x,x), \end{aligned}$$

it follow that $P(x,x) \leq 0$ and thus we must have $P(x,x) = 0$. This completes the proof. □

Now, we give an example to illustrate Theorem 4.3.

Example 4.4. Let $X = [0,1]$ and $D(x,y) = (x-y)^2$ with the wt-distance P on X defined by $P(x,y) = |y|^2$. We consider the following set:

$$X_{\leq} = \big\{(x,y) \in X \times X \,:\, x = y \text{ or } x, y \in \{0\} \cup \{\frac{1}{2^n} \,:\, n \geq 1\}\big\}$$

with the usual ordering. Let $T : X \to X$ be a mapping defined by

$$T(x) = \begin{cases} \frac{1}{2^{n+1}}, & \text{if } \ x = \dfrac{1}{2^n}, \ \ n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

for all $x \in X$. Obviously, T is nondecreasing. Also, T satisfies the condition (ii). Indeed, for any $n \in \mathbb{N}$, we have $\frac{1}{2^n} \neq T(\frac{1}{2^n})$. Moreover, for each $n \in \mathbb{N}$, we have

$$\inf\Big\{P\Big(\frac{1}{2^m}, \frac{1}{2^n}\Big) + P\Big(\frac{1}{2^m}, \frac{1}{2^m} - \frac{1}{2^{2m+1}}\Big) \,:\, m \in \mathbb{N}\Big\} = \frac{1}{2^{2n}} > 0.$$

Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ define by

$$\zeta(t,s) = \frac{s - Kt}{1 + Ks} \ \text{ for all } \ \ s, t \in [0,\infty).$$

Similarly, in Example 4.2, the function define as above is simulation function in the sense of Definition 4.1. Now, we show that T satisfies the condition (i). Let given $x = \frac{1}{2^n}$ with $(\frac{1}{2^n}, T(\frac{1}{2^n})) \in X_{\leq}$. Then we have

$$\begin{aligned}
\zeta(2P(Tx, T^2x), P(x,Tx)) &= \zeta(2P(\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}), P(\frac{1}{2^n}, \frac{1}{2^{n+1}})) \\
&= \zeta(2\frac{1}{2^{2n+4}}, \frac{1}{2^{2n+2}}) \\
&= \frac{\dfrac{1}{2^{2n+2}} - 2 \cdot \dfrac{1}{2^{2n+4}}}{1 + 2 \cdot \dfrac{1}{2^{2n+2}}} \\
&= \frac{2^{2n+3} - 2^{2n+2}}{(2^{2n+2})(2^{2n+3})} \cdot \frac{2^{2n+1}}{2^{2n+1} + 1} \\
&= \frac{2^{2n+2}(2-1)}{(2^{2n+4})(2^{2n+1}+1)} \\
&= \frac{2^{2n+2}}{(2^{2n+4})(2^{2n+1}+1)} \\
&> 0.
\end{aligned}$$

Therefore, all the hypothesis of Theorem 4.3 are satisfied and, further, $x = 0$ is a fixed point of T.

Corollary 4.5. *Let $(X, \leq)$ be a partially ordered set and (X, D) be a complete metric type space with constant $K \geq 1$ and P be a wt-distance on X. Suppose that $T : X \to X$ is a nondecreasing mapping satisfying the following conditions:*

(i) *there exists $\alpha \in [0, \frac{1}{K})$ such that*

$$P(Tx, T^2x) \leq \alpha P(x, Tx)$$

for all $x \leq Tx$;

(ii) *for all $x \in X$ with $x \leq Tx$,*

$$\inf\{P(x, y) + P(x, Tx)\} > 0$$

for all $y \in X$ with $y \neq Ty$;

(iii) *there exists $x_0 \in X$ such that $x_0 \leq Tx_0$.*

Then T has a fixed point in X.

Theorem 4.6. *Let $(X, \leq)$ be a partially ordered set and (X, D) be a complete b-metric space with constant $K \geq 1$ and P be a wt-distance on X. Suppose that $T : X \to X$ is a nondecreasing mapping and there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(KP(Tx, T^2x), P(x, Tx)) \geq 0$$

for all $(x, Tx) \in X_{\leq}$. Assume that one of the following conditions holds:

(i) *for all $x \in X$ with $(x, Tx) \in X_{\leq}$,*

$$\inf\{P(x, y) + P(x, Tx)\} > 0$$

for all $y \in X$ with $y \neq Ty$;

(ii) *if both $\{x_n\}$ and $\{Tx_n\}$ converge to z, then $z = Tz$;*

(iii) *T is continuous on X.*

If there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_{\leq}$, then T has a fixed point in X. Moreover, if $Tx = x$, then $P(x, x) = 0$.

Proof. In the case of T satisfying the condition (i), the conclusion was proved in Theorem 4.3. Let us prove that (ii) $\Longrightarrow$ (i). Suppose that the condition (ii) holds. Let $y \in X$ with $y \neq Ty$ such that

$$\inf\{P(x, y) + P(x, Tx) \, : \, (x, Tx) \in X_{\leq}\} = 0.$$

Then we can find a sequence $\{z_n\}$ such that $(z_n, Tz_n) \in X_{\leq}$ and

$$\inf\{P(z_n, y) + P(z_n, Tz_n)\} = 0.$$

So we have

$$\lim_{n\to\infty} P(z_n, y) = \lim_{n\to\infty} P(z_n, Tz_n) = 0.$$

Again, by Lemma 2.14, we have $\lim_{n\to\infty} Tz_n = y$. Moreover, $\lim_{n\to\infty} T^2z_n = y$. In fact, since

$$0 \leq \zeta(KP(Tz_n, T^2z_n), P(z_n, Tz_n)) \leq P(z_n, Tz_n) - KP(Tz_n, T^2z_n), \tag{4.14}$$

it follow from (4.14) and $K \geq 1$, we get that

$$\lim_{n\to\infty} P(Tz_n, T^2z_n) \leq \lim_{n\to\infty} KP(Tz_n, T^2z_n) \leq \lim_{n\to\infty} P(z_n, Tz_n) = 0.$$

Letting $x_n = Tz_n$, the sequences $\{x_n\}$ and $\{Tx_n\}$ converge to y. Hence, by the assumption (ii), $y = Ty$ and so (ii) $\Longrightarrow$ (i). Obviously, (iii) $\Longrightarrow$ (ii). This completes the proof. □

Now, we prove new theorems by replacing some conditions in Theorem 4.3 with other conditions.

Theorem 4.7. *Let $(X, \leq)$ be a partially ordered set and (X, D) be a complete b-metric space with constant $K \geq 1$ and P be a wt-distance on X. Suppose that $T : X \to X$ is a nondecreasing satisfying the following conditions:*

(i) *there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(KP(Tx, T^2x), P(x, Tx)) \geq 0$$

for all $(x, Tx) \in X_{\leq}$;

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_{\leq}$,*

(iii) *either T is orbitally continuous at x_0 or*

(iv) *T is orbitally $X_{\leq}$-continuous and there exists a subsequence $\{T^{n_k}x_0\}$ of $\{T^n x_0\}$ converges to some element $x_\star \in X$ such that $(T^{n_k}x_0, x_\star) \in X_{\leq}$ for any $k \in \mathbb{N}$.*

Then T has a fixed point in X. Moreover if $Tx = x$, then $P(x, x) = 0$.

Proof. If $Tx_0 = x_0$, then we are done. Suppose that the conclusion is not true. Then there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_{\leq}$. Since T is monotone, we have $(Tx_0, T^2x_0) \in X_{\leq}$. Continuing this process, we have a sequence $\{T^n x_0\}$ such that

$$(T^n x_0, T^m x_0) \in X_{\leq}$$

for any $n, m \in \mathbb{N}$. As in the same argument in Theorem 4.3, we can see that

$$\lim_{n\to\infty} P(T^n x_0, T^{n+1}x_0) = 0. \tag{4.15}$$

Moreover,

$$\lim_{m,n\to\infty} P(T^n x_0, T^m x_0) = 0. \tag{4.16}$$

and $\{T^n x_0\}$ is a Cauchy sequence converges to some element $z \in X$. Next, we prove that z is a fixed point of T. If the condition (iii) holds, then $T^{n+1}x_0 \to Tz$. By $P(x, \cdot)$ is K-lower semi-continuous and (4.16), we have

$$P(T^n x_0, z) \quad \leq \liminf_{m\to\infty} KP(T^n x_0, T^m x_0) \quad \leq \alpha_n^{'} \ \text{(say)} \tag{4.17}$$

and

$$P(T^n x_0, Tz) \quad \leq \liminf_{m\to\infty} KP(T^n x_0, T^{m+1} x_0) \quad \leq \beta_n^{'}, \ \text{(say)} \tag{4.18}$$

where the sequences $\{\alpha_n^{'} := \frac{\alpha_n}{K}\}$ and $\{\beta_n^{'} := \frac{\beta_n}{K}\}$ which converges to 0. By Lemma 2.14 (i), we conclude that $z = Tz$.

Suppose that the condition (iv) hold. From the fact that $\{T^{n_k}x_0\} \to z$ as $k \to \infty$, $(T^{n_k}x_0, z) \in X_{\leq}$ and T is orbitally $X_{\leq}$-continuous, it follows that $\{T^{n_k+1}x_0\} \to Tz$ as $k \to \infty$. Similarly, since $P(x, \cdot)$ is K-lower semi-continuous as above, we conclude that $z = Tz$ and the remaining part of the proof follow from the proof of Theorem 4.3. □

Corollary 4.8. *Let $(X, \leq)$ be a partially ordered set and (X, D) be a complete metric space and p be a w-distance on X. Suppose that $T : X \to X$ is a nondecreasing satisfying the following conditions:*

(i) *there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(p(Tx, T^2x), p(x, Tx)) \geq 0$$

for all $(x, Tx) \in X_{\leq}$;

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_{\leq}$,*

(iii) *either T is orbitally continuous at x_0 or*

(iv) *T is orbitally $X_{\leq}$-continuous and there exists a subsequence $\{T^{n_k}x_0\}$ of $\{T^n x_0\}$ converges to some element $x_\star \in X$ such that $(T^{n_k}x_0, x_\star) \in X_{\leq}$ for any $k \in \mathbb{N}$.*

Then T has a fixed point in X. Moreover if $Tx = x$, then $p(x, x) = 0$.

Corollary 4.9. *Let $(X, \leq)$ be a partially ordered set and (X, D) be a complete b-metric space with constant $K \geq 1$ and P be a wt-distance on X. Suppose that $T : X \to X$ is a nondecreasing satisfying the following conditions:*

(i) *there exists $\lambda \in [0, \frac{1}{K})$ such that*

$$P(Tx, T^2x) \leq \lambda P(x, Tx)$$

for all $(x, Tx) \in X_{\leq}$;

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_{\leq}$,*

(iii) *either T is orbitally continuous at x_0 or*

(iv) *T is orbitally $X_{\leq}$-continuous and there exists a subsequence $\{T^{n_k}x_0\}$ of $\{T^n x_0\}$ converges to some element $x_\star \in X$ such that $(T^{n_k}x_0, x_\star) \in X_{\leq}$ for any $k \in \mathbb{N}$.*

Then T has a fixed point in X. Moreover, if $Tx = x$, then $P(x, x) = 0$.

Example 4.10. Let $X = [0, 1]$ and $D(x, y) = (x - y)^2$ with the wt-distance P on X defined by $P(x, y) = |y|^2$. We consider the following set:

$$X_{\leq} = \big\{(x, y) \in X \times X \,:\, x = y \text{ or } x, y \in \{0\} \cup \{\frac{1}{n} \,:\, n \geq 1\}\big\},$$

where $\leq$ is the usual ordering. Let $T : X \to X$ be a mapping define by

$$T(x) = \begin{cases} x^2, & \text{if } x = \dfrac{1}{n},\ n \geq 2, \\ \dfrac{x}{2}, & \text{otherwise.} \end{cases}$$

Then T is a nondecreasing mapping. Also, $x = 0$ is an element in X such that $0 \leq T(0) = 0$ and so $(0, T(0)) \in X_{\leq}$. Hence T satisfies the condition (ii).

Next, we show that T satisfies the condition (i) of Theorem 4.7 with the simulation function in given in Example 4.4. If $x \neq \dfrac{1}{n}$ for all $n \geq 2$, then

$(x, T(x)) \in X_{\leq}$ and it is easy to see that T satisfies the condition (i). If $x = \dfrac{1}{n}$ for all $n \geq 2$, then $(\dfrac{1}{n}, T\dfrac{1}{n}) \in X_{\leq}$. Further, we have

$$\begin{aligned}
\zeta(2P(Tx, T^2x), P(x, Tx)) &= \zeta\Big(2P\Big(\frac{1}{n^2}, \frac{1}{n^4}\Big), P\Big(\frac{1}{n}, \frac{1}{n^2}\Big)\Big) \\
&= \zeta\Big(2\Big(\frac{1}{n^4}\Big)^2, \Big(\frac{1}{n^2}\Big)^2\Big) \\
&= \frac{\Big(\frac{1}{n^2}\Big)^2 - 2\Big(\frac{1}{n^4}\Big)^2}{1 + 2\cdot\Big(\frac{1}{n^2}\Big)^2} \\
&= \frac{n^8 - 2n^4}{n^{12}} \cdot \frac{n^4}{n^4+2} \\
&= \frac{n^8 - 2n^4}{n^8(n^4+2)} \\
&= \frac{n^4 - 2}{n^4(n^4+2)} \\
&> 0.
\end{aligned}$$

Hence T satisfies the condition (i). Furthermore, for each $x \in X$, $T^{n_i}(x) \to 0 \in X$ as $i \to \infty$, and also $T^{n_i+1}(x) \to T(0) \in X$ as $i \to \infty$. Hence all the conditions of Theorem 4.7 are satisfied. Furthermore, $x = 0$ is fixed points of T.

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