APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 19, no. 1 (2018), 1-7
doi:10.4995/agt.2018.7012

Partially topological group action

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Communicated by S. García-Ferreira

Abstract

The concept of partially topological group was recently introduced in [3]. In this article, we define partially topological group action on partially topological space and we generalize some fundamental results from topological group action theory.

2010 MSC: *Primary: 54A25; Secondary: 54B05.*

Keywords: *partially topological space; partially topological group; group action.*

1. Partially topological spaces

In this section, we recall definition of the category $\mathbf{GTS}_{pt}$ of partially topological spaces and strictly continuous mappings which was defined in [4].

Definition 1.1. Let X be any set, τ_X be a topology on X. A family of open families $\mathrm{Cov}_X \subseteq \mathcal{P}(\tau_X)$ will be called a **partial topology** if the following conditions are satisfied:

(i) if $\mathcal{U} \subseteq \tau_X$ and $\mathcal{U}$ is finite, then $\mathcal{U} \in \mathrm{Cov}_X$;
(ii) if $\mathcal{U} \in \mathrm{Cov}_X$ and $V \in \tau_X$, then $\{U \cap V : U \in \mathcal{U}\} \in \mathrm{Cov}_X$;
(iii) if $\mathcal{U} \in \mathrm{Cov}_X$ and, for each $U \in \mathcal{U}$, we have $\mathcal{V}(U) \in \mathrm{Cov}_X$ such that $\bigcup \mathcal{V}(U) = U$, then $\bigcup_{U\in\mathcal{U}} \mathcal{V}(U) \in \mathrm{Cov}_X$;
(iv) if $\mathcal{U} \subseteq \tau_X$ and $\mathcal{V} \in \mathrm{Cov}_X$ are such that $\bigcup \mathcal{V} = \bigcup \mathcal{U}$ and, for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$, then $\mathcal{U} \in \mathrm{Cov}_X$.

Received 09 December 2016 – Accepted 18 September 2017

Elements of τ_X are called **open sets**, and elements of Cov_X are called **admissible families**. We say that (X, Cov_X) is a **partially topological generalized topological space** or simply **partially topological space**. For simplicity, from now on, we shall denote a partially topological space (X, Cov_X) by X.

Let X and Y be partially topological spaces and let $f : X \to Y$ be a function. Then f is called **strictly continuous** if $f^{-1}(\mathcal{U}) \in \mathrm{Cov}_X$ for any $\mathcal{U} \in \mathrm{Cov}_Y$. A bijection $f : X \to Y$ is called a **strictly homeomorphism** if both f and f^{-1} are strictly continuous functions. If we have a strictly homeomorphism between X and Y we say that they are **strictly homeomorphic** and we denote that by $X \cong Y$.

Remark 1.2. The above notion of partial topology is a special case of the notion of generalized topology in the sense of H. Delfs and M. Knebusch considered in [2, 4, 5, 6, 7], when the family Op_X of open sets of the generalized topology forms a topology.

Definition 1.3. Let (X, Cov_X) be a partially topological space and let Y be a subset of X. Then the partial topology

$$\mathrm{Cov}_Y = (\langle \mathrm{Cov}_X \cap_2 Y \rangle_Y)_{pt},$$

that is: the smallest partial topology containing $\mathrm{Cov}_X \cap_2 Y$, is called a **subspace partial topology** on Y, and (Y, Cov_Y) is a **subspace** of (X, Cov_X). (It is also the smallest generalized topology containing $\mathrm{Cov}_X \cap_2 Y$.)

Fact 1.4. *Let $\varphi : X \to X'$ be a mapping between partially topological spaces and let Y be a subspace of X. Then the following are equivalent:*

a) *φ is strictly continuous,*
b) *the restriction mapping $\varphi|_Y : Y \to X'$ is strictly continuous.*

Definition 1.5. Let (X, Cov_X) and (Y, Cov_Y) be two partially topological spaces. The **product partial topology** on $X \times Y$ is the partial topology $\mathrm{Cov}_{X\times Y} = (\langle \mathrm{Cov}_X \times_2 \mathrm{Cov}_Y \rangle_{X\times Y})_{pt}$ in the notation of Definition 4.6 of [7]; in other words: the smallest partial topology in $X \times Y$ that contains $\mathrm{Cov}_X \times_2 \mathrm{Cov}_Y$.

Recall that a mapping $f : X \to Y$ is said to be an open mapping if for every open set U of X, the set $f(U)$ is open in Y. It is said to be a closed mapping if for every closed set A of X, the set $f(A)$ is closed in Y. Also, recall that a surjective mapping $f : X \to Y$ is said to be a quotient mapping provided a subset U of Y is open in Y if and only if $f^{-1}(U)$ is open in X.

2. Partially Topological Groups

In this section, we recall the definition of partially topological group. This notion was recently introduced in [3].

Definition 2.1. A **partially topological group** G is an ordered pair $((G, *), \mathrm{Cov}_G)$ such that $(G, *)$ is a group, while Cov_G is a generalized topology on G such that $\bigcup \mathrm{Cov}_G$ is a T_1 topology on G and the multiplication mapping of $(G \times G, \mathrm{Cov}_{G\times G})$ into (G, Cov_G), which sends ordered pair $(x, y) \in G \times G$

to $x * y$, is strictly continuous and the inverse mapping from (G, Cov_G) into (G, Cov_G), which sends each $x \in G$ to x^{-1}, is strictly continuous. For simplicity, from now on, we shall denote a partially topological group $((G, *), \mathrm{Cov}_G)$ by G.

Definition 2.2. Any subgroup H of a partially topological group G is a partially topological group and it is called a **partially topological subgroup of** G.

Definition 2.3. Let $\varphi : G \to G'$ be a function. Then φ is called a **morphism of partially topological groups** if φ is both strictly continuous and group homomorphism. Moreover, φ is an **isomorphism** if it is strictly homeomorphism and group isomorphism.

If we have an isomorphism between two partially topological groups G and G', then we say that they are isomorphic and we denote that by $G \cong G'$.

Remark 2.4. Obviously composition of two morphisms of partially topological groups is a morphism. In addition, the identity mapping is an isomorphism. So partially topological groups and their morphisms form a category **PTGr**.

3. Partially Topological Group Action On Partially Topological Space

In this section, we introduce partially topological group action on partially topological space and we extend some fundamental results in [1] of action of a topological group on a topological space to this new concept.

Definition 3.1. If G is a partially topological group with identity e and X is a partially topological space, then an **action** of G on X is a mapping $G \times X \to X$, with the image of (g, x) being denoted by $g(x)$, such that $(gh)(x) = g(h(x))$ and $e(x) = x$ for all $g, h \in G$ and $x \in X$.

If this mapping is strictly continuous, then the action is said to be strictly continuous.

The space X, with a given strictly continuous action of G on X, is called **partially** G**-space**.

For a point $x \in X$, the set $G(x) = \{gx : g \in G\}$ is called the **orbit** of x.

Definition 3.2. Let G be a partially topological group and X a partially topological space. Let G act on X. For a point x of X, the set

$$G_x = \{g \in G : gx = x\} \quad (or \quad G_x = \{g \in G : xg = x\})$$

is called the **stabilizer of** x.

Fact 3.3. *The stabilizer G_x of any point $x \in X$ is a subgroup of G.*

Definition 3.4. Let G be a partially topological group and X a partially topological space. Let G act on X. For a point x of X, we define a mapping

$$\mu_x : G \to X$$

by $\mu_x(g) = gx$ (or $\mu_x(g) = xg$).

Note that μ_x is strictly continuous by strictly continuity of the action. The action is called transitive if for each $x \in X$, $G_x = X$. Then Obviously we have the following fact.

Fact 3.5. *μ_x is surjective iff G acts transitively on X.*

Proposition 3.6. *Every strictly continuous action $\theta : G \times X \to X$ of a partially topological group G on a partially topological space X is an open mapping.*

Proof. It suffices to prove that the images under θ of the elements of some base for $G \times X$ are open in X. Let $O = U \times V \subset G \times X$, where U and V are open sets in G and X, respectively. Then $\theta(O) = \bigcup_{g \in G} \theta_g(V)$ is open in X since every θ_g is a strictly homeomorphism of X onto itself. Since the open sets $U \times V$ form a base for $G \times X$, the mapping θ is open. □

Proposition 3.7. *The strictly continuity of an action $\theta : G \times X \to X$ of a partially topological group G with identity e on a partially topological space X is equivalent to the strictly continuity of θ at the points of the set $\{e\} \times X \subset G \times X$.*

Proof. Let $g \in G$ and $x \in X$ be arbitrary and U be a neighborhood of gx in X. Since θ_h is a homeomorphism of X for each $h \in G$, the set $V = \theta_{g-1}(U)$ is a neighborhood of x in X. By the strictly continuity of θ at (e, x), we can find a neighborhood O of e in G and a neighborhood W of x in X such that $hy \in V$ for all $h \in O$ and $y \in W$. Clearly, if $h \in O$ and $y \in W$, then $(gh)(y) = g(hy) \in gV = \theta_g(V) = U$. Thus, $ky \in U$, for all $k \in gO$ and all $y \in W$, where $O' = gO$ is a neighborhood of g in G. Hence, the action θ is strictly continuous. □

Next we present two examples of strictly continuous actions of partially topological groups.

Example 3.8. Any partially topological group G acts on itself by left translations, that is, $\theta(x, y) = xy$ for all $x, y \in G$. The strictly continuity of this action follows from the strictly continuity of the multiplication in G.

Example 3.9. Let G be a partially topological group, H a closed subgroup of G, and let G/H be the corresponding left coset space. The action ϕ of G on G/H, defined by the rule $\phi(g, xH) = gxH$, is strictly continuous. Indeed, let $y_0 \in G/H$, and fix an open neighborhood O of y_0 in G/H. Choose $x_0 \in G$ such that $\pi(x_0) = y_0$, where $\pi : G \to G/H$ is the quotient mapping. There exist open neighborhoods U and V of the identity e in G such that $\pi(Ux_0) \subset O$ and $V^2 \subset U$. Clearly, $W = \pi(Vx_0)$ is open in G/H and $y_0 \in W$. By the choice of U and V, if $g \in V$ and $y \in W$, then $\phi(g, y) \in O$. Indeed, let $x_1 \in Vx_0$ with $\pi(x_1) = y$. Then $y = x_1H$ and $\phi(g, y) = gx_1H \in VVx_0H \subset \pi(Ux_0) \subset O$. Therefore, ϕ is continuous at $(e, y_0) \in G \times G/H$. Hence, ϕ is strictly continuous by Proposition 3.7.

Suppose that a partially topological group G acts strictly continuously on a partially topological space X and that X/G is the corresponding orbit set. Let

X/G have the partially quotient topology generated by the orbital projection $\pi : X \to X/G$ (a subset $U \subset X/G$ is open in X/G if and only if $\pi^{-1}(U)$ is open in X). The partially topological space X/G is called the orbit space or the orbit space of the partillay G-space X.

The following result shows that the orbital projection is always an open mapping.

Proposition 3.10. *If $\theta : G \times X \to X$ is a strictly continuous action of a partially topological group G on a partially topological space X, then the orbital projection $\pi : X \to X/G$ is an open mapping.*

Proof. For any open set $U \subset X$, consider the set $\pi^{-1}\pi(U) = GU$. Every left translation θ_g is a strictly homeomorphism of X onto itself, so the set $GU = \bigcup_{g \in G} \theta_g(U)$ is open in X. Since π is a quotient mapping, $\pi(U)$ is open in X/G. Hence, π is an open mapping. □

Theorem 3.11. *Suppose a compact partially topological group H acts strictly continuously on a Hausdorff partially space X, then the orbital projection $\pi : X \to X/H$ is both open and perfect mapping.*

Proof. First note that π is open by Proposition 3.10. Next we show that π is perfect. Let $y \in X/H$, choose $x \in X$ such that $\pi(x) = y$. Note that $\pi^{-1}(y) = Hx$ is the orbit of x in X. Since the mapping of H onto Hx assigning to every $g \in H$ the point $gx \in X$ is strictly continuous, the image Hx of the compact group H is also compact. Hence, all fibers of π are compact.

We show that the mapping π is closed. Let $y \in X/H$ and $x \in X$ such that $\pi(x) = y$. Let O be an open set in X containing $\pi^{-1}(y) = Hx$. Since the action of H on X is strict continuous, we can find, for every $g \in H$, open neighborhoods $g \in U_g$ and $x \in V_g$ in H and X, respectively, such that $U_g V_g \subset O$. By the compactness of H and of the orbit Hx, there exists a finite set $F \subset H$ such that $H = \bigcup_{g \in F} U_g$ and $Hx \subset \bigcup_{g \in F} gV_g$. Then $V = \bigcap_{g \in F} V_g$ is an open neighborhood of x in X, and we claim that $HV \subset O$. Indeed, if $h \in H$ and $z \in V$, then $h \in U_g$, for some $g \in F$, so that $hz \in U_g V \subset U_g V_g \subset O$. Thus, $W = \pi(V)$ is an open neighborhood of y in X/H, and we have that $\pi^{-1}\pi(V) = HV \subset O$. Hence, π is closed. □

Definition 3.12. Let X and Y be partially G-spaces with strictly continuous actions $\theta_X : G \times X \to X$ and $\theta_Y : G \times Y \to Y$. A strictly continuous mapping $f : X \to Y$ is called **partially G-equivariant** if $\theta_Y(g, f(x)) = f(\theta_X(g, x))$, that is, $gf(x) = f(gx)$, for all $g \in G$ and all $x \in X$. Clearly, f is partially G-equivariant if and only if the following diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\theta_X} & X \\ \Big\downarrow F & & \Big\downarrow f \\ G \times Y & \xrightarrow{\theta_Y} & Y \end{array}$$

commutes, where $F = id_G \times f$ is the product of the identity mapping id_G of G and the mapping f.

Example 3.13. Let H be a closed subgroup of a partially topological group G, and $Y = G/H$ be the left coset space. Denote by θ_G the action of G on itself by left translations, and by θ_Y the natural strictly continuous action of G on Y. Then the quotient mapping $\pi : G \to G/H$ defined by $\pi(x) = xH$ for each $x \in G$ is equivariant. Indeed, the equality $g(\pi(x)) = gxH = \pi(gx)$ holds for all $g, x \in G$. Equivalently, the following diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\theta_G} & G \\ \Big\downarrow \Pi & & \Big\downarrow \pi \\ G \times Y & \xrightarrow{\theta_Y} & Y \end{array}$$

commutes, where $\Pi = id_G \times \pi$.

Let $\eta = \{X_i : i \in I\}$ be a family of partially G-spaces. Then the product space $X = \prod_{i \in I} X_i$, if X is Hausdorff, is a partially G-space. To define an action of G on X, take any $g \in G$ and any $x = (x_i)_{i \in I} \in X$, and put $gx = (gx_i)_{i \in I}$. Thus, G acts on X coordinatewise.

The following result shows the strictly continuity of this action.

Proposition 3.14. *The coordinatewise action of G on the product $X = \prod_{i \in I} X_i$ of partially G-spaces is strictly continuous, that is, X is a partially G-space, if X is Hausdorff.*

Proof. By Proposition 3.7, it suffices to verify the continuity of the action of G on X at the neutral element $e \in G$. Let $x = (x_i)_{i \in I} \in X$ be an arbitrary point and $O \subset X$ a neighborhood of gx in X. Since canonical open sets form a base of X, we can assume that $O = \prod_{i \in I} O_i$, where each O_i is an open neighborhood of x_i in X_i and the set $F = \{i \in I : O_i \neq X_i\}$ is finite. Since all factors are partially G-spaces, we can choose, for every $i \in F$, open neighborhoods $e \in U_i$ and $x_i \in V_i$ in G and X_i, respectively, such that $U_i V_i \subset O_i$. Put $U = \bigcup\limits_{i \in F} V_i$ and $W = \prod_{i \in I} W_i$, where $W_i = V_i$ if $i \in F$ and $W_i = X_i$ otherwise. Therefore, it follows from the definition of the sets U and W that $UW \subset O$. Hence, the action of G on X is strictly continuous. $\square$

Theorem 3.15. *Let G be a partially topological group and X a partially topological space. Let G act on X. Suppose that both G and X/G are connected, then X is connected.*

Proof. Suppose that X is the union of two disjoint nonempty open subsets U and V. Now $\pi(U)$ and $\pi(V)$ are open in X/G. Since X/G is connected, $\pi(U)$ and $\pi(V)$ cannot be disjoint. If $\pi(x) \in \pi(U) \cup \pi(V)$, then both $U \cup O(x)$ and $V \cup O(x)$ are nonempty, where $O(x)$ is the orbit of x. It means $O(x)$ is a disjoint union of two nonempty open sets. But $O(x)$ is the image of G under the strictly continuous function $f : G \to X$ defined by $f(g) = g(x)$. Therefore, $O(x)$ is connected which is a contradiction. Hence, X is connected. □

Theorem 3.16. *If X is a compact partially topological group and G a closed subgroup acting on X by left translation, then X/G is regular.*

Proof. Since G is closed subgroup and the left translation mapping $L_x : X \to X$ is strictly homeomorphism then $\pi^{-1}\pi(x) = xG = L_x(G)$ is closed. Thus every point $\pi(x)$ of X/G is closed, and it follows that X/G is T_1 space.

Now we show that for a closed subset F of X/G and a point $p \notin F$ there are open sets U, V satisfying $p \in U, F \subset V, U \cap V = \varnothing$. Since X acts transitively on X/G, we may assume that p is an element of the class $eG = G$ of the identity element e. Since F is closed, there exists an open set U_0 such that $F \cap U_0 = \varnothing$ and $p \in U_0$. From the strictly continuity of group action of X, there is an open set W such that $e \in W$ and $W^{-1}W \subset \pi^{-1}(U_0)$. The set $W\pi^{-1}(F) = \bigcup_{x \in \pi^{-1}(F)} Wx$ is open. Since π is an open mapping, both $U = \pi(W)$ and $V = \pi(W\pi^{-1}(F))$ are open sets and $p \in U$ and $F \subset V$.

Next we show that $U \cap V = \varnothing$. Suppose that there exists $y \in U \cap V$. Then there exist $x_1, x_2 \in W$ and $x \in \pi^{-1}(F)$ such that $y = \pi(x_2) = \pi(x_1 x)$. Thus, we have $g \in G$ such that $x_2 g = x_1 x$, from which we deduce that $\pi(xg^{-1}) \in F \cap U_0 = \varnothing$ from $xg^{-1} = {x_1}^{-1} x_2 \in W^{-1}W \subset \pi^{-1}(U_0)$. Therefore, $U \cap V = \varnothing$. □

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