Applied General Topology

Appl. Gen. Topol. 18, no. 2 (2017), 317-330
doi:10.4995/agt.2017.7067

Multiple fixed point theorems for contractive and Meir-Keeler type mappings defined on partially ordered spaces with a distance

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Communicated by I. Altun

Abstract

We introduce and study a general concept of multiple fixed point for mappings defined on partially ordered distance spaces in the presence of a contraction type condition and appropriate monotonicity properties. This notion and the obtained results complement the corresponding ones from [M. Choban, V. Berinde, A general concept of multiple fixed point for mappings defined on spaces with a distance, Carpathian J. Math. 33 (2017), no. 3, 275–286] and also simplifies some concepts of multiple fixed point considered by various authors in the last decade or so.

2010 MSC: *47H10; 47H09.*

Keywords: *distance space; partial order; symmetric space; quasi-metric space; H-distance; contraction condition; multiple fixed point.*

1. Introduction

In a previous paper [32], the authors have introduced and studied a general concept of multidimensional fixed point for mappings defined on a distance space and satisfying a certain contraction condition.

Received 02 January 2017 – Accepted 12 July 2017

Several interesting new results that generalise, extend and unify corresponding related results from literature for the case of non ordered distance spaces were obtained. However, the great majority of the multidimensional fixed point theorems existing in literature were established in the setting of a partially ordered metric space or of a partially ordered generalised metric space. Therefore, the main aim of this paper is to study the concept of multidimensional fixed point introduced in [32] for the case of mappings defined on partially ordered distance space, thus extending and complementing most of the results established in [32].

We start by presenting a brief survey on the notion of *multidimensional fixed point*, which naturally emerged from the rich literature produced in the last four decades devoted to coupled fixed points. The concept itself of *coupled fixed point* has been first introduced and studied by Opoitsev, in a series of papers he published in the period 1975-1986, see [61]-[64]. Opoitsev has been inspired by some concrete problems arising in the dynamics of collective behaviour in mathematical economics and considered the coupled fixed point problem for mixed monotone nonlinear operators which also satisfy a nonexpansive type condition.

In 1987, Guo and Lakshmikantham [41], apparently not being aware of Opoitsev's previous results [61]-[64], have studied coupled fixed points in connection with coupled quasi-solutions of an initial value problem for ordinary differential equations. Amongst the subsequent developments we quote the following works: [40]; [30], containing coupled fixed point results of $\frac{1}{2}$-α-condensing and mixed monotone operators, where α denotes the Kuratowski's measure of non compactness, thus extending some previous results from [41] and [79]; [29], which discusses some existence results and iterative approximation of coupled fixed points for mixed monotone condensing set-valued operators; [28] where the authors obtained coupled fixed point results of $\frac{1}{2}$-α-contractive and generalized condensing mixed monotone operators.

More recently, Gnana Bhaskar and Lakshmikantham in [37] established coupled fixed point results for mixed monotone operators in partially ordered metric spaces in the presence of a Bancah contraction type condition. Essentialy, the results by Bhaskar and Lakshmikantham in [37] combined, in the context of bivariable mixed monotone mappings, the main fixed point results previously obtained by Nieto and Rodriguez-Lopez [58] and [59], for the case of one variable increasing and decreasing nonlinear operator, respectively. The last two papers are, in turn, a continuation of the hybrid fixed point theorem established in the seminal paper of Ran and Reurings [66], which has the merit to combine a metrical fixed point theorem (the contraction mapping principle) and an order theoretic fixed point result (Tarski's fixed point theorem).

Various applications of the theoretical results in coupled fixed point theory were also considered, for the case of: a) Uryson integral equations [63]; b) a system of Volterra integral equations [30], [28]; c) a class of functional equations arising in dynamic programming [29]; d) initial value problems for first order differential equations with discontinuous right hand side [41]; e) (two point) periodic boundary value problems [17], [37], [33], [82]; f) integral equations and systems of integral equations [3], [6], [9], [24], [39], [42], [78], [80], [85]; g) nonlinear elliptic problems and delayed hematopoesis models [84]; h) nonlinear Hammerstein integral equations [76]; i) nonlinear matrix and nonlinear quadratic equations [4], [24]; j) initial value problems for ODE [8], [75] etc.

For a very recent account on the developments of coupled fixed point theory, we also refer to [22].

On the other hand, in 2010, Samet and Vetro [74] apart of some coupled fixed point results they have established, considered a concept of *fixed point of m-order* as a natural extension of the notion of coupled fixed point. Then, in 2011, mainly inspired by [37], Berinde and Borcut [18] introduced the concept of *triple fixed point* and proved existence as well as existence and uniqueness triple fixed point theorems for three-variable mixed monotone mappings, while, in 2012, Karapinar and Berinde [47], have studied quadruple fixed points of nonlinear contractions in partially ordered metric spaces.

After these starting papers, a substantial number of articles were dedicated to the study of triple fixed points, quadruple fixed points, as well as to multiple fixed points (also called *fixed point of m-order*, or "a multidimensional fixed point", or "an m-tuplet fixed point", or "an m-tuple fixed point"), see [1], [2], [7], [48], [49], [50], [53], [60], [67]-[72], [81], [83], [86], which form a very selective list contributions.

Starting from this background, the main aim of the present paper is to study the concept of multidimensional fixed point introduced in [32] but for mappings defined on partially ordered distance space, in the presence of a contraction type condition and appropriate monotonicity properties, thus extending and complementing the results established in [32].

This approach is based on the idea to reduce the study of multidimensional fixed points and coincidence points to the study of usual one-dimensional fixed points for an associate operator. Note that, the first author who reduced the problem of finding a coupled fixed point of mixed monotone operators to the problem of finding a fixed point of an increasing one variable operator was Opoitsev, see for example [63].

2. Preliminaries

By a space we understand a topological T_0-space. We use the terminology from [36, 38, 73, 31].

Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ be a mapping such that:

(i_m) $d(x,y) \geq 0$, for all $x, y \in X$;
(ii_m) $d(x,y) + d(y,x) = 0$ if and only if $x = y$.

Then d is called a *distance* on X, while (X,d) is called a *distance space*.

Let d be a distance on X and $B(x,d,r) = \{y \in X : d(x,y) < r\}$ be the *ball* with the center x and radius $r > 0$. The set $U \subset X$ is called *d-open* if for any $x \in U$ there exists $r > 0$ such that $B(x,d,r) \subset U$. The family $\mathcal{T}(d)$ of all d-open subsets is the topology on X generated by d. A distance space is a *sequential space*, i.e., a space for which a set $B \subseteq X$ is closed if and only if together with any sequence it contains all its limits [36].

Let (X, d) be a distance space, $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. We say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is:

1) *convergent to* x if and only if $\lim_{n\to\infty} d(x, x_n) = 0$. We denote this by $x_n \to x$ or $x = \lim_{n\to\infty} x_n$ (really, we may denote $x \in \lim_{n\to\infty} x_n$);
2) *convergent* if it converges to some point x in X;
3) *Cauchy* or *fundamental* if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

A distance space (X, d) is called *complete* if every Cauchy sequence in X converges to some point x in X.

Let X be a non-empty set and d be a distance on X. Then:

- (X, d) is called a *symmetric space* and d is called a *symmetric* on X if

  (iii_m) $d(x, y) = d(y, x)$, for all $x, y \in X$;
- (X, d) is called a *quasimetric space* and d is called a *quasimetric* on X if

  (iv_m) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$;
- (X, d) is called a *metric space* and d is called a *metric* if d is a symmetric and a quasimetric, simultaneously.

Let X be a non-empty set and $d(x, y)$ be a distance on X with the following property:

(N) for each point $x \in X$ and any $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ such that from $d(x, y) \leq \delta$ and $d(y, z) \leq \delta$ it follows $d(x, z) \leq \varepsilon$.

Then (X, d) is called an *N-distance space* and d is called an *N-distance* on X. If d is a symmetric, then we say that d is an N-symmetric.

Spaces with N-distances were studied by Niemyzki [56] and by Nedev [55]. If d satisfies the condition

(F) for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that from $d(x, y) \leq \delta$ and $d(y, z) \leq \delta$ it follows $d(x, z) \leq \varepsilon$,

then d is called an *F-distance* or a *Fréchet distance* and (X, d) is called an *F-distance space*. Any F-distance d is an N-distance, too. If d is a symmetric and an F-distance on a space X, then we say that d is an F-symmetric.

Remark 2.1. If (X, d) is an F-symmetric space, then any convergent sequence is a Cauchy sequence. For N-symmetric spaces and for quasimetric spaces this assertion is not more true.

If $s > 0$ and $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all points $x, y, z \in X$, then we say that d is an *s-distance*. Any s-distance is an F-distance.

A distance space (X, d) is called an *H-distance space* if, for any two distinct points $x, y \in X$, there exists $\delta = \delta(x, y) > 0$ such that $B(x, d, \delta) \cap B(y, d, \delta) = \varnothing$.

We say that (X, d) is a C-distance space or a *Cauchy distance space* if any convergent Cauchy sequence has a unique limit point.

Remark 2.2. A distance space (X, d) is an H-distance space if and only if any convergent sequence in X has a unique limit point. Hence, any H-distance space is a C-distance space.

3. Ordering on Cartesian product of distance spaces

Let (X, d) be a distance space, $m \in \mathbb{N} = \{1, 2, ...\}$. On X^m consider the distances

$$d^m((x_1, ..., x_m), (y_1, ..., y_m)) = sup\{d(x_i, y_i) : i \leq m\}$$

and

$$\bar{d}^m((x_1, ..., x_m), (y_1, ..., y_m)) = \sum_{i=1}^{m} d(x_i, y_i).$$

Obviously, (X^m, d^m) and $(X^m, \bar{d}^m)$ are distance spaces, too.

Proposition 3.1 ([32])**.** *Let (X, d) be a distance space. Then:*

1. *If d is a symmetric, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are symmetric spaces, too.*
2. *If d is a quasimetric, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are quasimetric spaces, too.*
3. *If d is a metric, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are metric spaces, too.*
4. *If d is an F-distance space, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are F-distance spaces, too.*
5. *If d is an N-distance space, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are N-distance spaces, too.*
6. *If d is an H-distance space, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are H-distance spaces, too.*
7. *If (X, d) is a C-distance space, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are C-distance spaces, too.*
8. *If (X, d) is a complete distance space, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are complete distance spaces, too.*
9. *If d is an s-distance space, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are s-distance spaces, too.*
10. *The spaces (X^m, d^m) and $(X^m, \bar{d}^m)$ share the same convergent sequences and the same Cauchy sequences. Moreover, the distances d^m and $\bar{d}^m$ are uniformly equivalent, i.e., for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that:*
    - *from $d^m(x, y) \leq \delta$ it follows $\bar{d}^m(x, y) \leq \varepsilon$;*
    - *from $\bar{d}^m(x, y) \leq \delta$ it follows $d^m(x, y) \leq \varepsilon$.*

Let $\preceq$ be a (partial) order on a distance space (X, d). A sequence $\{x_n\}_{n \in \mathbb{N}}$ is called:

- *non-decreasing* if $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$;
- *non-increasing* if $x_n \succeq x_{n+1}$ for each $n \in \mathbb{N}$;
- *monotone* if it is either non-decreasing or non-increasing.

If $(X, d, \preceq)$ is an ordered distance space and $g : X \to X$ is a mapping, then $(X, d, \preceq)$ is said to have the sequential g-monotone property [37, 69] if it verifies:

i) if $\{x_n\}_{n\in\mathbb{N}}$ is a non-decreasing sequence and $\lim_{n\to\infty} d(x, x_n) = 0$, then $g(x_n) \preceq g(x)$ for all $n \in \mathbb{N}$;

ii) if $\{x_n\}_{n\in\mathbb{N}}$ is a non-increasing sequence and $\lim_{n\to\infty} d(x, x_n) = 0$, then $g(x_n) \succeq g(x)$ for all $n \in \mathbb{N}$.

An ordered distance space $(X, d, \preceq)$ is called *monotonically complete* if any monotone Cauchy sequence in X converges to some point in X.

Fix $m \in \mathbb{N}$ and a subset $L \subseteq \{1, 2, ..., m\}$. Like in [67, 69, 70], we introduce on X the ordering $\preceq_L$: $(x_1, ..., x_m) \preceq_L (y_1, ..., y_m)$ iff $x_i \preceq y_i$ for $i \in L$ and $y_j \preceq x_j$ for $j \notin L$.

By construction, $(X^m, d^m, \preceq_L)$ and $(X^m, \bar{d^m}, \preceq_L)$ are ordered distance spaces.

If $L \subseteq \{1, 2, ..., m\}$ and $M = \{1, 2, ..., m\} \setminus L$, then $x \preceq_L y$ if and only if $y \preceq_M x$ for $x, y \in X^m$. Hence $\preceq_M$ is the dual (inverse) order of the order $\preceq_L$.

Proposition 3.2. *Let $(X, d, \preceq)$ be a monotonically complete distance space. Then $(X^m, d^m, \preceq_L)$ and $(X^m, \bar{d^m}, \preceq_L)$ are ordered monotonically complete distance spaces, too.*

Proof. It is obvious. □

4. Multiple fixed point principles for monotone type operators

Fix $m \in \mathbb{N}$. Denote by $\lambda = (\lambda_1, ..., \lambda_m)$ a collection of mappings $\{\lambda_i : \{1, 2, ..., m\} \longrightarrow \{1, 2, ..., m\} : 1 \leq i \leq m\}$.

Let (X, d) be a distance space and $F : X^m \longrightarrow X$ be an operator. The operator F and the mappings λ generate the operator $\lambda F : X^m \longrightarrow X^m$, where

$$\lambda F(x_1, ..., x_m) = (y_1, ..., y_m) \text{ and } y_i = F(x_{\lambda_i(1)}, ..., x_{\lambda_i(m)}),$$

for each point $(x_1, ..., x_m) \in X^m$ and any index $i \in \{1, 2, ..., m\}$.

A point $a = (a_1, ..., a_m) \in X^m$ is called a *λ-multiple fixed point* of the operator F if $a = \lambda F(a)$, i.e., $a_i = F(a_{\lambda_i(1)}, ..., a_{\lambda_i(m)})$ for any $i \in \{1, 2, ..., m\}$.

Let $(X, d, \preceq)$ be a (partially) ordered distance space, $m \in \mathbb{N}$, $L \subseteq \{1, 2, ..., m\}$ and $F : X^m \longrightarrow X$ be an operator. In this context, we consider the following sets of assumptions that include a symmetric type contractive condition, similar to the symmetric contraction introduced and used by the second author in [14].

Conditions Ω_1:

1. $(X, \preceq)$ is a lattice;
2. If $x, y, z \in X$ and $x \preceq y \preceq z$, then $d(x, y) + d(y, x) \leq d(x, z) + d(z, x)$;
3. If $x, y \in X^m$, $x \neq y$ and $x \preceq_L y$, then $\lambda F(x) \preceq_L \lambda F(y)$ and $d^m(\lambda F(x), \lambda F(y)) + d^m(\lambda F(y), \lambda F(x)) < d^m(x, y) + d^m(y, x)$.

Conditions Ω_2:

1. $(X, \preceq)$ is a lattice;

2. If $x, y, z \in X$ and $x \preceq y \preceq z$, then $d(x, y) + d(y, x) \leq d(x, z) + d(z, x)$;
3. If $x, y \in X^m$, $x \neq y$ and $x \preceq_L y$, then $\lambda F(y) \preceq_L \lambda F(x)$ and $d^m(\lambda F(x), \lambda F(y)) + d^m(\lambda F(y), \lambda F(x)) < d^m(x, y) + d^m(y, x)$.

Conditions Ω_3:

1. $(X, \preceq)$ is a lattice;
2. If $x, y, z \in X$ and $x \preceq y \preceq z$, then $d(x, y) + d(y, x) \leq d(x, z) + d(z, x)$;
3. For any $i \in \{1, 2, \ldots, m\}$ the mapping λ_i is a surjection or, more generally, $| \cup \{\lambda_i^{-1}(j) : 1 \leq j \leq m\}| = m$, for each $i \in \{1, 2, \ldots, m\}$;
4. If $x, y \in X^m$, $x \neq y$ and $x \preceq_L y$, then $\lambda F(x) \preceq_L \lambda F(y)$ and $\bar{d}^m(\lambda F(x), \lambda F(y)) + \bar{d}^m(\lambda F(y), \lambda F(x)) < \bar{d}^m(x, y) + \bar{d}^m(y, x)$.

Conditions Ω_4:

1. $(X, \preceq)$ is a lattice;
2. If $x, y, z \in X$ and $x \preceq y \preceq z$, then $d(x, y) + d(y, x) \leq d(x, z) + d(z, x)$;
3. For any $i \in \{1, 2, \ldots, m\}$ the mapping λ_i is a surjection or, more generally, $| \cup \{\lambda_i^{-1}(j) : 1 \leq j \leq m\}| = m$, for each $i \in \{1, 2, \ldots, m\}$;
4. If $x, y \in X^m$, $x \neq y$ and $x \preceq_L y$, then $\lambda F(y) \preceq_L \lambda F(x)$ and $\bar{d}^m(\lambda F(x), \lambda F(y)) + \bar{d}^m(\lambda F(y), \lambda F(x)) < \bar{d}^m(x, y) + \bar{d}^m(y, x)$.

Now we can state concisely the following general and comprehensive multidimensional fixed point result.

Theorem 4.1. *Let $a \in X^m$ be a multidimensional fixed point of the operator of $F : X^m \longrightarrow X$. If any of the Conditions Ω_i, $i \in \{1, 2, 3, 4\}$, is satisfied, then the operator F has a unique multidimensional fixed point.*

Proof. Obviously, $(X^m, \preceq_L)$ is a lattice, too. Let $\rho = d^m$, for $i \in \{1, 2\}$, and $\rho = \bar{d}^m$, for $i \in \{3, 4\}$. Then for $x, y, z \in X^m$ and $x \preceq_L y \preceq_L z$ we have $\rho(x, y) + \rho(y, x) \leq \rho(x, z) + \rho(z, x)$. Assume that $b \in X^m$ is a multidimensional fixed point of the operator F with $b \neq a$.

Case 1. The points a and b are comparable.

Assume that $a \preceq_L b$. Then $\rho(a, b) + \rho(b, a) = \rho(\lambda F(a), \lambda F(b)) + \rho(\lambda F(b), \lambda F(a)) < \rho(a, b) + \rho(b, a)$, a contradiction.

Case 2. The points a and b are not comparable.

From any of the conditions Ω_i it follows that $(X, \preceq)$ is a lattice. Hence $(X^m, \preceq_L)$ is a lattice too, as the Cartesian product of lattices.

Fix $c = \max\{a, b\} \in X^m$. We put $d = \lambda F(\lambda F(c))$. By construction, $a \preceq_L d$ and $b \preceq_L d$. Hence, $c \preceq_L d$ and $\rho(a, c) \leq \rho(a, d)$, $\rho(b, c) \leq \rho(b, d)$. Therefore $\rho(a, c) + \rho(c, a) \leq \rho(a, d) + \rho(d, a)$. By virtue of the conditions Ω_i, we then have $\rho(a, d) + \rho(d, a) < \rho(a, c) + \rho(c, a)$, a contradiction. □

Now, according to [52], consider the following two classes of Meir-Keeler type assumptions.

Conditions MK_1:

1. For any two points $x, y \in X$ there exist an upper bound and a lower bound;

2 (Meir-Keeler monotone contraction condition). There exists a function $\delta : (0, +\infty) \longrightarrow (0, +\infty)$ such that from $r > 0$, $x, y \in X$, $d(x, y) < r + \delta(r)$ and $x \preceq y$ it follows that $d(x, y) < r$;
3. If $x, y \in X^m$, $x \preceq_L y$, then $\lambda F(x) \preceq_L \lambda F(y)$.

Conditions MK_2:

1. For any two points $x, y \in X$ there exist an upper bound and a lower bound;
2 (Meir-Keeler monotone contraction condition). There exists a function $\delta : (0, +\infty) \longrightarrow (0, +\infty)$ such that from $r > 0$, $x, y \in X$, $d(x, y) < r + \delta(r)$ and $x \preceq y$ it follows that $d(x, y) < r$;
3. If $x, y \in X^m$, $x \preceq_L y$, then $\lambda F(y) \preceq_L \lambda F(x)$.

Theorem 4.2. *Let (X, d) be an H-distance space and let $a \in X^m$ be a multidimensional fixed point of the operator of $F : X^m \to X$. Then in any of the Conditions MK_i, $i \in \{1, 2\}$, the operator F has a unique multidimensional fixed point.*

Proof. Obviously, in $(X^m, \preceq_L)$, for any two points $x, y \in X^m$, there exist an upper bound and a lower bound. Let $\rho = d^m$. In this case for any two points $x, y \in X^m$, from the condition $\rho(x, y) < r + \delta(r)$ and $x \preceq y$, it follows that $\rho(x, y) < r$.

Assume that $b \in X^m$ is a multidimensional fixed point of the operator of F and that $b \neq a$.

Case 1. The points a and b are comparable.

Assume that $a \preceq_L b$ and $\rho(a, b) = r > 0$. Since $\rho(a, b) < r + \delta(r)$, we have $r = \rho(a, b) = \rho(\lambda F(a), \lambda F(b)) < r$, a contradiction.

Case 2. The points a and b are not comparable. We put $r = \inf\{max\{\rho(a, c), \rho(b, c)\} : c \in X^m, a \preceq_L c, b \preceq_L c\}$. We claim that $r = 0$. Assume that $r > 0$. Then $\delta(r) > 0$ and there exists c such that $\max\{\rho(a, c), \rho(b, c)\} < r + \delta(r)$, $a \preceq_L c$, $b \preceq_L c$. We put $e = \lambda F(\lambda F(c))$. Then $a \preceq_L e$ and $b \preceq_L e$. Since $\lambda F(\lambda F(a)) = a$ and $\lambda F(\lambda F(b)) = b$, we have $\max\{\rho(a, e), \rho(b, e)\} < r$, a contradiction. Thus $r = 0$. For each $n \in \mathbb{N}$ there exists a point $c_n \in X^m$ such that $a \preceq_L c_n$, $b \preceq_L c_n$ and $max\{\rho(a, c_n), \rho(b, c_n)\} < 2^{-n}$. We can construct a sequence $\{c_n : n \in \mathbb{N}\}$ for which $a = \lim_{n\to\infty} c_n$ and $b = \lim_{n\to\infty} c_n$, a contradiction. □

In the particular case $m = 2$, the following theorems were proved in [21]. Their proofs in the general case are similar and we omit them.

Theorem 4.3. *Let $(X, d, \preceq)$ be an ordered metric space, $m \in \mathbb{N}$, $F : X^m \to X$ be an operator. Suppose that:*

a) *there exists a function $\delta : (0, +\infty) \longrightarrow (0, +\infty)$ such that from $r > 0$, $x, y \in X^m$, $d^m(x, y) < r + \delta(r)$ and $x \preceq y$ it follows that $d^m(\lambda F(x), \lambda F(y)) < r$;*
b) *for any two points $x, y \in X$ there exists an upper bound and a lower bound.*

Suppose also that one of the following sets of conditions is satisfied:

1. $(X, d, \preceq)$ *is monotonically complete; from* $x, y \in X^m$ *and* $x \preceq_L y$ *it follows that* $\lambda F(x) \preceq_L \lambda F(y)$; *there exists* $a \in X^m$ *such that* $a \preceq_L \lambda F(a)$.
2. $(X, d, \preceq)$ *is complete; from* $x, y \in X^m$ *and* $x \preceq_L y$ *it follows that* $\lambda F(y) \preceq_L \lambda F(x)$; *there exists* $a \in X^m$ *such that* $a \preceq_L \lambda F(a)$ *or* $F(a) \preceq_L a$.

Then there exists a unique multidimensional fixed point of the operator of F.

Theorem 4.4. *Let* $(X, d, \preceq)$ *be an ordered metric space,* $m \in \mathbb{N}$, $F : X^m \to X$ *be an operator. Suppose that:*

a) *there exists a function* $\delta : (0, +\infty) \longrightarrow (0, +\infty)$ *such that from* $r > 0$, $x, y \in X^m$, $\bar{d}^m(x, y) < r+\delta(r)$ *and* $x \preceq y$ *it follows that* $\bar{d}^m(\lambda F(x), \lambda F(y)) < r$;
b) *for any two points* $x, y \in X$, *there exist an upper bound and a lower bound; for any* $i \in \{1, 2, \ldots, m\}$, *the mapping* λ_i *is a surjection or, more generally,* $|\cup\{\lambda_i^{-1}(j) : 1 \leq j \leq m\}| = m$, *for each* $i \in \{1, 2, \ldots, m\}$.

Suppose also that one of the following sets of conditions is satisfied:

1. $(X, d, \preceq)$ *is monotonically complete; from* $x, y \in X^m$ *and* $x \preceq_L y$ *it follows that* $\lambda F(x) \preceq_L \lambda F(y)$; *there exists* $a \in X^m$ *such that* $a \preceq_L \lambda F(a)$.
2. $(X, d, \preceq)$ *is complete; from* $x, y \in X^m$ *and* $x \preceq_L y$ *it follows that* $\lambda F(y) \preceq_L \lambda F(x)$; *there exists* $a \in X^m$ *such that* $a \preceq_L \lambda F(a)$ *or* $F(a) \preceq_L a$.

Then there exists a unique multidimensional fixed point of the operator of F.

5. Some particular cases and a generic application of multiple fixed points

If we take concrete values of $m \in \mathbb{N}$ and consider various particular functions $\lambda = \{\lambda_i : \{1, ..., m\} \to \{1, ..., m\} : 1 \leq i \leq m\}$ then, most of the concepts of coupled, triple, quadruple,..., multiple fixed points existing in literature are obtained as particular cases of the concept of multiple fixed point considered in [32] and the present paper.

For example, if $m = 2$, $\lambda_1(1) = 1$, $\lambda_1(2) = 2$; $\lambda_2(1) = 2$, $\lambda_2(2) = 1$, we obtain the concept of coupled fixed point studied in [37] and in various subsequent papers. If $m = 3$, $\lambda_1(1) = 1$, $\lambda_1(2) = 2$, $\lambda_1(3) = 3$; $\lambda_2(1) = 2$, $\lambda_2(2) = 1$, $\lambda_2(3) = 2$; $\lambda_3(1) = 3$, $\lambda_3(2) = 2$, $\lambda_3(3) = 1$, then the concept of multiple fixed point studied in the present paper reduces to that of triple fixed point, first introduced in [18] and intensively studied in many other research works emerging from it.

We note that, as pointed out in [77], the notion of tripled fixed point due to Berinde and Borcut [18] is different from the one defined by Samet and Vetro [74] for $N = 3$, since in the case of ordered metric spaces, in order to keep the mixed monotone property working, it is necessary to take $\lambda_2(3) = 2$ and not $\lambda_2(3) = 3$.

It is also important to mention here that some cases of multidimensional coincidence point results (that extend multiple fixed point theorems) are not compatible with the mixed monotone property (see [7]).

For other concepts of multiple fixed points considered in literature the condition "λ_i is a surjection, for each $i \leq m$" is no more valid, see for example [18] and the research papers emerging from it, while the second condition, $|\cup \{\lambda_i^{-1}(j) : 1 \leq j \leq m\}| = m$, for each $i \leq m$, is satisfied.

Finally, we point out the fact that our approach in [32] and in this paper is based on the idea to obtain general multiple fixed point theorems by reducing this problem to a unidimensional fixed point problem and by simultaneously working in a more general and very reliable setting, i.e., that of distance space.

Many other related and relevant results could be obtained in the same way, by reducing the multidimensional fixed point problem to many other independent unidimensional fixed point principles, like the ones established in [5], [11], [12], [13], [15], [16], [19], [21], [23] etc.

We end the paper by indicating an interesting generic application of multiple fixed points in game theory.

Fix an orderable distance space $(X, d, \preceq)$ and a positive integer number $m \geq 2$. We put $\mathbb{N}_m = \{1, 2, ..., m\}$. For $L \subseteq \mathbb{N}_m$, we introduce on X^m the ordering $\preceq_L$: $(x_1, ..., x_m) \preceq_L (y_1, ..., y_m)$ iff $x_i \preceq y_i$ for $i \in L$ and $y_j \preceq x_j$ for $j \notin L$.

Denote by $\Lambda = (\lambda_1, ..., \lambda_m)$ a collection of mappings $\{\lambda_i : \mathbb{N}_m \longrightarrow \mathbb{N}_m : i \leq m\}$.

Let $F : X^m \longrightarrow X$ be an operator. The operator F and the mappings Λ generate the operator $\Lambda F : X^m \longrightarrow X^m$, where $\lambda F(x_1, ..., x_m) = (y_1, ..., y_m)$ and $y_i = F(x_{\lambda_i(1)}, ..., x_{\lambda_i(m)})$ for each point $(x_1, ..., x_m) \in X^m$ and any index $i \leq m$.

Assume now that $\mathbb{N}_m$ is the set of players and $i \in \mathbb{N}_m$ is the symbol of the ith player. In this case we say that:

- X is the space of the positions (decisions) of the players;
- $d(x, y)$ is the measure of the non-convenience of the position x relatively to the position y;
- ordering $\leq$ is the relation of domination of positions;
- a point $x = (x_1, x_2, ..., x_m) \in X^m$ is a selection of positions, where x_i is the position of the player i;
- the operator F is the operator of correction of the positions.

Every selection of positions $x = (x_1, x_2, ..., x_m) \in X^m$ determines the selection of positions $y = (y_1, y_2, ..., y_m) = \Lambda F(x) \in X^m$.

For any player i the number $d(x_i, y_i)$ is the measure of the non-convenience of the position x_i relatively to the position y_i for the i^{th} player.

One considers that the selection of the positions $x = (x_1, x_2, ..., x_m) \in X^m$ is optimal if $d(x_i, y_i)$ is minimal for each i.

In particular, if $\Lambda F(x) = x$, then the selection of positions x is optimal.

One can find distinct concrete examples of the above general model in [54], [51], [61], [62].

Acknowledgements. *This second author acknowledges the support provided by the Deanship of Scientific Research at King Fahd University of Petroleum and Minerals for funding this work through the projects IN151014 and IN141047.*

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