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Convergence theorems for finding the split common null point in Banach spaces

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Abstract

In this paper, we introduce a new iterative scheme for solving the split common null point problem. We then prove the strong convergence theorem under suitable conditions. Finally, we give some numerical examples for supporting our main results.

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1. Introduction

Let H_1 and H_2 be real Hilbert spaces and $T : H_1 \to H_2$ a bounded linear operator (we denote A^* by its adjoint) . Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. The split feasibility problem is to find $x \in C$ such that $Tx \in Q$. In order to solve the split feasibility problem (SFP), Byrne [5] proposed the following iterative algorithm in the framework of Hilbert spaces: $x_1 \in C$ and

$$x_{n+1} = P_C(x_n - \lambda T^*(I - P_Q)Tx_n),\ n \geq 1, \tag{1.1}$$

which is often called the CQ algorithm, where $\lambda > 0$, P_C and P_Q are the metric projections on C and Q, respectively. It was shown that the sequence

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$\{x_n\}$ converges weakly to a solution of SFP. Since then several iterations have been invented for solving the SFP (see, for example, [2, 11, 13, 17]).

Let $A : H_1 \to 2^{H_1}$ and $B : H_2 \to 2^{H_2}$ be set-valued mappings. Byrne et al. [6] considered the problem of finding a point z in H_1 such that

$$z \in A^{-1}0 \cap T^{-1}(B^{-1}0), \tag{1.2}$$

where the set of null points of A is defined by $A^{-1}0 = \{z \in H_1 : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex. This problem is called the split common null point problem and includes the spit feasibility problem as special cases; see also [8].

In 1953, Mann [10] introduced the following iteration process. Let C be a nonempty , closed and convex subset of a Banach space E. A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.3}$$

for all $x, y \in C$. We denote by $F(T)$ the fixed point set of T. For an initial point $x_1 \in C$, an iteration process $\{x_n\}$ is defined recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \; n \in \mathbb{N}, \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in [0,1] and T is a nonexpansive mapping on C.

In 1967, Halpern [7] defined an iteration process as follows: Take $x_0, x_1 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \; n \in \mathbb{N}, \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in [0, 1] and T is a nonexpansive mapping on C.

A mapping $f : C \to C$ is said to be a contraction if there exists $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \forall x, y \in C. \tag{1.6}$$

In 2000, Moudafi [12] introduced the following algorithm: For $x_1 \in C$, define the sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \; n \in \mathbb{N}, \tag{1.7}$$

where $\{\alpha_n\} \subset (0, 1)$ and T is a nonexpansive mapping. This method is called the viscosity approximation method.

Let H be a Hilbert space and let F be a strictly convex, reflexive and smooth Banach space. Let J_F be the duality mapping on F. Let C and D be nonempty, closed and convex subsets of H and F, respectively. Let P_C and P_D be the metric projections of H onto C and F onto D, respectively. Let $T : H \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $C \cap A^{-1}D \neq \varnothing$. In 2015, Alsulami and Takahashi [2] defined the following algorithm: For any $x_1 \in H$,

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)P_C(I - rT^* J_F(T - P_D T))x_n, \; n \in \mathbb{N}, \tag{1.8}$$

where $\{\beta_n\} \subset [0, 1]$ and $r \in (0, \infty)$. It was proved that if

$$0 < a \leq \beta_n \leq b < 1 \quad \textit{and} \quad 0 < r\|T\|^2 < 2 \tag{1.9}$$

for some $a, b \in \mathbb{R}$, then $\{x_n\}$ converges weakly to $z_0 \in C \cap T^{-1}D$, where $z_0 = \lim_{n\to\infty} P_{C\cap T^{-1}D}x_n$.

They introduced the following Halpern's type iteration: For any $x_1 \in H$,

(1.10)
$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u_n + (1-\alpha_n)P_C(I - rT^*J_F(I-P_D)T)x_n), n \in \mathbb{N},$$

where $r \in (0,\infty)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$. It was proved that if

$$0 < r\|T\|^2 < 2, \quad \lim_{n\to\infty} \alpha_n = 0, \tag{1.11}$$

$$\sum_{n=1}^{\infty} \alpha_n = \infty \quad \textit{and} \quad 0 < a \leq \beta_n \leq b < 1 \tag{1.12}$$

where $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 = C \cap A^{-1}D$, for some $z_0 = P_{C\cap A^{-1}D}u$.

Recently, using the idea of Halpern's iteration, Alofi et al. [1] proved the following strong convergence theorem for finding a solution of the split common null point problem in Banach spaces.

Theorem 1.1. *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let A and B be maximal monotone operators of H into 2^H and F into 2^{F^*} such that $A^{-1}0 \neq \varnothing$ and $B^{-1}0 \neq \varnothing$, respectively. Let J_λ be the resolvent of A for $\lambda > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$. Let $T : H \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \varnothing$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. Let $x_1 = x \in H$ and let $\{x_n\} \subset H$ be a sequence generated by*

(1.13)
$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u_n + (1-\alpha_n)J_{\lambda_n}(I - \lambda_n T^*J_F(I - Q_{\mu_n})T)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\}$, $\{\mu_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy the following conditions

$$0 < a \leq \lambda_n\|T\|^2 \leq b < 2, \quad 0 < k \leq \mu_n, \quad 0 < c \leq \beta_n \leq d < 1, \tag{1.14}$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty \tag{1.15}$$

for some $a, b, c, d, k \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $z_0 = P_{A^{-1}0\cap T^{-1}(B^{-1}0)}u$.

Motivated by the previous works, we introduce a new iterative scheme for solving the split common null point problem. We then prove the strong convergence theorem under suitable conditions. Finally, we give some numerical examples for supporting our main results.

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2. Preliminaries and lemmas

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we know from [15] that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle; \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.2}$$

Furthermore, for $x, y, u, v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.3}$$

The nearest point projection of a nonempty, closed and convex set C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know the metric projection P_C is firmly nonexpansive, *i.e.*,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \tag{2.4}$$

for all $x, y \in H$. Moreover $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [15].

Let E be a real Banach space with norm $\| \cdot \|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\} \tag{2.5}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \quad \textit{and} \quad \lim_{n\to\infty} \|x_n - y_n\| = 2, \tag{2.6}$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive.

The duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\} \tag{2.7}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.8}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J

is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J^* on E^*. For more details, see [14, 16].

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Lemma 2.1 ([16])**.** *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E, and let $x_1 \in E$ and $z \in C$ Then, the following conditions are equivalent:*
(1) $z = P_C x_1$;
(2) $\langle z - y, J(x_1 - z)\rangle \geq 0, \ \forall\, y \in C$.

Let E be a Banach space and let A be a mapping of E into 2^{E^*}. The effective domain of A is denoted by $\mathrm{dom}(A)$, that is, $\mathrm{dom}(A) = \{x \in E : Ax \neq \varnothing\}$. A multi-valued mapping A on E is said to be monotone if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $x, y \in \mathrm{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [4]; see also [14].

Lemma 2.2 ([4])**.** *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping on E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only for any $r > 0$,*

$$R(J + rA) = E^*, \tag{2.9}$$

where $R(J + rA)$ is the range of $J + rA$.

Let E be a uniformly convex and smooth Banach space with a Gâteaux differentiable norm and let A be a monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rAx_r. \tag{2.10}$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such J_r where $r > 0$ are called the metric resolvent of A. In a Hilbert space H, the metric resolvent J_r of A is simply called the resolvent of A. We also know the following lemmas:

Lemma 2.3 ([3, 18])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n \gamma_n + \beta_n \tag{2.11}$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 ([9])**.** *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense the there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which* *satisfies* $\Gamma_{n_i} < \Gamma_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Define the sequence* $\{\tau(n)\}_{n>n_0}$ *of integers as follows:*

$$\tau(n) = max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}, \tag{2.12}$$

where $n_0 \in \mathbb{N}$ *such that* $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \varnothing$. *Then, the following hold:*

(i) $\tau(n_0) \leq \tau(n_0+1) \leq \ldots$ *and* $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

3. Main results

In this section, we prove strong convergence theorems for finding a solution of the split common null point problem in Banach spaces.

Theorem 3.1. *Let* H *be a Hilbert space and let* F *be a uniformly convex and smooth Banach space. Let* J_F *be the duality mapping on* F. *Let* $f : H \to H$ *be a contraction. Let* A *and* B *be maximal monotone operators of* H *into* 2^H *and* F *into* 2^{F^*}, *respectively. Let* J_λ *be the resolvent of* A *for* $\lambda > 0$ *and let* Q_μ *be the metric resolvent of* B *for* $\mu > 0$. *Let* $T : H \to F$ *be a bounded linear operator such that* $T \neq 0$ *and let* T^* *be the adjoint operator of* T. *Suppose that* $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \varnothing$. *Let* $x_1 \in H$ *and let* $\{x_n\} \subset H$ *be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{\lambda_n}(I - \lambda_n T^* J_F(I - Q_{\mu_n})T)x_n \tag{3.1}$$

for all $n \in \mathbb{N}$, *where* $\{\mu_n\}, \{\lambda_n\} \subset (0,\infty), \{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$ *and* $\{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

$$0 < a \leq \lambda_n \|T^2\| \leq b < 2, \quad 0 < k \leq \mu_n, \quad 0 < c \leq \gamma_n \leq d < 1, \tag{3.2}$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad and \quad \sum_{n=1}^{\infty} \alpha_n = \infty \tag{3.3}$$

for some $a, b, c, d, k \in \mathbb{R}$. *Then* $\{x_n\}$ *converges strongly to* $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, *where* $z_0 = P_{A^{-1}0\cap T^{-1}(B^{-1}0)} f(z_0)$.

Proof. Put $z_n = J_{\lambda_n}(I - \lambda_n T^* J_F(I - Q_{\mu_n})T)x_n$ for all $n \in \mathbb{N}$ and let $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$. We have that $z = J_{\lambda_n} z$ and $Tz = Q_{\mu_n} Tz$ for all $n \in \mathbb{N}$.

Since J_{λ_n} is nonexpansive, we have

$$\begin{aligned}
\|z_n - z\|^2 &= \|J_{\lambda_n}(I - \lambda_n T^* J_F(I - Q_{\mu_n})T)x_n - J_{\lambda_n} z\|^2 \\
&\leq \|x_n - \lambda_n T^* J_F(I - Q_{\mu_n})T)x_n - z\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n \langle x_n - z, T^* J_F(I - Q_{\mu_n})Tx_n\rangle \\
&\quad + \lambda_n^2 \|T^* J_F(I - Q_{\mu_n})Tx_n\|^2 \\
&\leq \|x_n - z\|^2 - 2\lambda_n \langle Tx_n - Tz, J_F(I - Q_{\mu_n})Tx_n\rangle \\
&\quad + \lambda_n^2 \|T\|^2 \|(I - Q_{\mu_n})Tx_n\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n \langle Tx_n - Q_{\mu_n}Tx_n, J_F(I - Q_{\mu_n})Tx_n\rangle \\
&\quad - 2\lambda_n \langle Q_{\mu_n}Tx_n - Tz, J_F(I - Q_{\mu_n})Tx_n\rangle \\
&\quad + \lambda_n^2 \|T\|^2 \|(I - Q_{\mu_n})Tx_n\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n \|Tx_n - Q_{\mu_n}Tx_n\|^2 \\
&\quad - 2\lambda_n \langle Q_{\mu_n}Tx_n - Tz, J_F(I - Q_{\mu_n})Tx_n\rangle \\
&\quad + \lambda_n^2 \|T\|^2 \|(I - Q_{\mu_n})Tx_n\|^2 \\
&\leq \|x_n - z\|^2 - 2\lambda_n \|Tx_n - Q_{\mu_n}Tx_n\|^2 + \lambda_n^2 \|T\|^2 \|(I - Q_{\mu_n})Tx_n\|^2 \\
&= \|x_n - z\|^2 + \lambda_n(\lambda_n \|T\|^2 - 2)\|(I - Q_{\mu_n})Tx_n\|^2. \qquad (3.4)
\end{aligned}$$

Since $0 < \lambda_n \|T\|^2 < 2$, it follows that $\|z_n - z\| \leq \|x_n - z\|$ for all $n \in \mathbb{N}$. So we obtain

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n z_n - z\| \\
&\leq \alpha_n \|f(x_n) - z\| + \beta_n \|x_n - z\| + \gamma_n \|x_n - z\| \\
&\leq \alpha_n \alpha \|x_n - z\| + \alpha_n \|f(z) - z\| + (1 - \alpha_n)\|x_n - z\| \\
&= (1 - \alpha_n(1 - \alpha))\|x_n - z\| + \alpha_n \|f(z) - z\|. \qquad (3.5)
\end{aligned}$$

By induction, we conclude that $\{x_n\}$ is bounded. So are $\{Tx_n\}$, $\{z_n\}$ and $\{y_n\}$. Put $z_0 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} f(z_0)$. We see that

$$x_{n+1} - x_n = \alpha_n(f(x_n) - x_n) + \gamma_n(z_n - x_n), \qquad (3.6)$$

which implies that

$$x_{n+1} - x_n - \alpha_n(f(x_n) - x_n) = \gamma_n(z_n - x_n). \qquad (3.7)$$

It follows that

$$\begin{aligned}
\langle x_{n+1} - x_n - \alpha_n(f(x_n) - x_n), x_n - z_0\rangle &= \gamma_n \langle z_n - x_n, x_n - z_0\rangle \\
&= -\gamma_n \langle x_n - z_n, x_n - z_0\rangle. \qquad (3.8)
\end{aligned}$$

From (2.3), we obtain

$$\begin{aligned}
2\langle x_n - z_n, x_n - z_0\rangle &= \|x_n - z_0\|^2 + \|z_n - x_n\|^2 - \|z_n - z_0\|^2 \\
&\geq \|x_n - z_0\|^2 + \|z_n - x_n\|^2 - \|x_n - z_0\|^2 \\
&= \|z_n - x_n\|^2. \qquad (3.9)
\end{aligned}$$

From (3.8) and (3.9), we obtain

$$\begin{aligned} 2\langle x_{n+1}-x_n, x_n-z_0\rangle &= 2\alpha_n\langle f(x_n)-x_n, x_n-z_0\rangle - 2\gamma_n\langle x_n-z_n, x_n-z_0\rangle \\ &\le 2\alpha_n\langle f(x_n)-x_n, x_n-z_0\rangle - \gamma_n\|z_n-x_n\|^2. \end{aligned} \tag{3.10}$$

Using (2.3) and (3.10), we have

$$\|x_{n+1}-z_0\|^2-\|x_n-x_{n+1}\|^2-\|x_n-z_0\|^2 \le 2\alpha_n\langle f(x_n)-x_n, x_n-z_0\rangle-\gamma_n\|z_n-x_n\|^2. \tag{3.11}$$

Putting $\Gamma_n = \|x_n - z_0\|^2$ for all $n \in \mathbb{N}$, we see that

$$\Gamma_{n+1}-\Gamma_n-\|x_n-x_{n+1}\|^2 \le 2\alpha_n\langle f(x_n)-x_n, x_n-z_0\rangle-\gamma_n\|z_n-x_n\|^2. \tag{3.12}$$

We note that

$$\begin{aligned} \|x_{n+1}-x_n\| &= \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n z_n-x_n\| \\ &\le \alpha_n\|f(x_n)-x_n\|+\gamma_n\|z_n-x_n\|. \end{aligned} \tag{3.13}$$

This shows that

$$\begin{aligned} \|x_{n+1}-x_n\|^2 &\le (\alpha_n\|f(x_n)-x_n\|+\gamma_n\|z_n-x_n\|)^2 \\ &= \alpha_n^2\|f(x_n)-x_n\|^2+2\alpha_n\gamma_n\|f(x_n)-x_n\|\|z_n-x_n\| \\ &\quad + \gamma_n^2\|z_n-x_n\|^2. \end{aligned} \tag{3.14}$$

Hence by (3.12) and (3.14), we have

$$\begin{aligned} \Gamma_{n+1}-\Gamma_n &\le \alpha_n(\alpha_n\|f(x_n)-x_n\|^2+2\gamma_n\|f(x_n)-x_n\|\|z_n-x_n\|) \\ &\quad + \gamma_n^2\|z_n-x_n\|^2 \\ &\quad + 2\alpha_n\langle f(x_n)-x_n, x_n-z_0\rangle-\gamma_n\|z_n-x_n\|^2 \\ &= \alpha_n(\alpha_n\|f(x_n)-x_n\|^2+2\gamma_n\|f(x_n)-x_n\|\|z_n-x_n\|) \\ &\quad + \gamma_n(\gamma_n-1)\|z_n-x_n\|^2 \\ &\quad + 2\alpha_n\langle f(x_n)-z_0, x_n-z_0\rangle-2\alpha_n\|x_n-z_0\|^2. \end{aligned} \tag{3.15}$$

So we obtain

$$\begin{aligned} \Gamma_{n+1}-\Gamma_n+\gamma_n(1-\gamma_n)\|z_n-x_n\|^2 &\le \alpha_n(\alpha_n\|f(x_n)-x_n\|^2 \\ &\quad + 2\gamma_n\|f(x_n)-x_n\|\|z_n-x_n\|) \\ &\quad + 2\alpha_n\langle f(x_n)-z_0, x_n-z_0\rangle \\ &\quad - 2\alpha_n\|x_n-z_0\|^2. \end{aligned} \tag{3.16}$$

We next split the proof into two cases.

Case 1: Suppose that there exists a natural number N such that $\Gamma_{n+1} \le \Gamma_n$ for all $n \ge N$. In this case, $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1}-\Gamma_n) = 0$. Since $\lim_{n\to\infty}\alpha_n = 0$ and $0 < c \le \gamma_n \le d < 1$, by (3.16), we have

$$\lim_{n\to\infty}\|z_n-x_n\| = 0. \tag{3.17}$$

From (3.13) we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.18}$$

We next show that $\limsup_{n\to\infty}\langle f(z_0)-z_0, z_n-z_0\rangle \le 0$. Put

$$l = \limsup_{n\to\infty}\langle f(z_0)-z_0, z_n-z_0\rangle. \tag{3.19}$$

Then without loss of generality, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $l=\lim_{i\to\infty}\langle f(z_0)-z_0, z_{n_i}-z_0\rangle$ and $\{z_{n_i}\}$ converges weakly to some point $w\in H$. Since $\|x_n-z_n\|\to 0$, we also have that $\{x_{n_i}\}$ converges weakly to $w\in H$. On the other hand, from (3.4) we have

$$\begin{aligned}\lambda_n(2-\lambda_n\|T\|^2)\|(I-Q_{\mu n})Tx_n\|^2 &\le \|x_n-z_n\|^2-\|z_n-z\|^2\\ &\le \|x_n-z_n\|(\|x_n-z\|+\|z_n-z\|).\end{aligned} \tag{3.20}$$

Then since $\|x_n-z_n\|\to 0$ and $0<a\le\lambda_n\|T\|^2\le b<2$,

$$\lim_{n\to\infty}\|Tx_n-Q_{\mu_n}Tx_n\|=0. \tag{3.21}$$

Since $\{x_{n_i}\}$ converges weakly to $w\in H$ and T is bounded and linear, we also have $\{Tx_{n_i}\}$ converges weakly to Tw. Using this and $\lim_{n\to\infty}\|Tx_n - Q_{\mu_n}Tx_n\|=0$, we have that $Q_{\mu_{n_i}}Tx_{n_i}\rightharpoonup Tw$. Since Q_{μ_n} is the metric resolvent of B for $\mu_n>0$, we have that $\frac{J_F(Tx_n-Q_{\mu_n}Tx_n)}{\mu_n}\in BQ_{\mu_n}Tx_n$ for all $n\in\mathbb{N}$. By the monotonicity of B we obtain

$$0\le\left\langle u-Q_{\mu_{n_i}}Tx_{n_i}, v^*-\frac{J_F(Tx_{n_i}-Q_{\mu_{n_i}}Tx_{n_i})}{\mu_{n_i}}\right\rangle \tag{3.22}$$

for all $(u,v^*)\in B$. We observe that $\|J_F(Tx_{n_i}-Q_{\mu_{n_i}}Tx_{n_i})\| = \|Tx_{n_i} - Q_{\mu_{n_i}}Tx_{n_i}\|\to 0$ as $i\to\infty$. Since $0<k\le\mu_{n_i}$, it follows that $0\le\langle u-Tw, v^*-0\rangle$ for all $(u,v^*)\in B$. Because B is maximal monotone, we have $Tw\in B^{-1}0$. This implies that $w\in T^{-1}(B^{-1}0)$. Using $z_n=J_{\lambda_n}(x_n-\lambda_nT^*J_F(Tx_n - Q_{\lambda_n}Tx_n))$, we obtain

$$\begin{aligned} z_n &= J_{\lambda_n}(x_n-\lambda_nT^*J_F(Tx_n-Q_{\mu_n}Tx_n))\\ &\Leftrightarrow x_n-\lambda_nT^*J_F(Tx_n-Q_{\mu_n}Tx_n)\in z_n+\lambda_nAz\\ &\Leftrightarrow x_n-z_n-\lambda_nT^*J_F(Tx_n-Q_{\mu_n}Tx_n)\in\lambda_nAz_n\\ &\Leftrightarrow \frac{1}{\lambda_n}(x_n-z_n-\lambda_nT^*J_F(Tx_n-Q_{\mu_n}Tx_n))\in Az_n.\end{aligned} \tag{3.23}$$

Since A is monotone, we have that for $(u,v)\in A$,

$$\left\langle z_n-u, \frac{1}{\lambda_n}(x_n-z_n-\lambda_nT^*J_F(Tx_n-Q_{\mu_n}Tx_n))-v\right\rangle\ge 0 \tag{3.24}$$

which implies that

$$\left\langle z_n-u, \frac{x_n-z_n}{\lambda_n}-T^*J_F(Tx_n-Q_{\mu_n}Tx_n))-v\right\rangle\ge 0. \tag{3.25}$$

Replacing n by n_i, we have

$$\left\langle z_{n_i}-u, \frac{x_{n_i}-z_{n_i}}{\lambda_{n_i}}-T^*J_F(Tx_{n_i}-Q_{\mu_{n_i}}Tx_{n_i})-v\right\rangle\ge 0. \tag{3.26}$$

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Since $x_{n_i} - z_{n_i} \to 0$, $0 < a \leq \lambda_{n_i}\|T\|^2$, $z_{n_i} \rightharpoonup w$ and $T^*J_F(Tx_n - Q_{\mu_{n_i}}Tx_{n_i}) \to 0$, we get that $\langle w - u, -v\rangle \geq 0$. Since A is maximal, we have $0 \in Aw$. Therefore, $w \in A^{-1}0 \cap T^{-1}(B^{-1}0)$. Since $\{z_{n_i}\}$ converges weakly to $w \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, it follows that

$$l = \lim_{i\to\infty}\langle f(z_0) - z_0, z_{n_i} - z_0\rangle = \langle f(z_0) - z_0, w - z_0\rangle \leq 0. \tag{3.27}$$

On the other hand, we see that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \langle x_{n+1} - z_0, x_{n+1} - z_0\rangle \\
&= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n z_n - z_0, x_{n+1} - z_0\rangle \\
&= \langle \alpha_n(f(x_n) - z_0) + \beta_n(x_n - z_0) + \gamma_n(z_n - z_0), x_{n+1} - z_0\rangle \\
&= \alpha_n\langle f(x_n) - f(z_0) + f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\quad + \beta_n\langle x_n - z_0, x_{n+1} - z_0\rangle + \gamma_n\langle z_n - z_0, x_{n+1} - z_0\rangle \\
&= \alpha_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle + \alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\quad + \beta_n\langle x_n - z_0, x_{n+1} - z_0\rangle + \gamma_n\langle z_n - z_0, x_{n+1} - z_0\rangle \\
&\leq \alpha_n\alpha\|x_n - z_0\|\|x_{n+1} - z_0\| + \beta_n\|x_n - z_0\|\|x_{n+1} - z_0\| \\
&\quad + \gamma_n\|x_n - z_0\|\|x_{n+1} - z_0\| + \alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&= (\alpha_n\alpha + \beta_n + \gamma_n)\|\|x_n - z_0\|\|x_{n+1} - z_0\| \\
&\quad + \alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (\alpha_n\alpha + 1 - \alpha_n)\frac{1}{2}(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
&\quad + \alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&= \left(\frac{\alpha\alpha_n + 1 - \alpha_n}{2}\right)\|x_n - z_0\|^2 \\
&\quad + \left(\frac{\alpha\alpha_n + 1 - \alpha_n}{2}\right)\|x_{n+1} - z_0\|^2 \\
&\quad + \alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&= \left(1 - \frac{2(1-\alpha)\alpha_n}{1 + (1-\alpha)\alpha_n}\right)\|x_n - z_0\|^2 \\
&\quad + \left(\frac{2(1-\alpha)\alpha_n}{1 + (1-\alpha)\alpha_n}\right)\left(\frac{1}{1-\alpha}\right)\langle f(z_0) - z_0, x_{n+1} - z_0\rangle. \qquad (3.28)
\end{aligned}$$

Also, we have

$$\lim_{n\to\infty}\|z_n - x_{n+1}\| \leq \lim_{n\to\infty}(\|z_n - x_n\| + \|x_{n+1} - x_n\|) = 0. \tag{3.29}$$

Then

$$\limsup_{n\to\infty}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \leq 0. \tag{3.30}$$

Since $\sum_{n=1}^{\infty}\alpha_n = \infty$, by Lemma 2.3 we conclude that $x_n \to z_0$ as $n \to \infty$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_{n_i}\}$ such that $\Gamma_{n_i} \leq \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}. \tag{3.31}$$

Then by Lemma 2.6 we have $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. Thus by (3.16) we have for all $n \in \mathbb{N}$,

$$\begin{aligned}\gamma_{\tau(n)}(1-\gamma_{\tau(n)})\|z_{\tau(n)} - x_{\tau(n)}\|^2 \leq\ & \alpha_{\tau(n)}^2\|f(x_{\tau(n)}) - x_{\tau(n)}\|^2 \\ & + 2\alpha_{\tau(n)}\gamma_{\tau(n)}(\|f(x_{\tau(n)}) - x_{\tau(n)}\| \\ & \times \|z_{\tau(n)} - x_{\tau(n)}\|) \\ & + 2\alpha_{\tau(n)}\langle f(x_{\tau(n)}) - z_0, x_{\tau(n)} - z_0\rangle \\ & - 2\alpha_{\tau(n)}\|x_{\tau(n)} - z_0\|^2. \end{aligned} \tag{3.32}$$

Using $\lim_{n\to\infty} \alpha_n = 0$ and $0 < c \leq \gamma_n \leq d < 1$, we have

$$\lim_{n\to\infty} \|z_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{3.33}$$

As in the proof of Case 1, we can show that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0. \tag{3.34}$$

This gives

$$\lim_{n\to\infty} \|z_{\tau(n)} - x_{\tau(n)+1}\| = 0. \tag{3.35}$$

We next show that $\limsup_{n\to\infty}\langle f(z_0) - z_0, x_{\tau(n)+1} - z_0\rangle \leq 0$. Put

$$l = \limsup_{n\to\infty}\langle f(z_0) - z_0, x_{\tau(n)+1} - z_0\rangle. \tag{3.36}$$

So we have

$$l = \limsup_{n\to\infty}\langle f(z_0) - z_0, z_{\tau(n)} - z_0\rangle. \tag{3.37}$$

Without loss of generality, there exists a subsequence $\{z_{\tau(n_i)}\}$ of $\{z_{\tau(n)}\}$ such that

$$l = \lim_{i\to\infty}\langle f(z_0) - z_0, z_{\tau(n_i)} - z_0\rangle \tag{3.38}$$

and $\{z_{\tau(n_i)}\}$ converges weakly to some point $w \in H$. As in the proof of Case 1, we can show that $w \in A^{-1}0 \cap T^{-1}(B^{-1}0)$. Then it follows that

$$l = \lim_{i\to\infty}\langle f(z_0) - z_0, z_{\tau(n_i)} - z_0\rangle = \langle f(z_0) - z_0, w - z_0\rangle \leq 0. \tag{3.39}$$

As in the proof of Case 1, we also obtain

$$\begin{aligned}\|x_{\tau(n)+1} - z_0\|^2 \leq\ & \left(1 - \frac{2(1-\alpha)\alpha_{\tau(n)}}{1+(1-\alpha)\alpha_{\tau(n)}}\right)\|x_{\tau(n)} - z_0\|^2 \\ & + \left(\frac{2(1-\alpha)\alpha_n}{1+(1-\alpha)\alpha_n}\right)\left(\frac{1}{1-\alpha}\right)\langle f(z_0) - z_0, x_{\tau(n)+1} - z_0\rangle. \end{aligned} \tag{3.40}$$

Since $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$,

$$\left(\frac{2(1-\alpha)\alpha_{\tau(n)}}{1+(1-\alpha)\alpha_{\tau(n)}}\right)\|x_{\tau(n)}-z_0\|^2 \leq \left(\frac{2(1-\alpha)\alpha_n}{1+(1-\alpha)\alpha_n}\right) \times \left(\frac{1}{1-\alpha}\right)\langle f(z_0)-z_0, x_{\tau(n)+1}-z_0\rangle. \tag{3.41}$$

It is easily seen that $\left(\frac{2(1-\alpha)\alpha_{\tau(n)}}{1+(1-\alpha)\alpha_{\tau(n)}}\right) > 0$. Then we have

$$\|x_{\tau(n)}-z_0\|^2 \leq \left(\frac{1}{1-\alpha}\right)\langle f(z_0)-z_0, x_{\tau(n)+1}-z_0\rangle. \tag{3.42}$$

This shows that

$$\limsup_{n\to\infty} \|x_{\tau(n)}-z_0\|^2 \leq 0 \tag{3.43}$$

and hence $\|x_{\tau(n)}-z_0\| \to 0$ as $n\to\infty$. Thus $\|x_{\tau(n)+1}-z_0\| \to 0$ as $n\to\infty$. By Lemma 2.6, we obtain

$$\|x_n-z_0\| \leq \|x_{\tau(n)+1}-z_0\| \to 0 \tag{3.44}$$

as $n\to\infty$. This completes the proof. □

4. Examples and numerical results

In this section, we give examples including its numerical results for supporting our main theorem.

Example 4.1. Let $H=\mathbb{R}$. For $x\in\mathbb{R}$, we define $G:\mathbb{R}\to\mathbb{R}$ by

$$G(x) = \begin{cases} \omega x & if \;\; x \geq 0, \\ +\infty & otherwise. \end{cases}$$

Let $F:\mathbb{R}\to\mathbb{R}$ be defined by $F(x) = \omega|x| - \ln(1+\omega|x|)$.

Choose $x_1 = 2$, $\omega = 1$, $\alpha_n = \frac{1}{2n+1}$, $\beta_n = \frac{n}{2n+1}$, $\gamma_n = \frac{n}{2n+1}$ for all $n\in\mathbb{N}$. Let $f(x) = \frac{x}{2}$ and $Tx = x$. We aim to find the minimizers of F and G. Using algorithm (3.1), we have the following numerical results:

n	x_n	$\mid x_{n+1} - x_n \mid$
1	1.471404251	5.2859547×10^{-1}
2	1.265727840	7.8045565×10^{-1}
3	0.527004926	7.8045565×10^{-1}
4	0.250611252	1.3092799×10^{-1}
5	0.160205803	7.9460312×10^{-2}
6	0.114183169	5.3450423×10^{-2}
7	0.083166153	3.7671482×10^{-2}
8	0.061057394	2.7173989×10^{-2}
9	0.045012810	1.9840027×10^{-2}
10	0.033265124	1.4579710×10^{-2}
11	0.024621382	1.0752746×10^{-2}
12	0.018243176	7.9469695×10^{-3}
$\vdots$	$\vdots$	$\vdots$
50	0.000000260	1.1017075×10^{-7}

Table 1 Numerical results of Example 4.1 for iteration process (3.1)

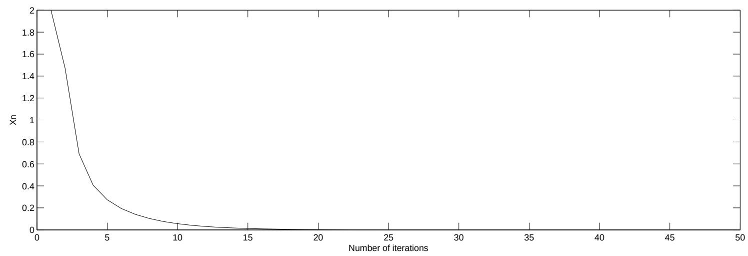

Figure 1: Convergence behavior of $\{x_n\}$ in Table 1.

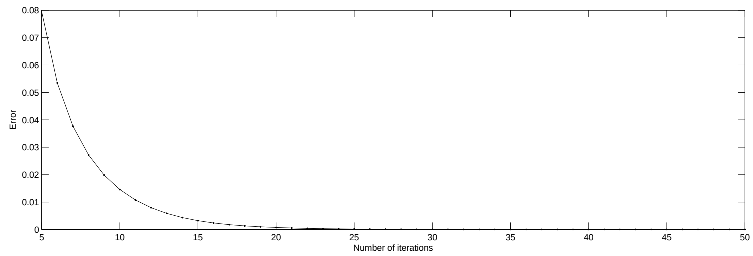

Figure 2: Error plots for all sequences $\{x_n\}$ in Table 1.

Example 4.2. Let $H = \mathbb{R}^3$. For $x \in \mathbb{R}^3$, define $G : \mathbb{R}^3 \to \mathbb{R}$ by $G(x) = \|Lx - y\|^2$, where
$L = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $y = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $F : \mathbb{R}^3 \to \mathbb{R}$ by $F(x) = 5\|x\|^2 + (15, 6, -7)x + 10$.

Let $T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. Find $x \in \mathbb{R}^3$ such that x is a minimizer of F and Tx also is a minimizer of G.

Choose $x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, $\alpha_n = \frac{1}{2n+1}$, $\beta_n = \frac{n}{2n+1}$, $\gamma_n = \frac{n}{2n+1}$ for all $n \in \mathbb{N}$ and let $f(x) = \frac{x}{2}$. Using algorithm (3.1), we have the following numerical results:

n	x_n	$\|x_{n+1} - x_n\|$
1	(0.1527,-0.6014,0.8512)	20.745517681470 $\times 10^{-1}$
2	(-0.5968,-0.6646,-0.0706)	11.896884171498 $\times 10^{-1}$
3	(-0.8523,-0.5312,0.7977)	9.149067074771 $\times 10^{-1}$
4	(-1.1354,-0.6011,0.3222)	5.577239966457 $\times 10^{-1}$
5	(-1.2027,-0.5415,0.7430)	4.302752764182 $\times 10^{-1}$
6	(-1.3156,-0.5838,0.4941)	2.766128927383 $\times 10^{-1}$
7	(-1.3323,-0.5554,0.7039)	2.124210254520 $\times 10^{-1}$
8	(-1.3822,-0.5792,0.5747)	1.406007674047 $\times 10^{-1}$
9	(-1.3869,-0.5656,0.6815)	1.078492388367 $\times 10^{-1}$
10	(-1.4115,-0.5788,0.6150)	7.214423578793 $\times 10^{-2}$
11	(-1.4139,-0.5724,0.6703)	5.574083184995 $\times 10^{-2}$
12	(-1.4273,-0.5798,0.6365)	3.720017649222 $\times 10^{-2}$
13	(-1.4295,-0.5769,0.6655)	2.926886211227 $\times 10^{-2}$
14	(-1.4375,-0.5811,0.6485)	1.930348053252 $\times 10^{-2}$
15	(-1.4397,-0.5800.0.6640)	1.568794063595 $\times 10^{-2}$
$\vdots$	$\vdots$	$\vdots$
100	(-1.4790,-0.5913,0.6743)	8.809288634531 $\times 10^{-5}$

Table 2 Numerical results of Example 4.2 for iteration process (3.1)

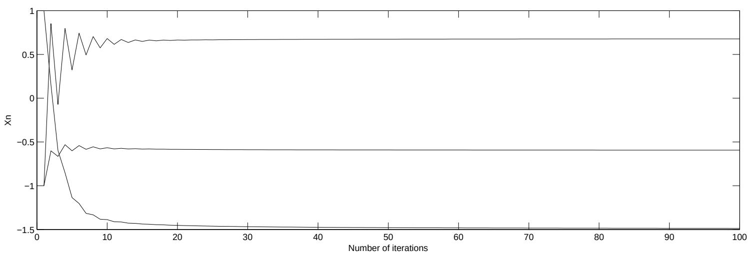

Figure 3: Convergence behavior of $\{x_n\}$ in Table 2 .

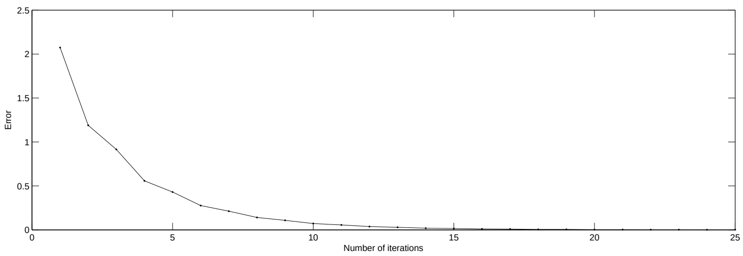

Figure 4: Error plots for all sequences $\{x_n\}$ in Table 2 .

From Table 2, we see that $\begin{bmatrix} -1.5 \\ -0.6 \\ 0.7 \end{bmatrix}$ is a minimizer of F such that $T \begin{bmatrix} -1.5 \\ -0.6 \\ 0.7 \end{bmatrix}$ $= \begin{bmatrix} -0.8 \\ 2 \\ -0.6 \end{bmatrix}$ is a minimizer of G.

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