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ORIGINAL ARTICLE

# Inverse problem of the Holling-Tanner model and its solution

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***Abstract*—In this paper we undertake to consider the inverse problem of parameter identification of nonlinear system of ordinary differential equations for a specific case of complete information about solution of the Holling-Tanner model for finite number of points for the finite time interval. In this model the equations are nonlinearly dependent on the unknown parameters. By means of the proposed transformation the obtained equations become linearly dependent on new parameters functionally dependent on the original ones. This simplification is achieved by the fact that the new set of parameters becomes dependent and the corresponding constraint between the parameters is nonlinear. If the conventional approach based on introduction of the Lagrange multiplier is used this circumstance will result in a nonlinear system of equations. A novel algorithm of the problem solution is proposed in which only one nonlinear equation instead of the system of six nonlinear equations has to be solved. Differentiation and integration methods of the problem solution are implemented and it is shown that the integration method produces more accurate results and uses less number of points on the given time interval.**

***Keywords*-Parameter estimation, Goal function, Absolute error curves, Inverse method, Holling-Tanner model, Least square method, Differentiation method, Integration method**

## I. Introduction

The numerical evaluation of known coefficient of a dynamical system i.e. the problem of dynamical system identification, is one of the most important problem of the mathematical biology [1], ecology [2], [3], [4], etc. Usually, to identify a dynamics of a system, it is necessary to have certain statistical information for time values about the unknown functions of this system. In the present paper we consider the inverse problem of parameter identification of the Holling-Tanner predator-prey model [5], [6]. This model is widely used in mathematical biology, for example, in the study of transmissible disease [7]. Several investigations have been done by various researchers on the mite-spider-mite, lynx-hare and sparrow-hawk-sparrow competition [8], [9], [10]. In [11], the authors proposed a method consisting in the direct integration of a given dynamical system with the subsequent application of quadrature rules and the least square method [12], [13] provided that there



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is complete statistical information about the unknown function. In this paper, we assume that the complete information about the competing species is available and the two methods of solution, differentiation and integration methods, are proposed. The problem of the Holling-Tanner model identification has its specifics, because it nonlinearly depends on the unknown parameters. It is possible to transform this model to a new form where the equations of the system linearly depends on the set of new parameters. These new parameters are not independent and we need to consider the constraint between the parameters, which are nonlinear. The Holling-Tanner model has only one constraint and hence, can be simply treated by a novel method developed by the authors. The theoretical considerations are accompanied by numerical examples where the developed algorithm is tested for both differentiation and integration methods of solution. It is shown that the integration methods is more accurate than the differentiation one and needs less amount of experimental information.

## II. Main results

In our paper we consider the Holling-Tanner model [9], [14] described by the following system of equations:

$$\begin{cases} \dot{x} = b_1 x - b_2 x^2 - b_3 \frac{x \cdot y}{b_4 + x}, \\ \dot{y} = b_5 y - b_6 \frac{y^2}{x}, \\ t = 0, \;\; x(0) = x_0, \;\; y(0) = y_0 \end{cases} \tag{1}$$

where $x = x(t)$, $y = y(t)$, $\dot{x} = \frac{dx(t)}{dt}$, $\dot{y} = \frac{dy(t)}{dt}$, $t$ is time and $b_1, \cdots b_6$ are positive constant parameters [15]. Initial conditions for this system are formulated so that at $t = 0 : x(t = 0) = x_0 > 0$ and $y(t = 0) = y_0 > 0$. The main results relating to solution of this initial value problem were obtained in [10], [16], [17], [18] as

- Solution of the initial value problem (1) $\{x(t), y(t)\}$ with positive initial conditions is positive, i.e. $x(t) > 0$ and $y(t) > 0$ for $t \geq 0$.
- Initial value problem (1) has the positive steady-state solution [15] $(\tilde{x}, \tilde{y})$ which corresponds to either stable focus or stable node critical point depending on $b_1, \cdots b_6$ so that:

$$\begin{aligned} \tilde{x} &= \frac{b_1 b_6 - b_3 b_5 - b_2 b_4 b_6 + \sqrt{\Delta}}{2 b_2 b_6} > 0, \\ \tilde{y} &= \frac{b_5 (b_1 b_6 - b_3 b_5 - b_2 b_4 b_6 + \sqrt{\Delta})}{2 b_2 b_6^2} > 0. \end{aligned} \tag{2}$$

where

$$\Delta = (b_1 b_6 - b_3 b_5 - b_2 b_4 b_6)^2 + 4 b_1 b_2 b_4 b_6^2.$$

- Initial value problem (1) has unstable steady-state solution

$$(\tilde{\tilde{x}}, \tilde{\tilde{y}}) = (\frac{b_1}{b_2}, 0),$$

which corresponds to the saddle critical point.

## III. On solvability of identification problem

Assume that solution of initial problem (1), $x(t)$ and $y(t)$ is given on the finite time interval $t \in [0, T]$ with initial $t = 0$ and terminal $t = T$ time instants in $N + 1$ equispaced time instants $t_i = \frac{T}{N} i \in [0, T]$:

$$x_i = x(t_i), \;\; y_i = y(t_i) \;\; (i = 0, \cdots, N) \tag{3}$$

Lets us formulate the identification problem for parameters $b_1, \cdots b_6$ from the known solution (3) This problem can be solved if the conditions of the following theorem are satisfied:

**Theorem 1.** *Parameters $b_1, b_2, b_3, b_4$ of model (1) can be identified by the least squares method if $(N + 1) \times 1$-vector columns $[x_i], [x_i^2], [x_i^3], [\dot{x}_i], [x_i, y_i]$ are linearly independent. Parameters $b_5$ and $b_6$ of the above mentioned model can be identified by the mentioned method if $(N + 1) \times 1$-vector columns $[y_i]$ and $[\frac{y_i^2}{x_1}]$ are linearly independent.*

*Proof:* By multiplying the first equation of system (1) by $(b_4 + x)$ and grouping the resulting terms we obtain

$$\begin{aligned} &C_1(-x^3(t)) + C_2(-x(t)y(t)) + C_3(-\dot{x}(t)) \\ &+ C_4(x(t)) + C_5(x^2(t)) + (-x(t)\dot{x}(t)) = 0, \end{aligned} \tag{4}$$

where $C_1 = b_2$, $C_2 = b_3$, $C_3 = b_4$, $C_4 = b_1 b_4$, $C_5 = b_1 - b_2 b_4$ are new unknown parameters. It is easy to check that the parameters $C_1, C_3, C_4, C_5$ satisfy the following constrains: $C_1 C_3^2 + C_3 C_5 - C_4$. Considering $x(t)$ and $y(t)$ in time instants $t = t_i$ we obtain the following overdetermined system of $N+1$ linear algebraic equations:

$$C_1\vec{f_1}+C_2\vec{f_2}+C_3\vec{f_3}+C_4\vec{f_4}+C_5\vec{f_5}-\vec{f_6}=0, \quad (5)$$

where

$$\begin{aligned}&\vec{f_1} = [f_{1i}] = \left[-x_i^3\right], \vec{f_2} = [f_{2i}] = [-x_i y_i],\\ &\vec{f_3} = [f_{3i}] = [-\dot{x}_i], \vec{f_4} = [f_{4i}] = [x_i],\\ &\vec{f_5} = [f_{5i}] = \left[x_i^2\right], \text{ and } \vec{f_6} = [f_{6i}] = [x_i \dot{x}_i]\end{aligned}$$

are $(N+1) \times 1$-vector columns. Hence, the unknown parameters $C_1$, $C_2$, $C_3$, $C_4$ and $C_5$ can be found by, for example, the least squares method [19] by means of the constrained minimization of function $G_1$:

$$\begin{aligned}&G_1 = G_1(C_1, C_2, C_3, C_4, C_5, \lambda) =\\ &= \frac{1}{2}(C_1\vec{f_1}+C_2\vec{f_2}+C_3\vec{f_3}+C_4\vec{f_4}+C_5\vec{f_5}-\vec{f_6})^T\\ &\quad(C_1\vec{f_1}+C_2\vec{f_2}+C_3\vec{f_3}+C_4\vec{f_4}+C_5\vec{f_5}-\vec{f_6})\\ &+ \lambda(C_1 C_3^2 + C_3 C_5 - C_4) \longrightarrow \min \qquad (6)\end{aligned}$$

This problem can be solved providing that vectors $\vec{f_1}, \cdots, \vec{f_5}$ are linearly independent in (6). The last term contains the Lagrange multiplier $\lambda$ and the constraint between coefficients $C_1, \cdots, C_5$. Moreover, the second equation of system (1) can be rewritten in time instants $t = t_i$ as the following overdetermined system of $N+1$ linear algebraic equations:

$$C_6\vec{f_7} + C_7\vec{f_8} - \vec{f_9} = 0, \quad (7)$$

where

$$\begin{aligned}&\vec{f_7} = [f_{7i}] = [y_i], \vec{f_8} = [f_{8i}] = \left[\frac{-y_i^2}{x_i}\right],\\ &\vec{f_9} = [f_{9i}] = [\dot{y}_i], C_6 = b_5, C_7 = b_6.\end{aligned}$$

That is why coefficients $C_6, C_7$ can be found by application of the least square method by means of minimization of function $G_2$

$$\begin{aligned}G_2 =&G_2(C_6, C_7) = \frac{1}{2}\left(C_6\vec{f_7} + C_7\vec{f_8} + \vec{f_9}\right)^T\\ &(C_6\vec{f_7} + C_7\vec{f_8} + \vec{f_9}) \longrightarrow \min \qquad (8)\end{aligned}$$

This problem can be solved providing that vectors $\vec{f_7}$ and $\vec{f_8}$ are linearly independent of (8). ■

**Remark 2.** *In vectors $\vec{f_3}, \vec{f_6}$ the component $\dot{x}_i$, and in vector $\vec{f_9}$ the components $\dot{y}_i$ are calculated by means of numerical differentiation of $x_i, y_i$ with respect to time $t$ and that is why the proposed method is called the differential method of identification.*

**Corollary 3.** *Parameters $b_1, b_2, b_3, b_4$ of the model (1) can be identified by the least square method [19] if $(N+1) \times 1$-vector columns*

$$\left[\int_0^{t_i} x(\tau)d\tau\right], \left[\int_0^{t_i} x^2(\tau)d\tau\right], \left[\int_0^{t_i} x^3(\tau)d\tau\right], [x_i - x_0], \left[\int_0^{t_i} x(\tau)y(\tau)d\tau\right]$$

*are linearly dependent. Parameters $b_5$ and $b_6$ of the abovementioned model can be identified by the abovementioned method if $(N+1) \times 1$-vector columns $\left[\int_0^{t_i} y(\tau)d\tau\right]$ and $\left[\int_0^{t_i} \frac{y^2(\tau)}{x(\tau)}d\tau\right]$ are linearly dependent.*

*Proof:* Integrating expression (4) with respect to time $t \in [0, T]$ we obtain

$$\begin{aligned}&C_1\left(-\int_0^t x^3(\tau)d\tau\right) + C_2\left(-\int_0^t x(\tau)y(\tau)d\tau\right)\\ &+ C_3\left(x_0 - x(t)\right) + C_4\left(\int_0^t x(\tau)d\tau\right)\\ &+ C_5\left(\int_0^t x^2(\tau)d\tau\right) - \left(\frac{1}{2}(x^2(t) - x_0^2)\right) = 0. \qquad (9)\end{aligned}$$

Integrating second equation of system 5 with respect to time $t \in [0, T]$ we have

$$\begin{aligned}&C_6\left(\int_0^t y(\tau)d\tau\right) + C_7\left(-\int_0^t \frac{y^2(\tau)}{x(\tau)}d\tau\right)\\ &\qquad -C_3\left(y(t) - y_0\right) = 0. \quad (10)\end{aligned}$$

Performing all integrations in (9) and (10) from 0 to $t_j \in [0, T]$ we obtain the following overdetermined systems of $N+1$ linear algebraic equations

$$\begin{aligned} &C_1\vec{g_1} + C_2\vec{g_2} + C_3\vec{g_3} + C_4\vec{g_4} + C_5\vec{g_5} - \vec{g_6} = 0, \\ &C_6\vec{g_7} + C_7\vec{g_8} - \vec{g_9} = 0, \end{aligned} \quad (11)$$

where

$$\begin{aligned} &\vec{g_1} = \left[-\int_0^{t_i} x^3(\tau)d\tau\right], \ \vec{g_2} = \left[-\int_0^{t_i} x(\tau)y(\tau)d\tau\right], \\ &\vec{g_3} = [x_0 - x_i], \ \vec{g_4} = \left[\int_0^{t_i} x(\tau)d\tau\right], \\ &\vec{g_5} = \left[\int_0^{t_i} x^2(\tau)d\tau\right], \ \vec{g_6} = \left[\frac{1}{2}(x_i^2 - x_0^2)\right], \\ &\vec{g_7} = \left[\int_0^{t_i} y(\tau)d\tau\right], \ \vec{g_8} = \left[-\int_0^{t_i} \frac{y^2(\tau)}{x(\tau)}d\tau\right], \\ &\vec{g_9} = [y_i - y_0] \end{aligned}$$

are the $(N+1)\times 1$-vector columns. Now applying the method used in Theorem 1 we prove the Corollary. ■

**Remark 4.** *In vector $\vec{g_1}, \vec{g_2}, \vec{g_4}, \vec{g_5}, \vec{g_7}, \vec{g_8}$ the integrals are calculated by means of numerical integration of $x_i, y_i$ and their combinations with respect to time $t$ and that is why the proposed method is called the integration method of identification.*

**Remark 5.** *Note that expressions (5), (7) and (11) are linear with respect to unknown constants $C_1, \cdots, C_7$. Direct use of the constraint minimization using the Lagrange multiplier with constraint:*

$$C_1C_3^2 + C_3C_5 - C_4 = 0 \quad (12)$$

*produces nonlinear system of equations for determination of six unknowns $C_1, C_2, C_3, C_4, C_5, \lambda$. Thus the search is performed in six-dimensional space of parameters and hence this method substantially complexifies the solution procedure. Determination of parameters and $C_6$ and $C_7$ needs solution of linear system of two algebraic equations. In the next section we describe an original problem solution algorithm reducing the search space dimension to one and using only linear matrix manipulations in the process of solution, which substantially simplifies and accelerates the problem solution.*

## IV. Solution of the parameter identification problem

There are four original independent parameters $(b_1, b_2, b_3, b_4)$ in the first equation of (1). First four $C$- parameters $(C_1, C_2, C_3, C_4)$ depend on $b$-parameters so that there is one-to-one correspondence between them. The parameter $C_5$ depends on the first four $C$-parameter as follows:

$$C_5 = \frac{C_4}{C_3} - C_1C_3^2. \quad (13)$$

Hence, it is possible to consider $(C_1, C_2, C_3, C_4)$ as independent parameters and introduce new name for the dependent parameter $C_5 = -\lambda$. The novel algorithm will be considered in detail for the differentiation method of solution, i.e. with $\vec{f}_{1,\cdots,9}$ - vector columns(see expression (5) and (7). The integration method of solution uses the same algorithm in which $\vec{f}_{1,\cdots,9}$ - vector columns are changed to $\vec{g}_{1,\cdots,9}$ -ones (see (11)). Parameter $\lambda$ will be selected from the given interval $\lambda \in [\lambda_{min}, \lambda_{max}]$ and substituted in goal function $G_3$ which is composed as follows

$$\begin{aligned} G_3 &= G_3(C_1, C_2, C_3, C_4, \lambda) \\ &= \frac{1}{2}\Big(C_1\vec{f_1}+C_2\vec{f_2}+C_3\vec{f_3}+C_4\vec{f_4}-(\lambda\vec{f_5}+\vec{f_6})\Big)^T \\ &\quad \Big(C_1\vec{f_1}+C_2\vec{f_2}+C_3\vec{f_3}+C_4\vec{f_4}-(\lambda\vec{f_5}+\vec{f_6})\Big) \end{aligned} \quad (14)$$

and subjected to minimization. In expression (14), parameter $\lambda$ is considered as constant at every minimization and minimization itself is performed with respect to parameters $C_1, C_2, C_3, C_4$. Solution of this problem is given by the following formula

$$\begin{aligned} C(\lambda) &= [C_1(\lambda),\ C_2(\lambda),\ C_3(\lambda),\ C_4(\lambda)]^T \\ &= \left((L_1^TL_1)^{-1}L_1^T\right)R(\lambda), \end{aligned} \quad (15)$$

where

$$L_1 = \left[\vec{f_1}\ \ \vec{f_2}\ \ \vec{f_3}\ \ \vec{f_4}\right]^T, \ R(\lambda) = \lambda\vec{f_5} + \vec{f_6}. \quad (16)$$

In expression (15) it is possible to calculate $1\times(N+1)$- vector row $\left((L_1^TL_1)^{-1}L_1^T\right)$ only once and after that perform its multiplication by $(N+1)\times 1$-vector row $R(\lambda)$, which is very fast operation. Components of vector $C(\lambda)$ and $C_5 = -\lambda$ are substituted in the constraint (12) to obtain the following nonlinear scalar equation

$$C_1(\lambda)C_3^2(\lambda) - \lambda C_3(\lambda) - C_4(\lambda) = 0, \qquad (17)$$

which is solved with respect to $\lambda$. All roots of Equation (17) are found (sometimes to find all the roots it is necessary to expand the interval $\lambda \in [\lambda_{min}, \lambda_{max}]$ to the left or to the right or to both sides). After finding a particular root $\lambda$ the corresponding $b$-parameters are calculated as follows:

$$b_1(\lambda) = \frac{C_4(\lambda)}{C_3(\lambda)},\ \ b_2(\lambda) = C_1(\lambda),$$
$$b_3(\lambda) = C_2(\lambda),\ \ b_4(\lambda) = C_3(\lambda). \qquad (18)$$

(See (4)). The estimations of $b$-parameters are obtained from the proper selection of root $\lambda = \bar{\lambda}$:

$$\bar{b_1} = b_1(\bar{\lambda}),\ \ \bar{b_2} = b_2(\bar{\lambda}),$$
$$\bar{b_3} = b_3(\bar{\lambda}),\ \ \bar{b_4} = b_4(\bar{\lambda}) \qquad (19)$$

(one of the criteria of the correct choice of $\bar{\lambda}$ must be positiveness of all estimated $\bar{b}$ parameters, see **Numerical Examples**). Parameters $b_5$ and $b_6$ are estimated by means of minimization of the goal function of Equation 8. Solution of this problem is given by the formulas:

$$\begin{bmatrix} \bar{b_5} \\ \bar{b_6} \end{bmatrix} = (L_2^T L_2)^{-1} L_2^T \vec{f_9} \qquad (20)$$

where $L_2 = \begin{bmatrix} \vec{f_7} & \vec{f_8} \end{bmatrix}$ is $(N+1)\times 2$- matrix, (See (7)).

Expression (15)-(20) give solution to the identification problem by means of the differentiation method. To find solution of the problem by the integral method it is necessary to consider vectors $\vec{g}_{1,\cdots,9}$ (See expression 11) instead of $\vec{f}_{1,\cdots,9}$. In the next section you will find more information about application of the differentiation and integration methods.

## V. Numerical examples

Let us solve the initial problem of Equation 1 with the following parameters:

$$\begin{array}{lll} b_1 = 0.2 & b_2 = 0.01 & b_3 = 0.05 \\ b_4 = 1 & b_5 = 0.062 & b_6 = 0.0223 \end{array} \qquad (21)$$

and initial conditions: $[x_0\ \ y_0]^T = [10\ \ 5]^T$. The stable critical point has coordinates $(\tilde{x}, \tilde{y}) \approx (7.77064, 21.4066)$ (see Equation 2) and it is the stable focus (eigenvalues of the linearized system in the vicinity of the critical point are $\nu_{1,2} \approx -0.0138 \pm 0.0735i$, where $i^2$=$-1$). The unstable saddle has coordinates $(\tilde{\tilde{x}}, \tilde{\tilde{y}})$=(20,0). Numerical solution $x = x(t)$ on the time interval $t \in [0, T = 150]$ in $N+1 = 25$ points is shown in Figure 1 and solution $y = y(t)$ is shown in Figure 2. Performing solution by means of the differential

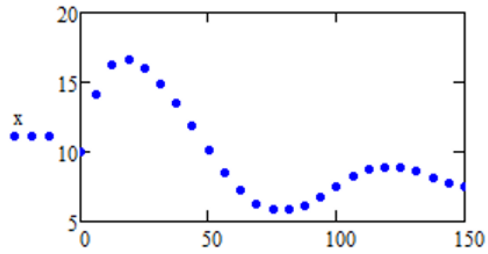


Fig. 1. Graph of solution $x = x(t)$

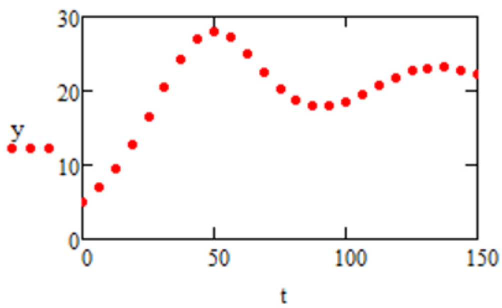


Fig. 2. Graph of solution $y = y(t)$

method in accordance with the described algorithm we obtain nonlinear Equation (12) from which the parameters are calculated: $\lambda_1 \approx -0.3282$, $\lambda_2 \approx -0.1091$ and $\lambda_3 \approx 0.2087$. As we see, only $\lambda_3$ parameter can be selected from three

TABLE I
VALUES OF $b$-PARAMETERS, CORRESPONDING TO DIFFERENT ROOTS OF EQUATION (12) FOR $N+1=25$ (DIFFERENTIATION METHOD)

| Original Values | $\lambda_1 \approx -0.3282$ | $\lambda_2 \approx -0.1091$ | $\lambda_3 \approx 0.2087$ |
|---|---|---|---|
| $b_1 = 0.2000$ | $-0.0580$ | $-0.0621$ | $0.2129$ |
| $b_2 = 0.0100$ | $-0.0150$ | $-0.0045$ | $0.0108$ |
| $b_3 = 0.0500$ | $-0.0295$ | $-0.0026$ | $0.0491$ |
| $b_4 = 1.0000$ | $-18.0380$ | $-10.5185$ | $0.3906$ |

roots, because $\lambda_1$ and $\lambda_2$ generate the negative values of $b$-parameters. The relative error of the $b$-parameters corresponding to $\lambda_3$-parameter are as follows:

$$\begin{aligned} ERROR\%(b_1) &\approx 6.437\% \\ ERROR\%(b_2) &\approx 7.725\% \\ ERROR\%(b_3) &\approx 1.832\% \\ ERROR\%(b_4) &\approx 60.943\% \end{aligned} \tag{22}$$

Estimation of parameters $b_5$ and $b_6$ gives coincidence with the original values of the parameters in four decimals with the following relative errors:

$$\begin{aligned} ERROR\%(b_5) &\approx 0.029\% \\ ERROR\%(b_6) &\approx 0.028\% \end{aligned} \tag{23}$$

Comparison of original graphs with graphs obtained by numberical solution of initial problem (1) with the same initial conditions but with estimated parameters is shown in Figure 3 and Figure 4.

As we see the estimated parameters gives quite good estimation of the process dynamics. The estimated values of the steady states are as follows $(\tilde{\tilde{x}}, \tilde{\tilde{y}}) \approx (7.7143, 21.4282)$ with relative errors:

$$\begin{aligned} ERROR\%(\tilde{x}) &\approx 0.102\% \\ ERROR\%(\tilde{y}) &\approx 0.101\% \end{aligned} \tag{24}$$

Estimation of the parameters with $N+1 = 49$ points gives $\lambda_1 \approx -0.2585$, $\lambda_2 \approx -0.0878$, $\lambda_3 \approx 0.1914$ and the following values of parameters (see Table 2)

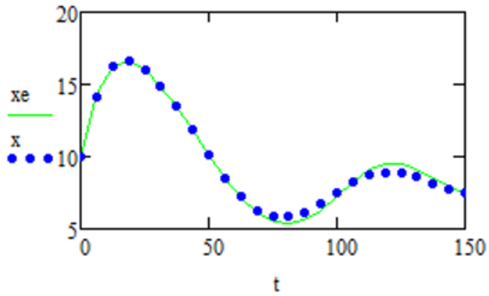


Fig. 3. Graph of original solution $x = x(t)$ (dots) and solution with estimated parameters (solid line)

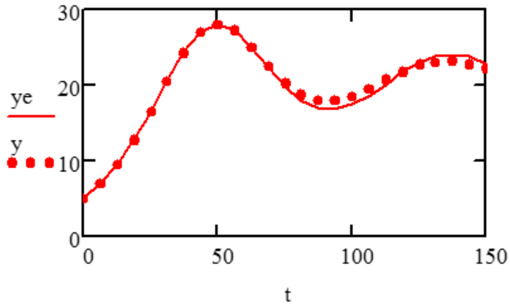


Fig. 4. Graph of original solution $y = y(t)$ (dots) and solution with estimated parameters (solid line)

As we see, only $\lambda_3$ parameter can be selected from the three roots, because $\lambda_1$ and $\lambda_2$ generate the negative values of $b$-parameters. One can see the substantial improvement of the parameters estimations. The relative errors of the $b$-parameters

TABLE II
VALUES OF $b$-PARAMETERS, CORRESPONDING TO DIFFERENT ROOTS OF EQUATION (12) FOR $N+1=49$ (DIFFERENTIATION METHOD)

| Original Values | $\lambda_1 \approx -0.2585$ | $\lambda_2 \approx -0.0878$ | $\lambda_3 \approx 0.1914$ |
|---|---|---|---|
| $b_1 = 0.2000$ | $-0.0405$ | $-0.0485$ | $0.2010$ |
| $b_2 = 0.0100$ | $-0.0119$ | $-0.0036$ | $0.0101$ |
| $b_3 = 0.0500$ | $-0.0271$ | $-0.0022$ | $0.0500$ |
| $b_4 = 1.0000$ | $-18.3099$ | $-10.9975$ | $0.9553$ |

corresponding to $\lambda_3$- parameter are as follows:

$$\begin{aligned} ERROR\%(b_1) &\approx 0.496\% \\ ERROR\%(b_2) &\approx 0.583\% \\ ERROR\%(b_3) &\approx 0.087\% \\ ERROR\%(b_4) &\approx 4.469\% \end{aligned} \tag{25}$$

Estimation of parameters $b_5$ and $b_6$ gives coincidence with the original ones in four decimals with the following relative errors:

$$\begin{aligned} ERROR\%(b_5) &\approx 0.002\% \\ ERROR\%(b_6) &\approx 0.002\% \end{aligned} \tag{26}$$

Comparison of original graphs with graphs obtained by numerical solution of initial problem (1) with the same initial conditions but with estimated parameters is shown in Figure 5 and Figure 6.

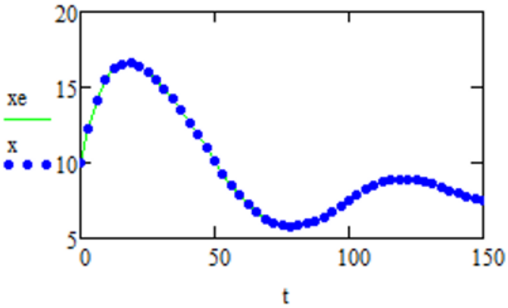


Fig. 5. Graph of original solution $x = x(t)$ (dots) and solution with estimated parameters (solid line)

As we see the estimated parameters give very good estimation of the process dynamics. The

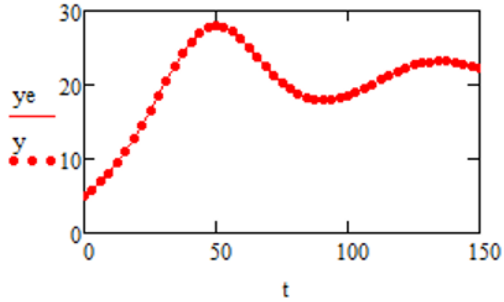


Fig. 6. Graph of original solution $y = y(t)$ (dots) and solution with estimated parameters (solid line)

estimated values of the steady states are as follows $(\tilde{\tilde{x}}, \tilde{\tilde{y}}) \approx (7.7077, 21.4102)$ with relative errors:

$$\begin{aligned} ERROR\%(\tilde{x}) &\approx 0.017\% \\ ERROR\%(\tilde{y}) &\approx 0.017\% \end{aligned} \tag{27}$$

Absolute errors in calculation of $x = x(t)$ and $y = y(t)$ in the differentiation method for $N+1=25$ and $N+1=49$ points are shown in Figure 7 and Figure 8. Performing solution by means of the integration method in accordance with the described algorithm we obtain three roots of nonlinear equation (12): $\lambda_1 \approx -0.2391$, $\lambda_2 \approx -0.0725$, $\lambda_3 \approx 0.1899$.

As we see, only $\lambda_3$ parameter can be selected from the three roots, because $\lambda_1$ and $\lambda_2$ generate the negative values of $b$-parameters. The relative errors of the $b$-parameters corresponding to $\lambda_3$-

TABLE III
VALUES OF $b$-PARAMETERS, CORRESPONDING DIFFERENT ROOTS OF EQUATION (12) FOR $N+1=25$ (INTEGRATION METHOD)

| Original Values | $\lambda_1 \approx -0.2391$ | $\lambda_2 \approx -0.0725$ | $\lambda_3 \approx 0.1899$ |
|---|---|---|---|
| $b_1 = 0.2000$ | $-0.0302$ | $-0.0386$ | $0.1999$ |
| $b_2 = 0.0100$ | $-0.0115$ | $-0.0032$ | $0.0100$ |
| $b_3 = 0.0500$ | $-0.0282$ | $-0.0022$ | $0.0500$ |
| $b_4 = 1.0000$ | $-18.1074$ | $-10.6869$ | $0.9997$ |

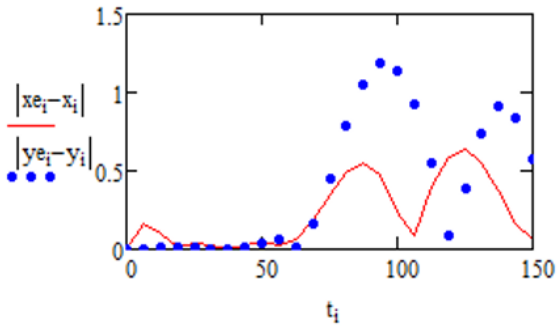


Fig. 7. Absolute Errors of Calculation for $N+1=25$ points (Differentiation method)

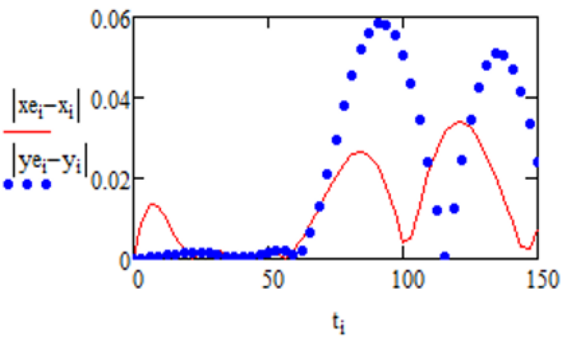


Fig. 8. Absolute Errors of Calculation for $N+1=49$ points (Differentiation method)

parameter are as follows:

$$\begin{aligned} ERROR\%(b_1) &\approx 0.052\% \\ ERROR\%(b_2) &\approx 0.044\% \\ ERROR\%(b_3) &\approx 0.059\% \\ ERROR\%(b_4) &\approx 0.033\% \end{aligned} \tag{28}$$

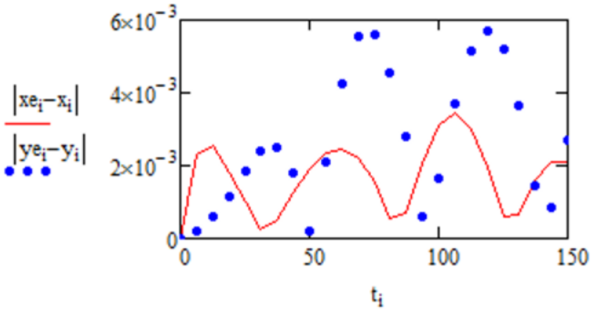


Fig. 9. Absolute errors of calculation for $N+1=25$ points (Integration method)

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Estimation of parameters $b_5$ and $b_6$ gives coincidence with the original values of $b$-parameter in four decimals with the following relative errors:

$$\begin{aligned} ERROR\%(b_5) &\approx 0.008\% \\ ERROR\%(b_6) &\approx 0.007\% \end{aligned} \tag{29}$$

Comparison of original graphs with graphs obtained by numerical solution of initial problem (1) with the same initial conditions but with estimated parameters are visually indistinguishable from Figure 5 and Figure 6. Absolute errors in calculation of $x = x(t)$ and $y = y(t)$ in the integration method for $N+1 = 25$ points are shown in Figure 9.

The parameters are estimated with very high accuracy at $N+1 = 25$ points. The estimated values of the steady states are as follows $(\tilde{\tilde{x}}, \tilde{\tilde{y}}) \approx (7.7062, 21.4061)$ with relative errors:

$$\begin{aligned} ERROR\%(\tilde{x}) &\approx 0.002\% \\ ERROR\%(\tilde{y}) &\approx 0.002\% \end{aligned} \tag{30}$$

## VI. Conclusion

Two methods of solution of the inverse problem on parameter identification of the Holling-Tanner model with complete information are discussed. These are the differentiation and integration methods of solution. The conditions are indicated at which all parameters of the model can be identified. The main disadvantage of the conventional method of constraint minimization by means of the Lagrange multipliers is that the method generates a system of six nonlinear equations with unknown initial guess values. Proposed is the novel method of the problem solution in which the six dimensional space of search is reduced to one dimensional space and the procedure of the initial guess value is performed by fast vector multiplication. Numerical examples of the proposed algorithm implementation are demonstrated for the differentiation and integration methods. It is shown that the integration method generates more accurate results than the differentiation one. The integration method also needs less number of points on the fixed time interval to produce accurate results than the differentiation method.

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