On Unique Common Fixed Point Theorems for Three and Four Self Mappings in Symmetric Fuzzy Metric Space

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Abstract
In this paper, we prove two unique common fixed point theorems for three and four self mappings in symmetric fuzzy metric spaces.

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1. Introduction

In 1965, Zadeh A.L. [3] introduced the concept of Fuzzy set as a new way to represent vagueness in our everyday life. However, when the uncertainty is due to fuzziness rather than randomness, as sometimes in the measurement of an ordinary length, it seems that the concept of a fuzzy metric space is more suitable. We can divide them into following two groups:

The first group involves those results in which a fuzzy metric on a set *X* is treated as a map where *X* represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between fuzzy objects.

On the other hand in the second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy.

Kramosil I. and Michalek J. [2] have introduced the concept of fuzzy metric spaces in different ways. In this paper, we prove two unique common fixed point theorems for three and four self mappings in symmetric fuzzy metric spaces.

2. Preliminaries

Definition 2.1: A binary operation $*$: $[0,1]\times[0,1] \rightarrow [0,1]$ is ***continuous t-norm,*** if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a\in [0, 1]$;
(iv) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0, 1]$.

Example 2.1: $a * b = \min\{a, b\}$ and $a * b = a . b$ are t-norms.
Kramosil I and Michalek J. [3] introduced the concept of fuzzy metric spaces as follows:

Definition 2.2: Fuzzy metric space: The 3-tuple (X, M, ¤) is called a fuzzy metric space (shortly, FM-space) if X is an arbitrary set, ¤ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x. t)$,
(FM-4) $M(x, y, t)$ ¤ $M(y, z, s) \leq M(x, z, t + s)$ (Triangular inequality) and
(FM-5) $M(x, y, .) : [0, 1) \rightarrow [0, 1]$ is left continuous $\forall$ $x, y, z \in X$ and $s, t > 0$.

Note that M(x, y, t) can be thought of as the degree of nearness between x and y with respect to t. If only (FM- 1, 2, 3) hold, the 3-tuple (X, M, ¤) is said to be ***fuzzy semi – metric (symmetric) space.***

We can fuzzyfy examples of metric spaces into fuzzy metric spaces in a natural way: Let (X, d) be a metric space. Define a ¤ b = a + b for all a, b in X. Define M(x, y, t) = t / (t + d(x, y)) for all x, y in X and t > 0. Then, (X,M, ¤) is a fuzzy metric space, and this fuzzy metric induced by a metric d is called the ***Standard fuzzy metric***.

Consider M to be a fuzzy metric space with the following condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$, for all x, y in X and $t > 0$.

Definition 2.3: Let (X, M, *) be fuzzy semi-metric space. Then,

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$,

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$$

and

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$,

$$\lim_{n\to\infty} M(x_n, x, t) = 1$$

Definition 2.4: A fuzzy semi-metric space (X, M, *) is said to be complete if and only if every Cauchy sequence in X is convergent.

Example 2.2: Let $X = \{1/n: n \in \mathbf{N}\} \cup \{0\}$ and let * be the continuous t-norm and defined by a * b = ab for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ and $x, y \in X$, define M, by

$$M(x, y, t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0, \\ 0 & t = 0 \end{cases}$$

Clearly, (X, M, *) is complete fuzzy semi-metric space.

Definition 2.5: A pair of self mappings (f , g) of a fuzzy semi-metric space (X, M, *) is said to be ***commuting*** if M (fgx, gfx , t) = 1, for all $x \in X$.

Definition 2.6: A pair of self mappings (f , g) of a fuzzy semi-metric space (X, M, *) is said to be ***weakly commuting*** if M (fgx , gfx , t) $\geq$ M (fx, gx, t), for all $x \in X$ and $t > 0$.

Definition 2.7: A pair of self mappings (f, g) of a fuzzy semi-metric space (X, M, *) is said to be ***compatible,*** if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u, \text{ for some u in X}$$

Definition 2.8: Let (X, M, *) be a fuzzy semi-metric space, and let f and g be self maps on X. A point x in X is called a ***coincidence point*** of f and g iff fx = gx. In this case, w = fx = gx is called a *point of coincidence of f and g.*

Definition 2.9: A pair of self mappings (f, g) of a fuzzy semi-metric space (X, M, *) is said to be ***weakly compatible,*** if they commute at the coincidence points, i.e.

if fu = gu, for some u in X , then fgu = gfu

It is easy to see that two compatible maps are weakly compatible, but converse is not true.

Definition 2.10: Two self mappings f and g of a fuzzy semi-metric space (X, M, *) are called ***occasionally weakly compatible*** (owc) iff there is a point x in X which is coincidence point of f and g at which f and g commute.

3. Main Results

3.1 A unique common fixed point theorem for three mappings

Theorem 3.1: Let (X, M, *) be symmetric Fuzzy metric space. Suppose f, g, and h are three self mappings of (X, M, *) satisfying the conditions:
(1) for all x, y in X

$$\int_0^{M(fx,gy,t)} \phi(t)dt \le \int_0^{\alpha M(hx,hy,t)+\beta[M(fx,hx,t)+M(gy,hy,t)]+\gamma[M(hx,gy,t)+M(hy,fx,t)]} \phi(t)dt$$

where ϕ : $\mathbf{R}^+ \to \mathbf{R}$ is a Lebesgue-integrable mapping which is summable, nonnegative and such that $\int_0^{\in} \phi(t)dt > 0$ for each $\in$ > 0, and α, β, γ are non-negative real numbers such that $\alpha + 2\beta + 2\gamma < 1$.
(2) the pair of mappings (f, h), or (g, h), is owc.

Then f, g and h have a unique common fixed point.

Proof: Suppose that f and h are owc then there is an element u in X such that fu = hu and fhu = hfu.

First, we prove that fu = gu. Indeed, by inequality (1), we get

$$\int_0^{M(fu,gu,t)} \phi(t)dt \le \int_0^{\alpha M(hu,hu,t)+\beta[M(fu,hu,t)+M(gu,hu,t)]+\gamma[M(hu,gu,t)+M(hu,fu,t)]} \phi(t)dt$$
$$= \int_0^{\beta[M(gu,fu,t)]+\gamma[M(fu,gu,t)]} \phi(t)dt$$
$$= \int_0^{\beta[M(fu,gu,t)]+\gamma[M(fu,gu,t)]} \phi(t)dt = \int_0^{(\beta+\gamma)M(fu,gu,t)} \phi(t)dt$$
$$< \int_0^{M(fu,gu,t)} \phi(t)dt$$

which is a contradiction, hence, gu = fu = hu.

Again, suppose that ffu ≠ fu. By the use of condition (1), we have

$$\int_0^{M(ffu,gu,t)} \phi(t)dt \le \int_0^{\alpha M(hfu,hu,t)+\beta[M(ffu,hfu,t)+M(gu,hu,t)]+\gamma[M(hfu,gu,t)+M(hu,ffu,t)]} \phi(t)dt$$
$$= \int_0^{\alpha M(ffu,gu,t)+2\gamma[M(ffu,gu,t)]} \phi(t)dt$$
$$= \int_0^{(\alpha+2\gamma)M(ffu,gu,t)} \phi(t)dt$$
$$< \int_0^{M(ffu,gu,t)} \phi(t)dt$$

this contradiction implies that ffu = fu = hfu.

Now, suppose that gfu ≠ fu. By inequality (1), we have

$$\int_0^{M(fu,gfu,t)} \phi(t)dt \le \int_0^{\alpha M(hu,hfu,t)+\beta[M(fu,hu,t)+M(gfu,hfu,t)]+\gamma[M(hu,gfu,t)+M(hfu,fu,t)]} \phi(t)dt$$

$$= \int_0^{\beta M(gfu,fu,t)+\gamma[M(fu,gfu,t)]} \phi(t)dt$$

$$= \int_0^{(\beta+\gamma)M(fu,gfu,t)} \phi(t)dt$$

$$< \int_0^{M(fu,gfu,t)} \phi(t)dt$$

this above contradiction implies that gfu = fu.

Put fu = gu = hu = t, so, t is a common fixed point of mappings f, g and h. Now, let p and z be two distinct common fixed points of f, g and h. I.e. fp = gp =hp = p, and fz = gz = hz = z . As p ≠ z, then from condition (1), we have

$$\int_0^{M(p,z,t)} \phi(t)dt = \int_0^{M(fp,gz,t)} \phi(t)dt \le \int_0^{\alpha M(hp,hz,t)+\beta[M(fp,hp,t)+M(gz,hz,t)]+\gamma[M(hp,gz,t)+M(hz,fp,t)]} \phi(t)dt$$

$$= \int_0^{\alpha M(p,z,t)+2\gamma M(p,z,t)]} \phi(t)dt$$

$$= \int_0^{(\alpha+2\gamma)M(p,z,t)} \phi(t)dt$$

$$< \int_0^{M(p,z,t)} \phi(t)dt$$

a contradiction, hence z = p. Thus, the common fixed point is unique.

If we put $\phi(t) = 1$ in the above theorem, we get the following result:

Corollary: Let : (X. <. *) be symmetric fuzzy-metric space. Suppose *f*, *g*, and *h* are three self-mapping of (X, M, *) satisfying the conditions:

(1) for all *x*, *y* in X

M(fx, gy, r) ≤ α*M(hx, hy, t)* + β [*M(fx, hx, t)*+*M(gy, hy, t)*]+ γ [*M(hx, gy, t)*+*M(hy, fx, t)*] and α, β, γ are non-negative reals such that α+2β+2γ<1

(2) pair of mappings (f,h) or (g,h) is owc

Then f, g and h have a unique common fixed point.

3. 2. A unique common fixed-point theorem for four mappings.

Now, we give our second main result:

<u>Theorem 3.2:</u> Let (X, M, *) be symmetric fuzzy-metric space. Suppose f, g , h and k are four self mappings of (X, M, *) satisfying the following conditions:

(1)

$$\int_0^{M(fx,gy,t)} \phi(t)dt \le \int_0^{\alpha M(hx,ky,t)+\beta[M(fx,hx,t)+M(gy,ky,t)]+\gamma[M(hx,gy,t)+M(ky,fx,t)]} \phi(t)dt$$

for all x and y in X, where ϕ : $R^+ \to R$ is a Lebesgue-integrable mapping which is summable, nonnegative and such that $\int_0^{\in} \phi(t)dt > 0$ for each ∈ > 0, and α, β, γ are non-negative real numbers such that $\alpha + 2\beta + 2\gamma < 1$

(2) pair of mappings (f, h) and (g, k) are owc.

Then, f, g, h and k have a unique common fixed point.

Proof: Since pairs of mappings (f, h) and (g , k) are owc, there exists two points, u and v, in X such that fu = hu and fhu = hfu, gv = kv and gkv = kgv.

First, we prove that fu = gv. Indeed, by inequality (1), we get

$$\int_0^{M(fu,gv,t)} \phi(t)dt \le \int_0^{\alpha M(hu,kv,t)+\beta[M(fu,hu,t)+M(gv,kv,t)]+\gamma[M(hu,gv,t)+M(kv,fu,t)]} \phi(t)dt$$
$$= \int_0^{\alpha[M(hu,kv,t)]+\gamma[M(fu,gv,t)]} \phi(t)dt$$
$$= \int_0^{(\alpha+\gamma)M(fu,gv,t)} \phi(t)dt$$
$$< \int_0^{M(fu,gv,t)} \phi(t)dt$$

which is a contradiction; hence, gv = fu = hu = kv.

Again, suppose that ffu = fhu = hfu $\neq$ fu. By the use of condition (1), we have

$$\int_0^{M(ffu,gv,t)} \phi(t)dt \le \int_0^{\alpha M(hfu,kv,t)+\beta[M(ffu,hfu,t)+M(gv,kv,t)]+\gamma[M(hfu,gv,t)+M(kv,ffu,t)]} \phi(t)dt$$
$$= \int_0^{\alpha[M(ffu,fu,t)]+2\gamma[M(ffu,gv,t)]} \phi(t)dt$$
$$= \int_0^{(\alpha+2\gamma)M(ffu,gv,t)} \phi(t)dt$$
$$< \int_0^{M(ffu,gv,t)} \phi(t)dt$$

this contradiction implies that ffu = fu = hfu = fhu.

Similarly, gfu = kfu = fu. Put fu = p, therefore p is a common fixed point of mappings f, g, h and k.

Now, let p and z be two distinct common fixed points of f, g, h and k. That is, fp = gp =hp = kp = p and fz = gz = hz = kz = z . As p $\neq$ z, then from condition (1), we have:

$$\int_0^{M(p,z,t)} \phi(t)dt = \int_0^{M(fp,gz,t)} \phi(t)dt \le \int_0^{\alpha M(hp,hz,t)+\beta[M(fp,hp,t)+M(gz,hz,t)]+\gamma[M(hp,gz,t)+M(hz,fp,t)]} \phi(t)dt$$
$$= \int_0^{\alpha M(p,z,t)+2\gamma M(p,z,t)]} \phi(t)dt$$
$$= \int_0^{(\alpha+2\gamma)M(p,z,t)} \phi(t)dt$$
$$< \int_0^{M(p,z,t)} \phi(t)dt$$

a contradiction; hence z = p. Thus, the common fixed point is unique. □

If we put $\phi(t) = 1$ in the above theorem, we get the following result:

Corollary: Let (X, M,*) be symmetric Fuzzy metric space.

Suppose f, g, h and k are four self mappings of (X, M,*) satisfying the following conditions:

(1) $M(fx,gy,t) \le \alpha M(hx,ky,t) + \beta[M(fx,hx,t) + M(gy,ky,t)] + \gamma[M(hx,gy,t) + M(ky,fx,t)]$ for all x and y in X, and α, β, γ are non-negative reals such that $\alpha + 2\beta + 2\gamma < 1$

(2) pair of mappings (f, h) and (g, k) are owc.

Then, f, g, h and k have a unique common fixed point.

Example: Let X = $[0,\infty)$ with the symmetric Fuzzy-metric:

$$M(x, y, t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0, \\ 0 & t = 0 \end{cases}$$

Define

$$f(x) = g(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x \in [1,\infty) \end{cases}, \quad h(x) = \begin{cases} 3 & x \in [0,1) \\ \dfrac{1}{x} & x \in [1,\infty) \end{cases}, \quad k(x) = \begin{cases} 9 & x \in [0,1) \\ \dfrac{1}{\sqrt{x}} & x \in [1,\infty) \end{cases}$$

Clearly (f, h) and (g, k) are owc.
By taking

$$\phi(x) = 3x^2, \alpha = \frac{1}{4}, \beta = \frac{1}{5}, \gamma = \frac{1}{6}$$

all the hypothesis of theorem 2.2 are satisfied, and x = 1 is the unique common fixed point of mappings f, g, h and k.

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3. References

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