

An Economic Model Predictive Approach for Inventory Routing and Control with Time Windows Constraints: Application in the Distribution of Industrial Gases

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In the process of industrial gas distribution, the uncertainty of future demand presents an important challenge for the production of efficient delivery plans, where the overall cost is minimized and product shortages must be avoided. The objective of this paper is to present a new method for producing future delivery plans that eliminate stock-out risk and minimize transportation and inventory holding costs, in the face of time window constraints. The proposed method follows a Economic Model Predictive Control (EMPC) approach, which makes use of a new mathematical programming model of the distribution process to predict future inventory levels. The method is illustrated using an example with a simple forecasting approach. It is shown that the stock-outs risk can be eliminated, but still balanced decisions between cost saving and risk aversion remain at the discretion of the planner. It is also shown that coordinated inventory routing and control through the EMPC approach brings economic benefits and hedges effectively against the market volatility, while opting for minimum overall costs may incur excessive shortage risk.

1. Introduction

In the present study the Vendor Managed Inventory (VMI) strategy is considered for the distribution of liquefied industrial gases, with a combined backhauling of the empty tanks. Following a VMI strategy a tool that optimizes dispatches produces short-term plans by solving Inventory Routing Problems (IRP) to decide when, how much to deliver to each customer, and what will be the delivery routes. Real problems involve inherently time constraints that appear in the form of hard or soft time windows, within which vehicles must reach the customers' sites and start the delivery. While deterministic planning models are useful tools to help reduce costs by taking into account customer synergies in the inventory-distribution planning, uncertain demand fluctuations may significantly affect the decision-making across the industrial gas supply chain. Thus, it is necessary to extend the deterministic planning models to address these uncertainties, and develop effective optimization and control algorithms for these problems (You et al, 2011b). Supply chain managers also wish to balance profit maximization versus minimization of the stock-out risk. One way to minimize risk is to hold some buffer stock so that the supply chain can respond to rush orders or demand spikes. The present work deals with demand uncertainties by employing an EMPC approach to control customer inventories along forecasted trajectories of safety stock levels. The literature on the application of MPC on distribution systems involving reverse logistics, time constraints, routing and inventory control at the customers' sites is absent. An MPC application to similar supply chain systems is that of Seferlis and Gianellos (2004), however, there were no routing decisions involved. Here, a multi-period deterministic model is first presented for the Inventory Routing with Simultaneous Pickup and Delivery and Time Windows problem (IRPSPDTW), which captures all attributes of reverse logistics, time constraints and inventory routing, as an extension of the single period model proposed by Nikolakopoulos (2015). Next, following a centralized EMPC strategy, the IRPSPDTW is solved with a rolling horizon. A solution is obtained over the entire time horizon but only the part of the solution related to the first time period is actually implemented before moving to the next period. Then the procedure is repeated with updated sequences. The methodology is illustrated using an example where an inventory

routing systems for the distribution of industrial gases is stabilized in the face of hard time constraints, and balanced decisions are produced between the minimization of the overall cost and the minimization of stock-out risk.

2. Mathematical programming model of the IRPSPDTW

In the multi-period IRPSPDTW an homogeneous fleet V with $K_{avl} = |V|$ available vehicles of equal capacity Q must serve a set N of n customers, $N = \{1, 2, \dots, n\}$. The planning horizon considers T periods, $t = 1, \dots, T$. There is a demand $d_{i,t}$ for every customer $i \in N$. Each customer i maintains its own inventory up to capacity C_i pays for inventory carrying cost of h_i units per period per stock keeping unit. The complete directed graph induced by the customers is $G = (N^*, A)$, where $N^* = N \cup \{0, n+1\}$ is the set of vertices. The depot is represented by vertices 0 and $n+1$. A is the set of arcs connecting any pair of vertices, while no arc terminates in vertex 0, and no arc originates from vertex $n+1$. A travelling cost $c_{i,j}$ is assigned to each arc $(i, j) \in A$ and $c_{\max} = \max(c_{i,j})$. Also, a traveling time $tr_{i,j}$ corresponds to each arc $(i, j) \in A$, while a time window $[e_i, l_i]$ is assigned to each vertex $i \in N^*$. This is in fact a time restriction, meaning that the time $Tr_{i,t}$ at which service to customer i in period t begins, must lie within the respective time window. If in period t a vehicle reaches customer i at time $Tr_{i,t} < e_i$ then it waits for a time period equal to $e_i - Tr_{i,t}$ until the service begins. The time window, in which a vehicle can leave from or arrive at the depot is $[e_0, l_0] = [e_{n+1}, l_{n+1}] = [E, L]$. Zero delivery demands and pickup demands are assigned to nodes $\{0, n+1\}$, that is, $d_0 = d_{n+1} = p_0 = p_{n+1} = 0$. By allowing some of the vehicles to remain at the depot there is no cost contributed. These vehicles do not serve any customer and their routes are represented by arc $(0, n+1)$ to which zero travelling cost and time are assigned: $c_{0,n+1} = tr_{0,n+1} = 0$. The objective is to minimize the overall transportation and inventory carrying cost incurred over a specific planning horizon. Each customer is serviced at most once in each period t , ($p_i, d_i \leq Q, \forall i \in N$), capacity constraints must be satisfied, and no split delivery or backlogging is permitted.

The next five sets of variables are considered for the problem:

$$z_{k,i,j,t} = \begin{cases} 1 & \text{if vehicle } k \text{ drives from } i \text{ to } j \text{ in period } t \\ 0 & \text{otherwise} \end{cases}, \quad (i,j) \in A, k \in V \text{ and } t = 1, \dots, T$$

$Y_{k,i,j,t}$ and $P_{k,i,j,t}$ are the vehicle loads of full and empty tanks on that trip respectively, and $I_{i,t}$ and $EI_{i,t}$ are the inventories of full and empty tanks at customer i , at the end of period t respectively. The IRPSPDTW can then be formally described as the following multi-period commodity network flow model with capacity and time window constraints:

$$\text{Min} \quad \sum_{t=1}^T \left\{ \sum_{k \in V} \sum_{(i,j) \in A} c_{i,j} z_{k,i,j,t} + \sum_{i \in N} h_i I_{i,t} \right\} \quad (1)$$

s.t.

$$\sum_{\substack{j=0 \\ j \neq i}}^N z_{k,i,j,t} \leq 1, \quad \forall i \in N, k \in V \text{ and } t = 1, \dots, T \quad (2)$$

$$\sum_{\substack{m=0 \\ m \neq i}}^{N \cup \{0\}} z_{k,i,m,t} - \sum_{\substack{l=0 \\ l \neq i}}^{N \cup \{n+1\}} z_{k,l,i,t} = 0 \quad \forall k \in V, \forall i \in N, t = 1, \dots, T \quad (3)$$

$$Y_{k,i,j,t} - z_{k,i,j,t} Q \leq 0 \quad \forall k \in V, \forall i, j \in N, i \neq j, t = 1, \dots, T \quad (4)$$

$$\sum_{\substack{l=0 \\ l \neq i}}^{N \cup \{0\}} Y_{k,l,i,t} - \sum_{\substack{m=0 \\ m \neq i}}^{N \cup \{n+1\}} Y_{k,i,m,t} \geq 0 \quad \forall k \in V, \forall i \in N, t = 1, \dots, T \quad (5)$$

$$P_{k,i,j,t} - z_{k,i,j,t} Q \leq 0 \quad \forall k \in V, \forall i, j \in N, i \neq j, t = 1, \dots, T \quad (6)$$

$$\sum_{\substack{m=0 \\ m \neq i}}^{N \cup \{n+1\}} P_{k,i,m,t} - \sum_{\substack{l=0 \\ l \neq i}}^{N \cup \{0\}} P_{k,l,i,t} \geq 0, \quad \forall k \in V, \forall i \in N, t = 1, \dots, T \quad (7)$$

$$I_{i,t-1} - I_{i,t} + \sum_{k=1}^V \left(\sum_{\substack{l=0 \\ l \neq i}}^N Y_{k,l,i,t} - \sum_{\substack{m=0 \\ m \neq i}}^N Y_{k,i,m,t} \right) = d_{i,t} \quad \forall i \in N, t = 1, \dots, T \quad (8)$$

$$EI_{i,t} = EI_{i,t-1} - \sum_{k=1}^V \left(\sum_{\substack{m=0 \\ m \neq i}}^N P_{k,i,m,t} - \sum_{\substack{l=0 \\ l \neq i}}^N P_{k,l,i,t} \right) + d_{i,t} \quad \forall i \in N, t=1, \dots, T \quad (9)$$

$$Y_{k,i,j,t} + P_{k,i,j,t} \leq Q, \quad \forall i \in N^+, \forall k \in V, t=1, \dots, T \quad (10)$$

$$I_{i,t} + EI_{i,t} \leq C_i, \quad \forall i \in N, t=1, \dots, T \quad (11)$$

$$Tr_{k,i,t} + z_{k,i,j,t}(st_i + t_{i,j}) - Tr_{k,j,t} \leq (1 - z_{k,i,j,t})T^*, \quad \forall (i,j) \in A, \forall k \in V, t=1, \dots, T \quad (12)$$

$$e_i \left(\sum_{j \in N^+} z_{k,i,j,t} \right) \leq Tr_{k,i,t} \leq l_i \left(\sum_{j \in N^+} z_{k,i,j,t} \right), \quad \forall i \in N^+, \forall k \in V, t=1, \dots, T \quad (13)$$

$$E \leq Tr_{k,i,t} \leq L, \quad \forall i \in \{0, n+1\}, \forall k \in V, t=1, \dots, T \quad (14)$$

$$Y_{k,i,(n+1),t} = 0, \quad \forall i \in N, \forall k \in V, t=1, \dots, T \quad (15)$$

$$P_{k,(0),j,t} = 0, \quad \forall j \in N, \forall k \in V, t=1, \dots, T \quad (16)$$

$$Y_{k,i,j,t} \geq 0, P_{k,i,j,t} \geq 0, I_{i,t} \geq 0, EI_{i,t} \geq 0, \quad \forall i, j \in N, \quad \forall k \in V, t=1, \dots, T \quad (17)$$

$$z_{k,i,j,t} \in \{0, 1\}, Y_{k,i,j,t}, P_{k,i,j,t}, I_{i,t}, EI_{i,t} \in \mathbb{Z}, \quad \forall (i,j) \in A, \quad \forall k \in V, t=1, \dots, T \quad (18)$$

The objective function (1) minimizes the inventory and the transportation costs. Constraints set (2) states that a vehicle will visit a location no more than once in a time period, and constraint set (3) ensures route continuity. Constraints (4) and (6) ensure that the number of tanks transported between two locations will always be zero whenever there is no vehicle moving between these locations, and that the amount transported is less than or equal to the vehicle's capacity. Constraint sets (5) and (7) along with the other elements of the model ensure that efficient solutions will not contain sub-tours. Constraint sets (8) and (9) are the balances of full and empty tanks at customer i at the end of period t . According to the capacity constraint set (10), when a vehicle leaves a customer, the total load must not exceed the capacity Q . Constraint set (11) ensures that the amount of tanks (empty and full) at a customer cannot exceed its' capacity to hold inventory C_i . Constraint set (12) forces continuity for the travel times and the times that vehicles reach the customers. The constant value T^* must be at least equal to $(L - E)$, otherwise feasible solutions could be potentially cut off. Constraint set (13) restrict the times at which vehicles reach the customers within their time windows. Constraint set (14) forbids vehicles to leave from or return at the depot in times outside the depot's operating time window. According to constraint sets (15) and (16), the vehicles must return to the depot empty of full tanks, and must leave the depot without empty tanks. Constraints (17) and (18) are the variable positivity and integrity constraints.

3. The Economic Model Predictive Inventory Control Strategy

In this work, the MPC strategy described in (Camacho and Bordons 1999) is adjusted for solving the inventory routing and control problem in the face of unknown future demand and the presence of time window constraints. According to the approach, the future outputs for the prediction horizon Np , are predicted at each instant t using the process model (2 - 18). The set of future control signals is calculated by optimizing an objective function, which balances the economic cost criterion of Eq(1) and a set-point tracking criterion, which is a quadratic function. If the control horizon is decided to be $Nc = 1$, the control signal $u(t|t)$ (deliveries) is applied by the distribution system for only one period ahead, whilst the future calculated deliveries are rejected, because at the next sampling instant the output of the system (i.e. the demand and the inventories one period ahead) $y(t+1)$ are already known and the procedure repeats itself with this new value and all the sequences are brought up to date. Thus $u(t+1|t+1)$ is calculated using the receding horizon concept.

The basic structure of the adopted strategy is presented in Figure 1. The desired trajectories of safety inventory levels are compared to the predicted outputs of the method and the errors are fed to the optimization model. The optimization model takes into account:

- The future customers' demands predicted by the forecasting model,
- The capacity and routing constraints and
- The cost function, where the future safety inventory tracking and the economic cost error are balanced.

The future deliveries plan is produced by the optimization model (Eqs(2) - (18) and (20) - (23)) and used as future input signals for model (2 - 18) which predicts the future inventory levels. The procedure iterates to cover the whole planning period, by using the control rule of applying to the real system only the first input result for the deliveries: $u(t)$ at each iteration and using the receding horizon concept to update forecasts and decision variables.

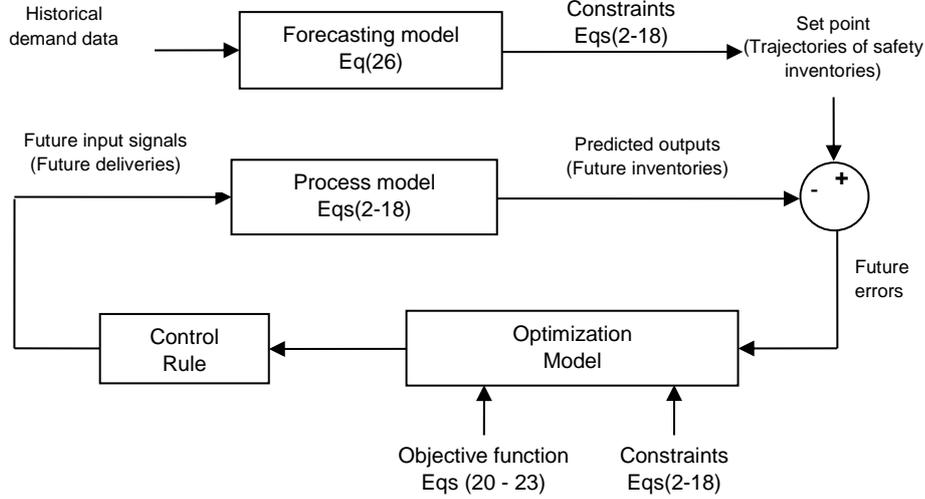


Figure 1: Structure of the Model Predictive strategy for Inventory Routing and Control

We use a state space representation of the system:

$$x^+ = Ax + Bu - B_d d \quad (19)$$

in which $x \in \mathbb{Z}^N$ are the system states i.e. the customer inventories $[I_1 I_2 \dots I_N]^T$; ($x_{i,t} \equiv I_{i,t}$). The manipulated input is $u \in \mathbb{Z}^N$, i.e. the quantities to be delivered $[u_1 u_2 \dots u_N]^T$, where $u_i = \sum_{k=1}^V \left(\sum_{l=0}^N Y_{k,l,i} - \sum_{m=0}^N Y_{k,i,m} \right)$. The

disturbance to the system is the customers' demand $d \in \mathbb{R}^N$, or $[d_1 d_2 \dots d_N]^T$. The outputs of the system are the states $y \equiv x$.

3.1 The EMPC model

For the nominal disturbance (demand) d_s , assuming stability of the system (Amrit and Rawlings, 2011) (A is the identity matrix), we adopt the model of the EMPC problem (20).

$$\left. \begin{aligned} \min_u \quad & \sum_{t=0}^{Np-1} [ag^c(x(t)) + (1-a)g^r(x(t))] \\ \text{s.t.} \quad & x(t+1) = Ax(t) + Bu(t) - B_d d(t), \quad t \in \{0, \dots, Np-1\} \end{aligned} \right\} \quad (20)$$

The transportation and inventory cost are formalized by:

$$g^c(x(t)) = \sum_{k \in V} \sum_{(i,j) \in A} c_{i,j} z_{k,i,j,t} + \sum_{i \in N} h_i x_{i,t} \quad (21)$$

The safety inventory tracking cost is:

$$g^r(x(t)) = \frac{1}{2} (x_t - x_{t,s})^T P (x_t - x_{t,s}) \quad (22)$$

where matrix P introduces the cost of tracking the safety stock x_s . A , B and B_d are identity matrices, and $z_{k,i,j,t}$, $x(t)$ and $u(t)$ derive from Eq(8), with respect to Eqs(2-7) and (9-18) for period $t = 1, \dots, Np$. Coefficients a and $(1-a)$, where $a \in [0, 1]$, are relative weightings assigned to the economic and the tracking costs respectively. Linearization of the quadratic term of the objective function

Since P is diagonal, the non-linear term $\frac{1}{2} (x_t - x_{t,s})^T P (x_t - x_{t,s})$, can be replaced by

$$\frac{1}{2} \sum_{i=1}^N \left[\text{diag}(P)_i (x_{i,t}^2 - 2x_{i,t}x_{i,j,s} + x_{i,j,s}^2) \right].$$

Then by substituting $w_{1,t} = x_t^2$ $w_{2,t} = x_t$ Eq(20) becomes:

$$g^{tr}(x(t)) = \frac{1}{2} \sum_{j=1}^N \left[\text{diag}(P_{j,t}) (w_{1,j,t} - 2w_{2,j,t}x_{j,s} + x_{j,s}^2) \right]$$

also let $w_{1,t} = y_{1,t} + 4y_{2,t} + \dots + Q^2 y_{Q,t}$ and $w_{2,t} = y_{1,t} + 2y_{2,t} + \dots + Qy_{Q,t}$ ($x_t \leq Q$)

$$\sum_{k=1}^Q y_{k,t} \leq 1, y_{k,t} \in \{0,1\}$$
(23)

Model (20) with the linearization constraints of (23) is a mixed integer linear model, which can be solved to optimality within reasonable computational times for moderate instances.

4. Illustrative Example

The methodology is illustrated using an example that involves 1 supplier, 3 customers, and 3 vehicles. The vehicles' capacity is $Q = 20$. The initial inventories of full and empty tanks are $l_0 = [2 \ 15 \ 20]$ and $P_0 = [11 \ 5 \ 2]$ respectively and the unit inventory holding cost is $h = 1$. The planning horizon spans 12 time periods. The unit transportation costs of node pairs are: $c_{0,1} = 15$, $c_{0,2} = 18$, $c_{0,3} = 22$, $c_{1,2} = 32$, $c_{1,3} = 14$, $c_{2,3} = 34$. The travelling times between node pairs are: $t_{0,1} = 1.8$, $t_{0,2} = 2.1$, $t_{0,3} = 2.6$, $t_{1,2} = 3.8$, $t_{1,3} = 1.6$, $t_{2,3} = 4$. Both cost and travelling time tables are symmetric. The average service time at each customer is set to $st = 0.5$. The earliest and latest times at which the service can begin at each customer are: $[e_i, l_i] = [6, 8]$ for $i = 1, 2$ and $[6, 9]$ for $i = 3$. The time window for the depot is $[e_0, l_0] = [e_4, l_4] = [5, 14]$; no vehicle can start before 5 and return later than 14 time units (hours in a day). The targeting cost of the safety stock is a diagonal matrix P , where $P_{i,j} = 2$ for $i = j$.

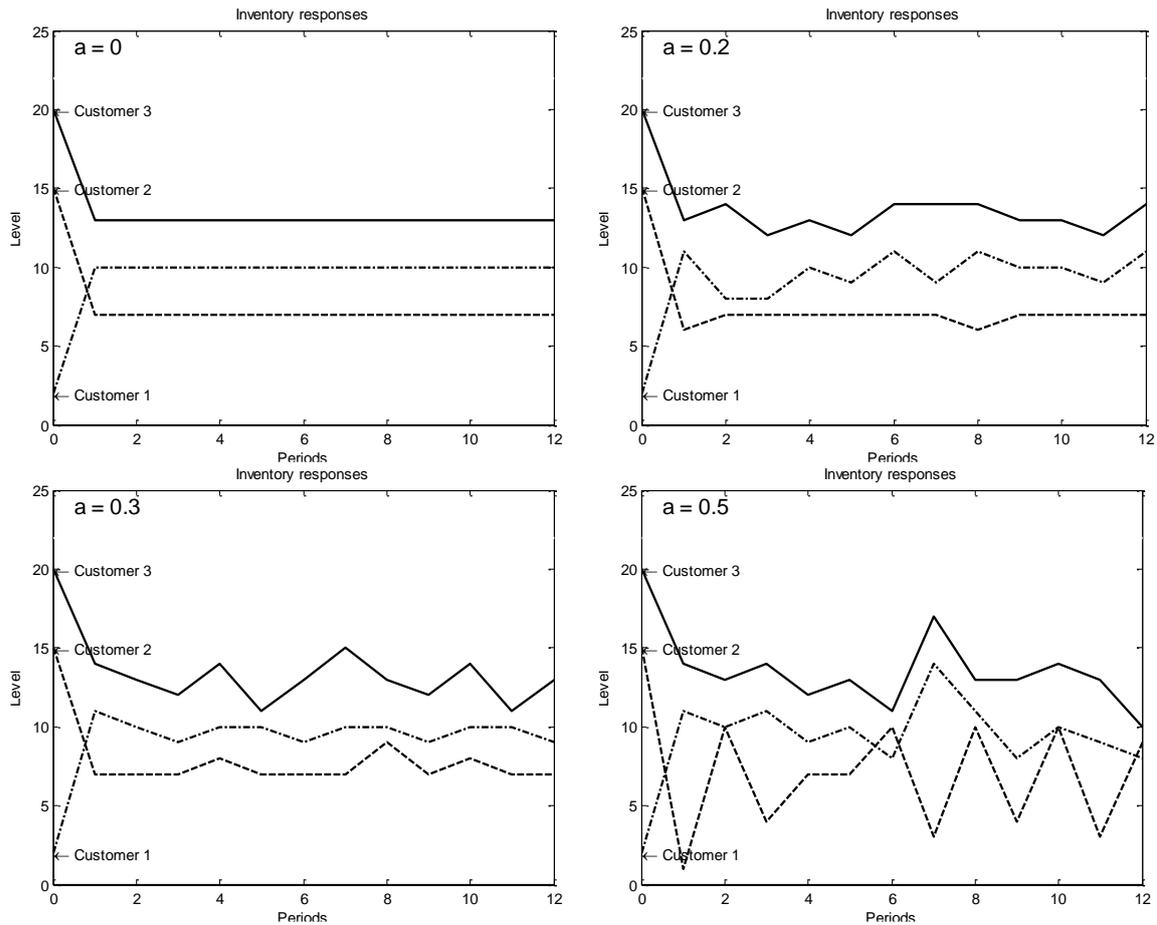


Figure 2: Inventory responses for $a = 0, 0.2, 0.3$ and 0.5

The following demand forecast is adopted for predicting the future demands as disturbances of the dynamical system: perfect demand information is assumed for one period ahead, thus $Nc = 1$, ($Np = 12$). For the

remainder of the horizon, the demand forecast is set equal to the nominal demand $d_s = [10 \ 7 \ 13]$. It is assumed that the customers actual demand, reported in Table 1, is normally distributed around the nominal demand with variance $\sigma^2 = 0.5$. The targeted safety stock is the solution of $x(t+1) = Ax(t) + Bu(t) - B_d d(t)$ for one period delivery delay: $x_s = [10 \ 7 \ 13]$. Figure 2, presents the response of the inventories with α varying from 0 (i.e. strategy for maximum shortage risk aversion) to $\alpha = 0.5$ (i.e. opt for the economic criterion). In the first case, the results show that the safety stock levels are reached promptly and maintained throughout the whole planning horizon. For $\alpha = 0.2$, the response of the closed-loop system, immediately settles to the safety steady state. For increasing values of α (0.3 and 0.5) the deviation of the inventory levels from the safety trajectory grows larger. This translates into reduced transportation and inventory holding costs.

Table 3 reports the distribution of costs when α varies within $[0, 1]$, where tradeoffs can be observed between the cost of tracking the safety inventories and the total, transportation and inventory costs. It is shown that balanced decisions can be made based on the change of a single parameter α ; additionally the desired inventory tracking can be matched with the best possible routing schedules. The tracking cost grows exponentially with α , while the cost reduction does not follow the same pattern. For values of α near 1 the supply chain is exposed to excessive stock-out risk without equivalent economic rewards.

Table 1: Cost distribution with varying α

α	0	0.2	0.3	0.4	0.5	0.8	0.95	1
Total cost	1,680	1,562	1,542	1,449	1,333	1,199	1,080	1,045
Transp. cost	1,320	1,205	1,182	1,100	979	906	878	855
Inv. cost	360	357	360	349	354	293	202	190
Track. cost	0	25	28	93	192	481	1,654	1,902

5. Conclusions

The paper presents a linear deterministic model for the multi-period IRPSPDTW and a dynamical predictive inventory control strategy for the industrial gas distribution business. It is shown that by using the proposed economic MPC strategy, the supply chain can be stabilized. The method offers a useful tool for producing balanced decisions between economic costs and tracking of the safety inventory while the produced schedules are always feasible and with optimal routing. It is also shown that balanced decisions can be made based on the results of the EMPC model by manipulating a single parameter. However, the supply chain is exposed to excessive stock-out risk without equivalent economic rewards for extreme values of this parameter. Whichever is the case, the model produces operationally feasible plans at minimum possible transportation cost. Future research may consider the development of custom algorithms for the solution of problem instances with higher cardinality that will be required since the IRP is a problem known to be NP-hard.

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