



# Quadratically Constrained Quadratic Optimal Problem in Robust Relay Networks

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The paper investigates the problem of a distributed robust relay beamforming in one-way relay networks with imperfect channel state information. The aim is to maximize the SNR (signal-to-noise ratio) of the receiver with the constraint of the total transmits power of relays is less than a certain predefined threshold. Because the channel is imperfect and the objective function is fractional quadratic form, the optimal problem is a non-convex problem which is hard to be solved. We first change the fractional problem into a non-fractional one. Second, we consider the semi-definite programming (SDP) method efficiently by exploiting S-procedure theorem and relaxing the rank-one constrain to obtain the robust beamforming vector. The paper gives the schematic total algorithm too. Numerical results show that the proposed algorithm can solve the maximum problem with robust features efficiently and obtain satisfactory results.

## 1. Introduction

In this paper, we consider the problem of minimizing a fractional uncertainty quadratic function in the complex field as follows:

$$\min \frac{f_1(x)}{f_2(x)} \quad (1)$$

$$s.t. g(x) \leq 1.$$

Where  $f_i(x) = x^H A_i x + c_i, i = 1, 2, c_i \in R, A_i = \{A_i^0 + \sum_{j=1}^k u_j A_i^j | uu^H < 1\}, A_i^0, A_i^j \in \mathbb{C}^{n \times n}$  are complex and

Hermitian matrix.  $g(x) = \|Bx\|^2, B \in \mathbb{C}^{m \times n}, B = \{B^0 + \sum_{j=1}^k u_j B^j | uu^H < 1\}$ , the vectors  $u = (u_1, u_2, \dots, u_k)^H$ . Obviously, the disturbance vector  $u$  change in a certain set  $U = \{u \in \mathbb{C}^k | u^H u \leq 1\}$ ; and the disturbance matrix  $B = B^0 + \sum_{j=1}^k u_j B^j$  is affine for  $u$ .

The superscript " $H$ " denotes the conjugate transpose. Furthermore, we require that the denominator of the objective function is nonzero, that is  $|f_2(x)| > N$ .  $F^0$  is feasible region of problem (1).

Levinson studied first the complex optimization problem of linear optimization in complex space in 1966 (Levinson, 1966). The linear fractional optimization in complex space was discussed by Swarup and Sharma in 1970 (Swarup and Sharma, 1970). Later, more and more authors focus to the complex fractional optimization problems due to the wide range of applications in different areas including signal processing, wireless communications and so on (Ubaidulla et.al, 2007, Safavi et.al, 2012).

## 2. A parametric schematic algorithm

Because the object function of optimization problem (1) includes the disturbance vector  $u$ , which make the problem unsolvable. By introducing an auxiliary variable  $t$ , problem (1) can be translated into the following form.

$$\begin{aligned} \max_{x,t} \quad & t \\ \text{s.t.} \quad & x^H [C^0 + \sum_{j=1}^k u_j C^j] x \geq 1 - c_1 + t c_2, \\ & \|(B^0 + \sum_{j=1}^k u_j B^j) x\|^2 \leq 1, \\ & u u^H < 1. \end{aligned} \quad (2)$$

Where,  $C^j = t A_2^j - A_1^j$ ,  $j = 0, 1, \dots, k$ . Let  $F(x) = (B^0 x, B^1 x, \dots, B^k x)$ , so

$$\|(B^0 + \sum_{j=1}^k u_j B^j) x\|^2 = \begin{bmatrix} 1 \\ u \end{bmatrix}^H F(x)^H F(x) \begin{bmatrix} 1 \\ u \end{bmatrix} \quad u u^H \leq 1 \Leftrightarrow \begin{bmatrix} 1 & 0 \\ u & -I \end{bmatrix}^H \begin{bmatrix} 1 \\ u \end{bmatrix} \geq 0$$

$$x^H [C^0 + \sum_{j=1}^k u_j C^j] x = \begin{bmatrix} 1 \\ u \end{bmatrix}^H (x^H C^0 x, x^H C^1 x, \dots, x^H C^k x)$$

Let  $d(x) = (x^H C^0 x, x^H C^1 x, \dots, x^H C^k x)$ , we have

$$x^H [C^0 + \sum_{j=1}^k u_j C^j] x \geq 1 - c_1 + t c_2 \Leftrightarrow \begin{bmatrix} 1 \\ u \end{bmatrix}^H d(x) d(x)^H \begin{bmatrix} 1 \\ u \end{bmatrix} \geq (1 - c_1 + t c_2)^2$$

Problem (2) can be rewritten as

$$\begin{aligned} \max_{x,t} \quad & t \\ \text{s.t.} \quad & \begin{bmatrix} 1 \\ u \end{bmatrix}^H (d(x) d(x)^H - \begin{bmatrix} (1 - c_1 + t c_2)^2 & 0 \\ 0 & 0 \end{bmatrix}) \begin{bmatrix} 1 \\ u \end{bmatrix} \geq 0, \\ & \begin{bmatrix} 1 \\ u \end{bmatrix}^H \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - F(x)^H F(x) \right) \begin{bmatrix} 1 \\ u \end{bmatrix} \geq 0, \\ & \begin{bmatrix} 1 \\ u \end{bmatrix}^H \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix} \geq 0. \end{aligned} \quad (3)$$

To proceed, we will introduce a theorem S-procedure which can translate (3) into a certain SDP problem.

**Lemma 1** Let  $P$  and  $Q$  are both symmetric matrix, there is a vector  $u_0$  which satisfies  $(u_0)^T P(u_0) > 0$ , then “ $u^T P u \geq 0 \Rightarrow u^T Q u \geq 0$ ” iff “ $\exists \lambda \geq 0$ , s.t.  $Q \succeq \lambda P$ ”. (Ye, 2004; Boyd and Vandenberghe, 2004; Zhang et.al, 2011)

We can translate the constraint conditions of the problem (3) into the semi-definite cone optimization problem.

In fact, we let:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - F(x)^H F(x)$$

then the last two constraint conditions of problem (3) can be explained as:

$$\begin{bmatrix} 1 \\ u \end{bmatrix}^H P \begin{bmatrix} 1 \\ u \end{bmatrix} \geq 0 \Rightarrow \begin{bmatrix} 1 \\ u \end{bmatrix}^H Q \begin{bmatrix} 1 \\ u \end{bmatrix} \geq 0$$

Let  $u^* = (1, 0, \dots, 0)^T \in \mathbb{C}^{k+1}$ , then  $u^{*H} P u^* = 1 > 0$ , by the lemma 1 we know:  $\exists \lambda_1 \geq 0$ , such that

$Q \succeq \lambda_1 P$ , namely

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - F(x)^H F(x) \succeq \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 - \lambda_1 & 0 \\ 0 & \lambda_1 I \end{bmatrix} - F(x)^H F(x) \succeq 0, \lambda_1 > 0$$

In the same way, we have:  $\exists \lambda_2 \geq 0$ , such that

$$d(x)d(x)^H - \begin{bmatrix} (1-c_1+tc_2)^2 & 0 \\ 0 & 0 \end{bmatrix} \succeq \lambda_2 \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix} \Leftrightarrow d(x)d(x)^H - \begin{bmatrix} (1-c_1+tc_2)^2 + \lambda_2 & 0 \\ 0 & -\lambda_2 I \end{bmatrix} \succeq 0$$

For given  $t$ , problem (3) translates into a certain SDP problem as follow:

$$\begin{aligned} & \max_{x,t} \quad t \\ & \text{s.t.} \quad d(x)d(x)^H - \begin{bmatrix} (1-c_1+tc_2)^2 + \lambda_2 & 0 \\ 0 & -\lambda_2 I \end{bmatrix} \succeq 0, \\ & \quad \begin{bmatrix} 1-\lambda_1 & 0 \\ 0 & \lambda_1 I \end{bmatrix} - F(x)^H F(x) \succeq 0, \\ & \quad \lambda_1, \lambda_2 > 0. \end{aligned} \quad (4)$$

The major motivation for the problem (1) in this paper is from a practical issue in the communication system. Today, communications over wireless channels continue to be major challenges in technologies. More and more scholars study the collaborative use of amplify and forward (AF) or decode and forward (DF) protocols. Now most of the research works is based on the known ideal channel state information (CSI) which is hard to obtain in practice, however, non-ideal CSI will affect the performance of the network seriously [8]. So more and more scholars are interested in the robust beamforming algorithms in the non-cognitive or cognitive relay network and have published many papers about this topic (Cai and Wang, 2014; Veria et.al, 2008).

### 3. Robust distributed beamforming problem

We study a wireless network which includes two parts as Figure 1: a transmitter, a receiver, and  $N$  relay nodes. All nodes in network have been configured with one antenna. We assume that there is no direct link between the sources and the destinations, the help of relays is necessary to establish the communication link. Here we use two-step amplify-and-forward (AF) protocol. Assuming a flat fading scenario, denote  $f_i$  the channel coefficient from the transmitter to the  $i$ th relay and  $g_i$  represent the channel coefficient from the  $i$ th relay to the receiver.

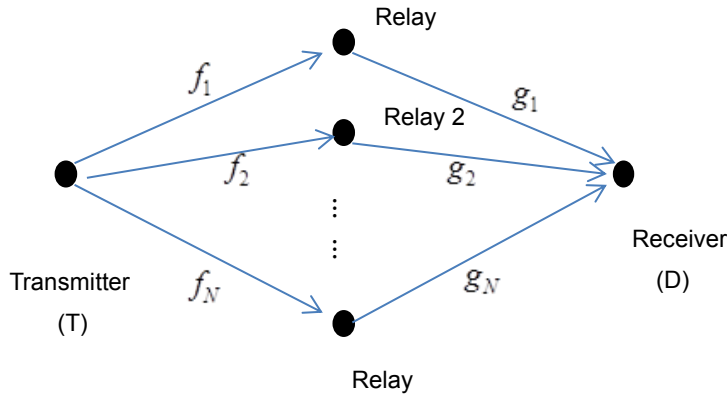


Figure 1: System model

During the MAC, transmitter T transmits message  $s$  to relays with the power  $P_0$ . Then the received signal at  $i$ th relay is given as follows:

$$r_i = \sqrt{P_0} f_i s + n_{R_i} \quad (5)$$

where,  $n_{R_i} \sim N(0, N_{R_i})$  represents the complex Gaussian noise at the  $i$ th relay.

During the BC, the retransmit signal  $y_i$  of the  $i$ th relay is

$$y_i = w_i r_i \quad (6)$$

Where,  $w_i \in \mathbb{C}$  is the complex beamforming weight. Then the received signals at receiver D can be written as:

$$y_s = \sum_{i=1}^N g_i y_i + n_d \quad (7)$$

where,  $n_d \sim N(0, N_d)$  represent the complex Gaussian noise at D. Substituting (5) and (6) into (7), we can obtain (8):

$$y_s = \sum_{i=1}^N w_i g_i (\sqrt{P_0} f_i s + n_{R_i}) + n_d = \underbrace{\sum_{i=1}^N \sqrt{P_0} w_i f_i g_i s}_{\text{signal}, s_T} + \underbrace{\sum_{i=1}^N w_i g_i n_{R_i} + n_d}_{\text{noise}, n_T} \quad (8)$$

$S_D$ ,  $N_D$  are the received signal power and the noise power of D respectively, so we have

$$S_D = E\{|s_T|^2\} = E\left\{\left|\sum_{i=1}^N \sqrt{P_0} w_i f_i g_i s\right|^2\right\} = P_0 g^H (w \odot f)(w \odot f)^H g$$

$$N_D = E\{|n_T|^2\} = E\left\{\left|\sum_{i=1}^N w_i g_i n_{R_i} + n_d\right|^2\right\} = g^H (w \odot n_R)(w \odot n_R)^H g + N_d$$

where,

Let  $w = (w_1, w_2, \dots, w_N)^H$ ,  $g = (g_1, g_2, \dots, g_N)$ ,  $f = (f_1, f_2, \dots, f_N)$ ,  $n_R = (n_{R_1}, \dots, n_{R_N})$ .  $\odot$  denotes the Hadamard product.

The total transmitted power of relays is:

$$P_T = \sum_{i=1}^N E\{|y_i|^2\} = \sum_{i=1}^N |w_i|^2 E\{|r_i|^2\} = w^H D w \quad (9)$$

Where  $D = \text{diag}\{P_0|f_1|^2, P_0|f_2|^2, \dots, P_0|f_N|^2\} + N_R \mathbf{I}$ .

In this paper, we assume the central processor is placed the inside of relays or the near of relays for searching CSI vector  $f_i, i = 1 \dots N$ . Because there is quantization error and feedback time-delay error in the actual transmission, ideal CSI vector  $g_i, i = 1, \dots, N$  can't be obtained, CSI error model may be expressed as:

$g = \bar{g} + \Delta g$ ,  $\bar{g}$  denotes the observed channel vector and the error vector  $\Delta g$  is bounded, given by  $\Delta g \in \{a \in \mathbb{C}^N \mid \|a\|_2 \leq \rho\}$ .

The SNR of SD is defined as  $SINR = \frac{S_D}{N_D}$ .

In the section, our aim is to obtain the optimal robust beamforming weight coefficient and maximize the SNR of D.

So the optimal model can write as following:

$$\max \frac{P_0 g^H (w \odot f)(w \odot f)^H g}{g^H (w \odot n_R)(w \odot n_R)^H g + N_d} \quad (10)$$

$$s.t. \quad w^H D w \leq P,$$

$$\Delta g^H \Delta g \leq \rho.$$

Because of  $(a \odot b)(a \odot b)^H = aa^H \odot bb^H$  and so the optimization (10) can be rewritten as:

$$\begin{aligned}
& \max \frac{P_0 g^H (ff^H \odot ww^H) g}{g^H (n_R n_R^H \odot ww^H) g + N_d} \\
& \text{s.t. } w^H D w \leq P, \\
& \quad \Delta g^H \Delta g \leq \rho.
\end{aligned} \tag{11}$$

Let  $W = ww^H$ , so  $\text{rank}(W) = 1$ , and rewrite the optimization problem(11):

$$\begin{aligned}
& \max \frac{P_0 g^H (ff^H \odot W) g}{g^H (n_R n_R^H \odot W) g + N_d} \\
& \text{s.t. } D \bullet W \leq P, \\
& \quad \Delta g^H \Delta g \leq \rho, \\
& \quad \text{Rank}(W) = 1.
\end{aligned} \tag{12}$$

Due to the rank-one constraint, the optimization problem (12) is hard to be solved. Therefore we resort to relaxing (13) by deleting the rank-one constraint  $\text{Rank}(W) = 1$ , namely

$$\begin{aligned}
& \max \frac{P_0 g^H (ff^H \odot W) g}{g^H (n_R n_R^H \odot W) g + N_d} \\
& \text{s.t. } D \bullet W \leq P, \\
& \quad \Delta g^H \Delta g \leq \rho,
\end{aligned} \tag{13}$$

By using the method that introduced in the second section, we can obtain optimal solution  $W^*$  of the optimization (13) by MATLAB. If the optimal solution  $W^*$  of (13) is rank-one, the optimal solution  $w^*$  of (12) can be obtained by using eigenvalue decomposition. But if  $W^*$  is not rank-one, we can use randomization and eigenvector approximation method to obtain the optimum  $w^*$  of (12). (Huang and Zhang, 2010)

#### 4. Numerical solution and analysis

In this section, set  $P_0 = 10\text{dB}$ , relay number  $N = 5$ . The channel coefficient Vectors  $f$ ,  $g$  and the CSI error  $\Delta g$  can be generated following complex Gaussian distribution  $CN(0, I)$ . The complex Gaussian noise power of all nodes are 1, that is  $N_R, N_D, N_d = 1$ .

In Figure 2, we illustrate the achieved SNR of D as the total transmit power of relays  $P$ , consider the channel error bounds  $\rho = 0.1$ ,  $\rho = 0.7$  and the non-robust case. It is observed that SNR of D increase obviously with the total relay power enlarging no matter whether  $\rho = 0.1$ ,  $\rho = 0.7$  or the non-robust case. It is obvious that the proposed robust algorithm outperforms the non-robust scheme.

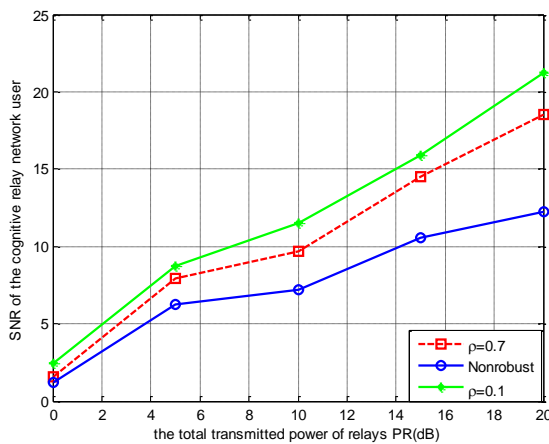


Figure 2: Output SNR maximum versus total relay power

## 5. Conclusions

In this paper, we proposed the robust relay beamforming for RN using semi-definite programming method for maximizing the SNR of D. The optimization problem is non-convex because of the channel information imperfect. We introduce a robust algorithm for this uncertain problem by exploiting S-procedure and relaxing the rank-one constrain. Numerical result shows that the proposed algorithm can solve the maximum problem with robust features efficiently and obtain satisfactory results too. This paper consider the channel uncertainty at the second process only, then future work will focus on channel errors in the two transmitted phases.

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