CUBO A Mathematical Journal
Vol.19, $N^{\underline{o}}$01, (79–87). March 2017

Three dimensional f-Kenmotsu manifold satisfying certain curvature conditions

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ABSTRACT

The purpose of the present paper is to study pseudosymmetry conditions on f-Kenmotsu manifolds.

RESUMEN

El propósito del presente artículo es estudiar condiciones de pseudosimetría en variedades f-Kenmotsu.

Keywords and Phrases: f-Kenmotsu manifold, cyclic parallel Ricci tensor, almost pseudo Ricci symmetry, pseudosymmetry, Ricci pseudosymmetry, Ricci generalized pseudosymmetry.

2010 AMS Mathematics Subject Classification: 47A63; 47A99.

1 Introduction

Let M^n be an almost contact manifold with an almost contact metric structure (ϕ, ξ, η, g) [1]. We denote by Φ, the fundamental 2-form of M^n i.e., $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields $X, Y \in \chi(M^n)$, where $\chi(M^n)$ being the Lie algebra of differentiable vector fields on M^n. Furthermore, we recollect the following definitions [1, 3, 8].
The manifold M^n and its structure (ϕ, ξ, η, g) is said to be:

i) normal if the almost complex structure defined on the product manifold $M^n \times R$ is integrable (equivalently, $[\phi, \phi] + 2d\eta \otimes \xi = 0$),

ii) almost cosymplectic if $d\eta = 0$ and $d\Phi = 0$,

iii) cosymplectic if it is normal and almost cosymplectic (equivalently, $\nabla\phi = 0$, where ∇ is covariant differentiation with respect to the Levi-Civita connection).

The manifold M^n is called locally conformal almost cosymplectic (respectively, locally conformal cosymplectic) if M^n has an open covering $\{U_t\}$ endowed with differentiable functions $\sigma_t : U_i \longrightarrow R$ such that over each U_t the almost contact metric structure $(\phi_t, \xi_t, \eta_t, g_t)$ defined by

$$\phi_t = \phi, \quad \xi_t = e^{\sigma_t}\xi, \quad \eta_t = e^{-\sigma_t}\eta, \quad g_t = e^{-2\sigma_t}g$$

is almost cosymplectic (respectively, locally conformal cosymplectic).

Normal locally conformal almost cosymplectic manifold were studied by Olszak and Rosca [7]. An almost contact metric manifold is said to be f-Kenmotsu if it is normal and locally conformal almost cosymplectic. The same type of manifold was also studied by Yildiz et al. [9] using the projective curvature tensor. Olszak and Rosca [7] also gave a geometric interpretation of f-Kenmotsu manifolds and studied some curvature restrictions. Among others, they proved that a Ricci symmetric f-Kenmotsu manifold is an Einstein manifold.

Our work is structured in the following way: After introduction, we have given some basic equations of f-Kenmotsu manifold in section 2. Section 3 deals with the study of 3-dimensional f-Kenmotsu manifold with cyclic parallel Ricci tensor. And we study almost pseudo Ricci symmetric, pseudosymmetric, Ricci pseudosymmetric and Ricci generalized pseudosymmetric 3-dimensional f-Kenmotsu manifolds in sections 4, 5, 6 and 7, respectively.

2 f-Kenmotsu manifolds

Let M^n be a smooth $(2n+1)$-dimensional manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) which satisfy

$$\phi^2 = -id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \cdot \phi = 0, \tag{2.1}$$

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any vector fields $X, Y \in \chi(M^n)$ where id is the identity of the tangent bundle TM^n, ϕ is a tensor field of type $(1, 1)$, ξ is a vector field, η is a 1-form and g is a Riemannian metric.

We say that $(M^n, \phi, \xi, \eta, g)$ is an f-Kenmotsu manifold if the Levi-Civita connection ∇ of ϕ satisfies the condition [6]

$$(\nabla_X \phi)(Y) = f[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.3}$$

where $f \in C^\infty(M^n)$ is strictly positive and $df \wedge \eta = 0$. If $f = 0$, then the manifold is cosymplectic [5]. An f-Kenmotsu manifold is called regular if $f^2 + f' \neq 0$ where $f' = \xi f$.
In an f-Kenmotsu manifold, from (2.3) we have

$$\nabla_X \xi = f[X - \eta(X)\xi]. \tag{2.4}$$

The condition $df \wedge \eta = 0$ holds if $\dim M^n \geq 5$ but it does not hold if $\dim M^n = 3$ [7].

$$(\nabla_X \eta)(Y) = f[g(X, Y) - \eta(X)\eta(Y)]. \tag{2.5}$$

In a 3-dimensional Riemannian manifold, we have

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y \\ &\quad -\frac{r}{2}\{g(Y, Z)X - g(X, Z)Y\}. \end{aligned} \tag{2.6}$$

In a 3-dimensional f-Kenmotsu manifold, we see that [7]

$$\begin{aligned} R(X, Y)Z &= (\frac{r}{2} + 2f^2 + 2f')(X \wedge Y)Z - (\frac{r}{2} + 3f^2 + 3f')\{\eta(X)(\xi \wedge Y)Z \\ &\quad + \eta(Y)(X \wedge \xi)Z\}, \end{aligned} \tag{2.7}$$

$$S(X, Y) = (\frac{r}{2} + f^2 + f')g(X, Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y), \tag{2.8}$$

where R, S, Q and r are the Riemannian curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature, respectively.
Now from(2.7), we have the following:

$$R(X, Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y], \tag{2.9}$$

$$R(\xi, Y)Z = -(f^2 + f')[g(Y, Z)\xi - \eta(Z)Y], \tag{2.10}$$

$$\eta(R(X, Y)Z) = -(f^2 + f')[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \tag{2.11}$$

And from (2.8), we get

$$S(X, \xi) = -2(f^2 + f')\eta(X), \tag{2.12}$$

and

$$Q\xi = -2(f^2 + f')\xi. \tag{2.13}$$

3 3-dimensional f-Kenmotsu manifold with cyclic parallel Ricci tensor

Suppose the manifold M^n under consideration satisfies the cyclic parallel Ricci tensor condition [4]. Then we have

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0, \tag{3.1}$$

for all $X, Y, Z \in \chi(M^n)$.
From the above equation, it is seen that r is constant. And we have

$$\begin{aligned}(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) &= -(\frac{r}{2} + 3f^2 + 3f')[(\nabla_X \eta)(Y)\eta(Z) \\ &\quad +\eta(Y)(\nabla_X \eta)(Z) + (\nabla_Y \eta)(Z)\eta(X) \\ &\quad +\eta(Z)(\nabla_Y \eta)(X) + (\nabla_Z \eta)(X)\eta(Y) \\ &\quad +\eta(X)(\nabla_Z \eta)(Y)].\end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we get

$$\begin{aligned}&(\frac{r}{2} + 3f^2 + 3f')[(\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z) + (\nabla_Y \eta)(Z)\eta(X) \\ &+\eta(Z)(\nabla_Y \eta)(X) + (\nabla_Z \eta)(X)\eta(Y) + \eta(X)(\nabla_Z \eta)(Y)] = 0.\end{aligned} \tag{3.3}$$

Using (2.5) in (3.3), we get

$$\begin{aligned}&(\frac{r}{2} + 3f^2 + 3f')[g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + g(Y, Z)\eta(X) + g(Y, X)\eta(Z) \\ &+g(Z, X)\eta(Y) + g(Z, Y)\eta(X) - 6\eta(X)\eta(Y)\eta(Z)] = 0, \qquad \text{since } f \neq 0.\end{aligned} \tag{3.4}$$

On substituting $X = Y = e_i$ in (3.4), where e_i is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq 3$, which gives

$$4\{\frac{r}{2} + 3f^2 + 3f'\}\eta(Z) = 0. \tag{3.5}$$

Hence, we get $\eta(Z) = 0$, which is a contradiction. Therefore, from (3.5) we have

$$r = -6(f^2 + f'). \tag{3.6}$$

Conversely, if $r = -6(f^2 + f')$ then from (3.2), we obtain

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0. \tag{3.7}$$

From the above discussions we have the following:

Theorem 3.1. *A 3-dimensional f-Kenmotsu manifold satisfies cyclic parallel Ricci tensor if and only if the scalar curvature* $r = -6(f^2 + f')$, *provided* $f \neq 0$.

4 Almost pseudo Ricci symmetric 3-dimensional f-Kenmotsu manifold satisfying cyclic Ricci tensor

Chaki and Kawaguchi [2] introduced the concept of almost pseudo Ricci symmetric manifolds as an extended class of pseudo symmetric manifolds. A Riemannian manifold (M^n, g) is called an almost pseudo Ricci symmetric manifold $(APRS)_n$, if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the following condition

$$(\nabla_U S)(V,W) = [A(U) + B(U)]S(V,W) + A(V)S(U,W) + A(W)S(U,V), \tag{4.1}$$

where A and B are two non-zero 1-forms defined by

$$A(U) = g(U, P_1), B(U) = g(U, P_2). \tag{4.2}$$

By taking the cyclic sum of (4.1), we see that

$$\begin{aligned}(\nabla_U S)(V,W) + (\nabla_V S)(W,U) + (\nabla_W S)(U,V) &= [3A(U) + B(U)]S(V,W) \\ +[3A(V) + B(V)]S(U,W) + [3A(W) + B(W)]S(U,V).\end{aligned} \tag{4.3}$$

Let M^n admit a cyclic Ricci tensor, then (4.3) becomes

$$\begin{aligned}&[3A(U) + B(U)]S(V,W) + [3A(V) + B(V)]S(U,W) + \\ &[3A(W) + B(W)]S(U,V) = 0.\end{aligned} \tag{4.4}$$

Replacing W by ξ in the above equation and using (2.12) and (4.2), we get

$$\begin{aligned}&-\{2(f^2 + f')\}[3A(U) + B(U)]\eta(V) - \{2(f^2 + f')\}[3A(V) + B(V)]\eta(U) \\ &+[3\eta(P_1) + \eta(P_2)]S(U,V) = 0.\end{aligned} \tag{4.5}$$

In (4.5), substituting $V = \xi$ and using (2.12) and (4.2), we have

$$-\{2(f^2 + f')\}[3A(U) + B(U)] - 4\{2(f^2 + f')\}[3\eta(P_1) + \eta(P_2)]\eta(U) = 0. \tag{4.6}$$

Again treating U by ξ and using (4.2) in (4.6), we obtain

$$\{f^2 + f'\}[3\eta(P_1) + \eta(P_2)] = 0, \tag{4.7}$$

which implies

$$[3\eta(P_1) + \eta(P_2)] = 0, \tag{4.8}$$

since $\{f^2 + f'\} \neq 0$.
From (4.8) and (4.6), it follows that

$$3A(U) + B(U) = 0. \tag{4.9}$$

Thus, we can state:

Theorem 4.1. *There is no almost pseudo Ricci symmetric 3-dimensional f-Kenmotsu manifold admitting cyclic Ricci tensor, unless $3A + B$ vanishes everywhere.*

5 Pseudosymmetric 3-dimensional f-Kenmotsu manifold

Let M^n be an pseudosymmetric 3-dimensional f-Kenmotsu manifold. Then we have,

$$(R(X,Y)\cdot R)(U,V)W = f_R Q(g,R)(U,V,W;X,Y), \tag{5.1}$$

for all $X, Y, U, V, W \in \chi(M^n)$.
From the above relation it follows that

$$\begin{aligned} &R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W \\ &-R(U,V)R(X,Y)W = f_R[(X\wedge_g Y)R(U,V)W - R((X\wedge_g Y)U,V)W \\ &-R(U,(X\wedge_g Y)V)W - R(U,V)(X\wedge_g Y)W], \end{aligned} \tag{5.2}$$

where

$$(X\wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{5.3}$$

Substituting X by ξ and using (2.10) and (5.3), (5.2) yields

$$\begin{aligned} &[(f^2+f')+f_R]\{g(Y,R(U,V)W)\xi - \eta(R(U,V)W)Y - g(Y,U)R(\xi,V)W \\ &+\eta(U)R(Y,V)W - g(Y,V)R(U,\xi)W + \eta(V)R(U,Y)W - g(Y,W)R(U,V)\xi \\ &+\eta(W)R(U,V)Y\} = 0. \end{aligned} \tag{5.4}$$

Taking inner product of (5.4) with ξ, we get

$$\begin{aligned} &[(f^2+f')+f_R]\{R(U,V,W,Y) - \eta(Y)\eta(R(U,V)W) - g(Y,U)\eta(R(\xi,V)W) \\ &+\eta(U)\eta(R(Y,V)W) - g(Y,V)\eta(R(U,\xi)W) + \eta(V)\eta(R(U,Y)W) \\ &-g(Y,W)\eta(R(U,V)\xi) + \eta(W)\eta(R(U,V)Y)\} = 0. \end{aligned} \tag{5.5}$$

By using (2.11), (5.5) becomes

$$\begin{aligned} &[(f^2+f')+f_R]\{R(U,V,W,Y) - (f^2+f')[-g(V,W)\eta(Y)\eta(U) \\ &+g(U,W)\eta(Y)\eta(V) - g(Y,U)g(V,W) + g(Y,U)\eta(V)\eta(W) + g(V,W)\eta(U)\eta(Y) \\ &-g(Y,W)\eta(U)\eta(V) - g(Y,V)\eta(W)\eta(U) + g(Y,V)g(U,W) + g(Y,W)\eta(V)\eta(U) \\ &-g(U,W)\eta(Y)\eta(V) + g(V,Y)\eta(W)\eta(U) - g(U,Y)\eta(V)\eta(W)]\} = 0. \end{aligned} \tag{5.6}$$

Contracting the above equation, we obtain

$$[(f^2+f')+f_R]\{S(V,W) + 2(f^2+f')g(V,W)\} = 0. \tag{5.7}$$

The above equation can hold only if either

(i) $(f^2+f') = -f_R$, or

(ii) $S(V, W) = \alpha g(V, W)$, where $\alpha = -2(f^2 + f')$.

This leads to the following:

Theorem 5.1. *A 3-dimensional pseudosymmetric f-Kenmotsu manifold with never vanishing function* $\{(f^2 + f') = -f_R\}$ *is an Einstein manifold.*

6 Ricci pseudosymmetric 3-dimensional f-Kenmotsu manifold

Suppose (M^n, g) be a 3-dimensional Ricci pseudosymmetric f-Kenmotsu manifold. Then we have,

$$(R(X, Y) \cdot S)(U, V) = f_S Q(g, S)(U, V; X, Y), \tag{6.1}$$

for all $X, Y, U, V, W \in \chi(M^n)$. From the above relation it follows that

$$(R(X, Y) \cdot S)(U, V) = f_S((X \wedge_g Y) \cdot S)(U, V),$$

or

$$\begin{aligned} -S(R(X, Y)U, V) - S(U, R(X, Y)V) = f[&-g(Y, U)S(X, V) + g(X, U)S(Y, V) \\ &-g(Y, V)S(U, X) + g(X, V)S(U, Y)]. \end{aligned} \tag{6.2}$$

Replacing X and U by ξ and using (2.1), (2.10) and (2.12) in the above equation, we get

$$[(f^2 + f') + f_S]\{S(Y, V) + 2(f^2 + f')g(Y, V)\} = 0, \tag{6.3}$$

which follows that either $[(f^2 + f') + f_S] = 0$ or

$$S(Y, V) = \alpha g(Y, V), \tag{6.4}$$

where $\alpha = -2(f^2 + f')$.
Thus we can state:

Theorem 6.1. *If a 3-dimensional f-Kenmotsu manifold* M^n *is Ricci pseudosymmetric with restrictions* $X = U = \xi$*, then either* $[(f^2 + f') + f_S] = 0$ *or the manifold is an Einstein manifold.*

7 Ricci generalized pseudosymmetric 3-dimensional f-Kenmotsu manifold

Consider a Ricci generalized pseudosymmetric 3-dimensional f-Kenmotsu manifold. Then we have

$$(R(X, Y) \cdot R)(U, V)W = f((X \wedge_S Y) \cdot R)(U, V)W, \tag{7.1}$$

for all $X, Y, U, V, W \in \chi(M^n)$.
We can write the above form as

$$\begin{aligned} & R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W \\ & -R(U,V)R(X,Y)W = f[S(Y,R(U,V)W)X - S(X,R(U,V)W)Y \\ & -S(Y,U)R(X,V)W + S(X,U)R(Y,V)W - S(Y,V)R(U,X)W \\ & +S(X,V)R(U,Y)W - S(Y,W)R(U,V)X + S(X,W)R(U,V)Y]. \end{aligned} \tag{7.2}$$

On substituting $X = U = \xi$ and using (2.10) and (2.12), (7.2) reduces to

$$\begin{aligned} & -(f^2+f')[(f^2+f')g(V,W)Y + R(Y,V)W - (f^2+f')g(Y,W)V] \\ & = f[(f^2+f')S(Y,V)\eta(W)\xi - 2(f^2+f')^2 g(V,W)Y - 2(f^2+f')R(Y,V)W \\ & +2(f^2+f')^2 g(Y,W)\eta(V)\xi + (f^2+f')S(Y,W)\eta(V)\xi - (f^2+f')S(Y,W)V \\ & +2(f^2+f')^2 g(V,Y)\eta(W)\xi]. \end{aligned} \tag{7.3}$$

Taking inner product of the above equation with Z, we get

$$\begin{aligned} & -(f^2+f')[(f^2+f')g(V,W)g(Y,Z) + g(R(Y,V)W,Z) - (f^2+f')g(Y,W)g(V,Z)] \\ & = f[(f^2+f')S(Y,V)\eta(W)\eta(Z) - 2(f^2+f')^2 g(V,W)g(Y,Z) \\ & -2(f^2+f')g(R(Y,V)W,Z) + 2(f^2+f')^2 g(Y,W)\eta(V)\eta(Z) \\ & +(f^2+f')S(Y,W)\eta(V)\eta(Z) - (f^2+f')S(Y,W)g(V,Z) \\ & +2(f^2+f')^2 g(V,Y)\eta(W)\eta(Z)]. \end{aligned} \tag{7.4}$$

Contracting (7.4) and simplifying gives

$$(f^2+f')(3f-1)[S(Y,Z) + 2(f^2+f')g(Y,Z)] = 0, \tag{7.5}$$

which means that either $(f^2+f')(3f-1) = 0$ or $S(Y,Z) = \alpha g(Y,Z)$, where $\alpha = -2(f^2+f')$.
Hence we can state the following:

Theorem 7.1. *If a 3-dimensional f-Kenmotsu manifold is Ricci generalized pseudosymmetric then either*

(i) $(f^2+f')(3f-1) = 0$, *or*

(ii) it is an Einstein manifold.

Acknowledgement: The second author is thankful to UGC for financial support in the form of Rajiv Gandhi National Fellowship (F1-17.1/2015-16/RGNF-2015-17-SC-KAR-26367).

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