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Hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$invexity frameworks and efficiency conditions for multiobjective fractional programming problems

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ABSTRACT

The parametrically generalized sufficient efficiency conditions for multiobjective fractional programming based on the hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$invexities are developed and then efficient solutions to the multiobjective fractional programming problems are established. Plus, the obtained results on sufficient efficiency conditions are generalized to the case of the $\epsilon-$efficient solutions. The results thus obtained generalize and unify a wider range of investigations on the theory and applications to the multiobjective fractional programming based on the hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$invexity frameworks.

RESUMEN

Se desarrollan las condiciones de eficiencia suficiente generalizadas paramétricamente de programación multifraccional multiobjetivo basado en las invexidades-$(\Phi, \Psi, \rho, \zeta, \theta)-$híbridas y luego se establecen las soluciones eficientes a los problemas de programación fraccional multiobjetivo. Además, los resultados obtenidos sobre condiciones de eficiencia suficiente se generalizan al caso de soluciones $\epsilon-$-eficientes. Los resultados obtenidos generalizan y unifican una amplia gama de investigaciones en la teoría y aplicaciones de la programación fraccional multiobjetivo basado en el marco de trabajo de las invexidades-$(\Phi, \Psi, \rho, \zeta, \theta)-$.

Keywords and Phrases: Generalized invexity, Multiobjective fractional programming, Efficient solutions, $\epsilon-$efficient solutions, Parametric sufficient efficiency conditions.

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1 Introduction

Among recent developments on higher order generalized invexties and duality models for mathematical programming, we begin with the work of Kawasaki [5] on some second order necessary conditions of the Kuhn - Tucker type under new weaker constraint qualifications for twice continuously differentiable functions, while Mishra and Rueda [11] introduced higher order generalized invexity and duality models in mathematical programming. Mangasarian [8] focused on the second order duality for a conventional nonlinear programming problem, where the approach is based on constructing a second order dual problem by taking linear and quadratic approximations of the objective and constraint functions for an arbitrary but fixed point leading to the Wolfe dual model for the approximated problem, while letting the fixed point to vary. Verma [24] introduced and studied the second order $(\rho, \eta, \theta)-$invexities to the context of parametrically sufficient optimality conditions in semiinfinite discrete minimax fractional programming. Zalmai and Zhang [37] have established a set of efficiency conditions and a fairly large number of global nonparametric sufficient efficiency results under various frameworks for generalized $(\eta, \rho)-$invexity for the semiinfinite discrete minimax fractional programming. Just recently, Verma [22] investigated a general framework for a class of $(\rho, \eta, \theta)-$invex functions to examine some parametric sufficient efficiency conditions for multiobjective fractional programming problems for weakly $\epsilon-$efficient solutions. Inspired by these research advances, we first introduce the hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$invexities as well as the second order hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$invexities, second, introduce some parametrically sufficient efficiency conditions for multiobjective fractional programming, and finally, explore the efficient solutions to multiobjective fractional programming problems. In addition, we generalize the obtained results based on the hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$invexities regarding the efficient solutions to the multiobjective fractional programming problems to the case of the $\epsilon-$efficient solutions to the multiobjective fractional programming problems. The results established in this communication, not only generalize (and unify) the results on general sufficient efficiency conditions for multiobjective fractional programming problems based on the hybrid invexity of functions, but also generalize second order invexity results in more general settings. There exists an enormous literature on higher order generalized invexity and duality models in mathematical programming.

We consider, based on the generalized $(\Phi, \Psi, \rho, \zeta, \theta)-$invexities of functions, the following multiobjective fractional programming problem:

(P)

$$\text{Minimize}\Big(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\Big)$$

subject to $x \in Q = \{x \in X : H_j(x) \leq 0, j \in \{1, 2, \cdots, m\}\}$,
where X is an open convex subset of Re^n (n-dimensional Euclidean space), f_i and g_i for $i \in \{1, \cdots, p\}$ and H_j for $j \in \{1, \cdots, m\}$ are real-valued functions defined on X such that $f_i(x) \geq 0$, $g_i(x) > 0$ for $i \in \{1, \cdots, p\}$ and for all $x \in Q$. Here Q denotes the feasible set of (P).

Next, we observe that problem (P) is equivalent to the nonfractional programming problem:

(Pλ)

$$Minimize\Big(f_1(x) - \lambda_1 g_1(x), \cdots, f_p(x) - \lambda_p g_p(x)\Big)$$

subject to $x \in Q$ with

$$\lambda = \Big(\lambda_1, \lambda_2, \cdots, \lambda_p\Big) = \Big(\frac{f_1(x^*)}{g_1(x^*)}, \frac{f_2(x^*)}{g_2(x^*)}, \cdots, \frac{f_p(x^*)}{g_p(x^*)}\Big),$$

where x^* is an efficient solution to (P).
We all can agree that general Mathematical programming problems offer a great opportunity for applications to other fields, for instance, applications to game theory, statistical analysis, engineering design (including design of control systems, design of earthquakes-resistant structures, digital filters, and electronic circuits), random graphs, boundary value problems, wavelet analysis, environmental protection planning, decision and management sciences, optimal control problems, continuum mechanics, robotics, and beyond. For more details on generalized efficiency and ϵ−efficiency results and applications, we recommend the reader [1 - 40].
This submission is organized as follows: the introductory section deals with a brief historical development for the multiobjective fractional mathematical programming, while emphasizing the roles of the generalized first (and second) order $(\Phi, \Psi, \rho, \zeta, \theta)$−invex functions as well as the first (and second) order generalized $(\Phi, \Psi, \rho, \zeta, \theta)$−invex functions. In Section 2, the hybrid $(\Phi, \Psi, \rho, \zeta, \theta)$−invex functions of higher orders are introduced, and Section 3 deals with sufficient efficiency conditions leading to the solvability of the problem (P) using the hybrid $(\Phi, \Psi, \rho, \zeta, \theta)$−invexities. In Section 4, general remarks are presented.

2 Hybrid Invexities

In this section, we introduce and develop some concepts and notations for the problem on hand. Let X be an open convex subset of Re^n (n-dimensional Euclidean space). Let $\langle\cdot,\cdot\rangle$ denote the inner product, and let $z \in Re^n$. Suppose that $f : X \to Re$ is a real-valued twice continuously differentiable function defined on X, and that $\bigtriangledown f(y)$ and $\nabla^2 f(y)$ denote, respectively, the gradient and Hessian of f at y. Recall a function $\Psi : Re^n \to Re$ is sublinear (super linear) if $\Psi(x + y) \le (\ge)\Psi(x) + \Psi(y)$ for all $x, y \in Re^n$, and $\Psi(ax) = a\Psi(x)$ for all $x \in Re^n$ and $a \in Re_+ = [0, \infty)$. Let $x^* \in X$.

Definition 2.1. A twice differentiable function $f : X \to Re$ is said to be hybrid $(\Phi, \Psi, \rho, \zeta, \theta)$−invex at x^* of second order if there exists a function $\Phi : Re \to Re$ such that for each $x \in X$, $\rho : X \times X \to Re$, $\Psi : Re^n \to Re$, $\zeta, \theta : X \times X \to Re^n$ and $z \in Re^n$,

$$\Phi\Big(f(x) - f(x^*)\Big) \ge \Psi\Big(\langle\bigtriangledown f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \zeta(x, x^*)\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2.$$

Definition 2.2. A twice differentiable function $f : X \to \mathrm{Re}$ is said to be hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$pseudo-invex at x^* of second order if there exists a function $\Phi : \mathrm{Re} \to \mathrm{Re}$ and $z \in \mathrm{Re}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathrm{Re}$, $\Psi : \mathrm{Re}^n \to \mathrm{Re}$, and $\zeta, \theta : X \times X \to \mathrm{Re}^n$,

$$\Psi\Big(\langle \triangledown f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \zeta(x, x^*)\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0$$
$$\Rightarrow \Phi\Big(f(x) - f(x^*)\Big) \geq 0.$$

Definition 2.3. A twice differentiable function $f : X \to \mathrm{Re}$ is said to be strictly hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$pseudo-invex at x^* of second order if there exists a function $\Phi : \mathrm{Re} \to \mathrm{Re}$ and $z \in \mathrm{Re}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathrm{Re}$, $\Psi : \mathrm{Re}^n \to \mathrm{Re}$, and $\zeta, \theta : X \times X \to \mathrm{Re}^n$,

$$\Psi\Big(\langle \triangledown f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \zeta(x, x^*)\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0$$
$$\Rightarrow \Phi\Big(f(x) - f(x^*)\Big) > 0.$$

Definition 2.4. A twice differentiable function $f : X \to \mathrm{Re}$ is said to be prestrictly hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$pseudo-invex at x^* of second order if there if there exists a function $\Phi : \mathrm{Re} \to \mathrm{Re}$ and $z \in \mathrm{Re}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathrm{Re}$, $\Psi : \mathrm{Re}^n \to \mathrm{Re}$, and $\theta, \zeta : X \times X \to \mathrm{Re}^n$,

$$\Psi\Big(\langle \triangledown f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \zeta(x, x^*)\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 > 0$$
$$\Rightarrow \Phi\Big(f(x) - f(x^*)\Big) \geq 0.$$

Definition 2.5. A twice differentiable function $f : X \to \mathrm{Re}$ is said to be hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$quasi-invex at x^* of second order if there exists a function $\Phi : \mathrm{Re} \to \mathrm{Re}$ such that for each $x \in X$, $\rho : X \times X \to \mathrm{Re}$, $\Psi : \mathrm{Re}^n \to \mathrm{Re}$, and $\theta, \zeta : X \times X \to \mathrm{Re}^n$,

$$\Phi\big(f(x) - f(x^*)\big) \leq 0$$
$$\Rightarrow \Psi\Big(\langle \triangledown f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \zeta(x, x^*)\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0.$$

Definition 2.6. A twice differentiable function $f : X \to \mathrm{Re}$ is said to be strictly hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$quasi-invex at x^* of second order if there exist a function $\Phi : \mathrm{Re} \to \mathrm{Re}$, and $z \in \mathrm{Re}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathrm{Re}$, $\Psi : \mathrm{Re}^n \to \mathrm{Re}$, and $\theta, \zeta : X \times X \to \mathrm{Re}^n$,

$$\Phi\big(f(x) - f(x^*)\big) \leq 0$$
$$\Rightarrow \Psi\Big(\langle \triangledown f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \zeta(x, x^*)\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 < 0.$$

Definition 2.7. A twice differentiable function $f : X \to \mathrm{Re}$ is said to be prestrictly hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$ quasi-invex at x^* of second order if there exist a function $\Phi : \mathrm{Re} \to \mathrm{Re}$, and $z \in \mathrm{Re}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathrm{Re}$, $\Psi : \mathrm{Re}^n \to \mathrm{Re}$, and $\theta, \zeta : X \times X \to \mathrm{Re}^n$,

$$\begin{aligned}&\Phi\Big(f(x) - f(x^*)\Big) < 0\\ &\Rightarrow \Psi\Big(\langle \bigtriangledown f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \zeta(x, x^*)\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0.\end{aligned}$$

Definition 2.8. A point $x^* \in Q$ is an efficient solution to (P) if there exists no $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)} \,\forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} < \frac{f_j(x^*)}{g_j(x^*)} \text{ for some } j \in \{1, \cdots, p\}.$$

Next to this context, we have the following auxiliary problem:

$(P\bar{\lambda})$

$$minimize_{x \in Q}(f_1(x) - \bar{\lambda}_1 g_1(x), \cdots, f_p(x) - \bar{\lambda}_p g_p(x)),$$

subject to $x \in Q$,

where $\bar{\lambda}_i$ for $i \in \{1, \cdots, p\}$ are parameters, and $\bar{\lambda}_i = \frac{f_(x^*)}{g_i(x^*)}$.

Next, we introduce the efficiency solvability conditions for $(P\bar{\lambda})$ problem.

Definition 2.9. A point $x^* \in Q$ is an efficient solution to $(P\bar{\lambda})$ if there does not exist an $x \in Q$ such that

$$f_i(x) - \bar{\lambda}_i g_i(x) \leq f_i(x^*) - \bar{\lambda}_i g_i(x^*) \,\forall\, i = 1, \cdots, p,$$

$$f_j(x) - \bar{\lambda}_j g_j(x) < f_j(x^*) - \bar{\lambda}_j g_j(x^*) \text{ for some } j \in \{1, \cdots, p\},$$

where $\bar{\lambda}_i = \frac{f_i(x^*)}{g_i(x^*)}$ for $i = 1, \cdots, p$.

Next, we recall the following result (Verma [24]) that is crucial to developing the results for the next section based on second order $(\Phi, \Psi, \rho, z, \theta)-$invexities.

Theorem 2.1. Let $x^* \in \mathbb{F}$ and $\lambda^* = \max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)$, for each $i \in \underline{p}$, let f_i and g_i be twice continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $z \to G_j(z, t)$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z, s)$ be twice continuously differentiable at x^* for all $s \in S_k$. If x^* is an efficient solution of (P), if the second order generalized Guignard constraint qualification holds at x^*, and if for any critical direction y, the set cone

$$
\begin{aligned}
&(\{\Big(\nabla G_j(x^*,t), \langle y, \nabla^2 G_j(x^*,t)y\rangle\Big) : t \in \hat{T}_j(x^*), j \in \underline{q}\} \cup \{\nabla f_i(x^*) - \lambda_i^* \nabla g_i(x^*) : i \in \underline{p}, i \neq i_0\})\\
+\quad & span(\{\Big(\nabla H_k(x^*,s), \langle y, \nabla^2 H_k(x^*,s)y\rangle\Big) : s \in S_k, k \in \underline{r}\}),\\
& where\ \hat{T}_j(x^*) \equiv \{t \in T_j \ : \ G_j(x^*,t) = 0\})
\end{aligned}
$$

is closed, then there exist $u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0,\ \sum_{i=1}^p u_i = 1\}$ and integers ν_0^* and ν^*, with $0 \leq \nu_0^* \leq \nu^* \leq n+1$, such that there exist ν_0^* indices j_m, with $1 \leq j_m \leq q$, together with ν_0^* points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0^*}$, $\nu^* - \nu_0^*$ indices k_m, with $1 \leq k_m \leq r$, together with $\nu^* - \nu_0^*$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu^*} \backslash \underline{\nu_0^*}$, and ν^* real numbers ν_m^*, with $\nu_m^* > 0$ for $m \in \underline{\nu_0^*}$, with the property that

$$
\begin{aligned}
&\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^*(\nabla g_i(x^*))] + \sum_{m=1}^{\nu_0^*} \nu_m^*[\nabla G_{j_m}(x^*, t^m)\\
+\quad & \sum_{m=\nu_0^*+1}^{\nu^*} \nu_m^* \nabla H_k(x^*, s^m) = 0,
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
&\langle y, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0^*} \nu_m^* \nabla^2 G_{j_m}(x^*, t^m)\\
+\quad & \sum_{m=\nu_0^*+1}^{\nu^*} \nu_m^* \nabla^2 H_k(x^*, s^m)\Big] y\rangle \geq 0,
\end{aligned}
\tag{2.2}
$$

$$
u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \ \ i \in \underline{p}, \tag{2.3}
$$

where $\underline{\nu} \setminus \underline{\nu_0}$ is the complement of the set $\underline{\nu_0}$ relative to the set $\underline{\nu}$.

3 Efficiency Conditions for Problem (P)

This section deals with some parametrically sufficient efficiency conditions for problem (P) under the hybrid frameworks for $(\Phi, \Psi, \rho, \zeta, \theta)-$invexities. We begin with real-valued functions $E_i(., x^*, u^*)$ and $B_j(., \nu)$ defined by

$$
E_i(x, x^*, u^*) = u_i[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)}\Big) g_i(x)],\ i \in \{1, \cdots, p\}
$$

and

$$
B_j(., \nu) = \nu_j H_j(x),\ j = 1, \cdots, m.
$$

Theorem 3.1. Let $x^* \in Q$, let f_i, g_i for $i \in \{1,\cdots,p\}$ with $\frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $g_i(x^*) > 0$ and H_j for $j \in \{1,\cdots,m\}$ be twice continuously differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \mathrm{Re}^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v^* \in \mathrm{Re}_+^m$ such that

$$\Sigma_{i=1}^p u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)] + \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*) = 0, \tag{3.1}$$

$$\Big\langle \zeta(x,x^*), \Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)] + \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0, \tag{3.2}$$

and

$$v_j^* H_j(x^*) = 0,\ j \in \{1,\cdots,m\}. \tag{3.3}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x,x^*) \geq 0$):

(i) $E_i(.;x^*,u^*)\ \forall i \in \{1,\cdots,p\}$ are hybrid $(\Phi,\Psi,\rho,\zeta,\theta)-$pseudo-invex at x^* with $\bar{\Phi}(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.,v^*)\ \forall j \in \{1,\cdots,m\}$ are hybrid $(\tilde{\Phi},\Psi,\bar{\rho},\zeta,\theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing with $\tilde{\Phi}(0) = 0$, and Ψ sublinear.

(ii) $E_i(.;x^*,u^*)\ \forall i \in \{1,\cdots,p\}$ are prestrictly hybrid $(\Phi,\Psi,\rho,\zeta,\theta)-$pseudo-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.,v^*)\ \forall j \in \{1,\cdots,m\}$ arestrictly hybrid $(\tilde{\Phi},\Psi,\rho,\zeta,\theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing with $\tilde{\Phi}(0) = 0$, and Ψ sublinear.

(iii) $E_i(.;x^*,u^*)\ \forall i \in \{1,\cdots,p\}$ are prestrictly hybrid $(\Phi,\Psi,\rho,\eta,\theta)-$quasi-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.,v^*)\ \forall j \in \{1,\cdots,m\}$ are strictly hybrid $(\tilde{\Phi},\Psi,\bar{\rho},\zeta,\theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing with $\tilde{\Phi}(0) = 0$, and Ψ sublinear.

(iv) For each $i \in \{1,\cdots,p\}$, f_i is hybrid $(\Phi,\Psi,\rho_1,\zeta,\theta)-$invex and $-g_i$ is hybrid $(\Phi,\Psi,\rho_2,\zeta,\theta)-$invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, $H_j(.,v^*)\ \forall j \in \{1,\cdots,m\}$ is hybrid $(\bar{\Phi},\Psi,\rho_3,\zeta,\theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing with $\bar{\Phi}(0) = 0$, Ψ sublinear, and $\Sigma_{j=1}^m v_j^*\rho_3 + \rho^* \geq 0$ for $\rho^* = \Sigma_{i=1}^p u_i^*(\rho_1 + \phi(x^*)\rho_2)$ and for $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)}$.

Then x^* is an efficient solution to (P).

Proof. If (i) holds, and if $x \in Q$, then using the sublinearity of Ψ, it follows from (3.1) and (3.2) that

$$\begin{aligned} &\Psi\Big(\Big\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)] \\ &+\frac{1}{2}\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \zeta(x,x^*)\Big\rangle\Big) \\ + \quad &\Psi\Big(\langle \Sigma_{j=1}^{m} \nu_j^* \bigtriangledown H_j(x^*) + \frac{1}{2}\nabla^2 H_j(x^*)z, \zeta(x,x^*)\Big\rangle\Big) \geq 0. \end{aligned} \tag{3.4}$$

Since $\nu^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m} \nu_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} \nu_j^* H_j(x^*),$$

and in light of the hybrid $(\tilde{\Phi}, \Psi, \bar{\rho}, \zeta, \theta)$−quasi-invexity of $B_j(.,\nu^*)$ at x^*, and assumptions on $\tilde{\Phi}$, we find

$$\tilde{\Phi}\Big(\Sigma_{j=1}^{m} \nu_j^* H_j(x) - \Sigma_{j=1}^{m} \nu_j^* H_j(x^*)\Big) \leq 0,$$

which results in

$$\Psi\Big(\langle \bigtriangledown H_j(x^*) + \frac{1}{2}\nabla^2 H_j(x^*)z, \zeta(x,x^*)\rangle\Big) + \bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2 \leq 0. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} &\Psi\Big(\Big\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)] \\ &+\frac{1}{2}\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \zeta(x,x^*)\Big\rangle\Big) \\ \geq \quad &\bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2 \geq -\rho(x,x^*)\|\theta(x,x^*)\|^2. \end{aligned} \tag{3.6}$$

Since $\rho(x,x^*) \geq 0$, applying the hybrid $(\Phi, \Psi, \rho, \zeta, \theta)$−pseudo-invexity at x^* to (3.6) and assumptions on Φ, we have

$$\Phi\Big(\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)})g_i(x)] - \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]\Big) \geq 0,$$

which implies

$$\begin{aligned} &\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \\ \geq \quad &\Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]) \\ = \quad &0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \geq 0. \tag{3.7}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)}) \leq 0 \ \forall i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)}) < 0 \ \text{ for some } \ j \in \{1, \cdots, p\}.$$

Hence, x^* is an efficient solution to (P).

Next, If (ii) holds, and if $x \in Q$, then using the sublinearity of Ψ, it follows from (3.1) and (3.2) that

$$\begin{aligned} & \Psi\Big(\langle \Sigma_{i=1}^{p} u_i^* [\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)] \\ & + \frac{1}{2}\sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \zeta(x, x^*)\Big\rangle\Big) \\ + \quad & \Psi\Big(\langle \Sigma_{j=1}^{m} \nu_j^* \bigtriangledown H_j(x^*) + \frac{1}{2}\nabla^2 H_j(x^*)z, \zeta(x, x^*)\Big\rangle\Big) \geq 0. \end{aligned} \tag{3.8}$$

Since $\nu^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m} \nu_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} \nu_j^* H_j(x^*),$$

which results (using assumptions on $\tilde{\Phi}$) in

$$\tilde{\Phi}\Big(\Sigma_{j=1}^{m} \nu_j^* H_j(x) - \Sigma_{j=1}^{m} \nu_j^* H_j(x^*)\Big) \leq 0.$$

Now, in light of the strictly hybrid $(\tilde{\Phi}, \Psi, \bar{\rho}, \zeta, \theta)-$quasi-invexity of $B_j(., \nu^*)$ at x^*, we find

$$\Psi\Big(\langle \bigtriangledown H_j(x^*) + \frac{1}{2}\nabla^2 H_j(x^*)z, \eta(x, x^*)\rangle\Big) + \bar{\rho}(x, x^*)\|\theta(x, x^*)\|^2 < 0. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\begin{aligned} & \Psi\Big(\langle \Sigma_{i=1}^{p} u_i^* [\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)] \\ + \quad & \frac{1}{2}\sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \zeta(x, x^*)\rangle \\ > \quad & \bar{\rho}(x, x^*)\|\theta(x, x^*)\|^2 > -\rho(x, x^*)\|\theta(x, x^*)\|^2. \end{aligned} \tag{3.10}$$

As a result, since $\rho(x, x^*) \geq 0$, applying the prestrictly hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$pseudo-invexity at x^* to (3.10) and assumptions on Φ, we have

$$\Phi\Big(\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)})g_i(x)] - \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]\Big) \geq 0,$$

which implies

$$\begin{aligned} & \Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \\ \geq\ & \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]) \\ =\ & 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \geq 0. \tag{3.11}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)}) \leq 0 \ \forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)}) < 0 \text{ for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an efficient solution to (P).

The proof applying (iii) is similar to that of (ii), and we just need to include the proof using (iv) as follows: since $x \in Q$, it follows that $H_j(x) \leq H_j(x^*)$, which implies $\bar{\Phi}\Big(H_j(x) - H_j(x^*)\Big) \leq 0$. Then applying the hybrid $(\bar{\Phi}, \Psi, \rho_3, \zeta, \theta)-$quasi-invexity of H_j at x^* and $v^* \in R_+^m$, we have

$$\Psi\Big(\langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), \zeta(x, x^*)\rangle + \frac{1}{2}\Big\langle \zeta(x, x^*), \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) z\Big\rangle\Big) \leq -\Sigma_{j=1}^{m} v_j^* \rho_3 \|\theta(x, x^*)\|^2.$$

Since $u^* \geq 0$ and $\frac{f_i(x^*)}{g_i(x^*)} \geq 0$, it follows from the hybrid $(\Phi, \Psi, \rho_3, \zeta, \theta)-$invexity assumptions that

$$
\begin{aligned}
& \Phi\Big(\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)]\Big) \\
= \;& \Phi\Big(\Sigma_{i=1}^{p} u_i^*\{[f_i(x) - f_i(x^*)] - (\frac{f_i(x^*)}{g_i(x^*)})[g_i(x) - g_i(x^*)]\}\Big) \\
\geq \;& \Psi\Big(\Sigma_{i=1}^{p} u_i^*\{\langle \nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla g_i(x^*), \zeta(x, x^*)\rangle\} \\
+ \;& \frac{1}{2}\langle \zeta(x, x^*), \Sigma_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z\rangle]\Big) \\
+ \;& \Sigma_{i=1}^{p} u_i^*[\rho_1 + \phi(x^*)\rho_2]\|\theta(x, x^*)\|^2 \\
\geq \;& -\Psi\Big(\big[\langle \Sigma_{j=1}^{m} v_j^* \nabla H_j(x^*), \zeta(x, x^*)\rangle + \frac{1}{2}\Big\langle \zeta(x, x^*), \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z\Big\rangle\big]\Big) \\
+ \;& \Sigma_{i=1}^{p} u_i^*[\rho_1 + \phi(x^*)\rho_2]\|\theta(x, x^*)\|^2 \\
\geq \;& (\Sigma_{j=1}^{m} v_j^*\rho_3 + \Sigma_{i=1}^{p} u_i^*[\rho_1 + \phi(x^*)\rho_2])\|\theta(x, x^*)\|^2 \\
= \;& (\Sigma_{j=1}^{m} v_j^*\rho_3 + \rho^*)\|\theta(x, x^*)\|^2 \\
\geq \;& 0,
\end{aligned}
$$

where $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)}$ and $\rho^* = \Sigma_{i=1}^{p} u_i^*(\rho_1 + \phi(x^*)\rho_2)$.
This implies that

$$\Phi\Big(\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)]\Big) \geq 0.$$

□

Theorem 3.2. Let $x^* \in Q$, let f_i, g_i for $i \in \{1, \cdots, p\}$ with $\frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be continuously differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \mathrm{Re}^p : u > 0, \Sigma_{i=1}^{p} u_i = 1\}$ and $v^* \in \mathrm{Re}_+^m$ such that

$$\Big\langle \Sigma_{i=1}^{p} u_i^*[\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla g_i(x^*)] + \Sigma_{j=1}^{m} v_j^* \nabla H_j(x^*), z)\Big\rangle \geq 0 \tag{3.12}$$

and

$$v_j^* H_j(x^*) = 0, \; j \in \{1, \cdots, m\}. \tag{3.13}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$):

(i) $E_i(.;x^*,u^*)$ $\forall i \in \{1,\cdots,p\}$ are first-order hybrid $(\Phi,\Psi,\rho,\zeta,\theta)-$pseudo-invex at x^* for $\Phi(a)\geq 0 \Rightarrow a\geq 0$, and $B_j(.,\nu^*)$ $\forall j\in\{1,\cdots,m\}$ are first-order hybrid $(\bar{\Phi},\Psi,\bar{\rho},\zeta,\theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing with $\bar{\Phi}(0)=0$, and Ψ sublinear.

(ii) $E_i(.;x^*,u^*)$ $\forall i \in \{1,\cdots,p\}$ are first-order hybrid prestrictly $(\Phi,\Psi,\rho,\zeta,\theta)-$pseudo-invex at x^* for $\Phi(a)\geq 0 \Rightarrow a\geq 0$, and $B_j(.,\nu^*)$ $\forall j\in\{1,\cdots,m\}$ are first-order strictly hybrid $(\bar{\Phi},\Psi,\bar{\rho},\zeta,\theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing with $\bar{\Phi}(0)=0$, and Ψ sublinear.

(iii) $E_i(.;x^*,u^*)$ $\forall i \in \{1,\cdots,p\}$ are first-order prestrictly hybrid $(\Phi,\Psi,\rho,\zeta,\theta)-$quasi-invex at x^* $\Phi(a)\geq 0 \Rightarrow a\geq 0$, and $B_j(.,\nu^*)$ $\forall j\in\{1,\cdots,m\}$ are first-order strictly hybrid $(\bar{\Phi},\Psi,\bar{\rho},\zeta,\theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing with $\bar{\Phi}(0)=0$, and Ψ sublinear.
and z

(iv) For each $i\in\{1,\cdots,p\}$, f_i is first-order hybrid $(\Phi,\Psi,\rho_1,\zeta,\theta)-$invex and $-g_i$ is first-order hybrid $(\Phi,\Psi,\rho_2,\zeta,\theta)-$invex at x^* for $\Phi(a)\geq 0\Rightarrow a\geq 0.$ $H_j(.,\nu^*)$ $\forall j\in\{1,\cdots,m\}$ is hybrid $(\bar{\Phi},\Psi,\bar{\rho_3},\zeta,\theta)-$quasi-invex at x^*, and $\Sigma_{j=1}^m \nu_j^*\rho_3+\rho^*\geq 0$ for $\bar{\Phi}$ increasing with $\bar{\Phi}(0)=0$, $\rho^*=\Sigma_{i=1}^p u_i^*(\rho_1+\phi(x^*)\rho_2)$ for $\phi(x^*)=\frac{f_i(x^*)}{g_i(x^*)}$, and Ψ sublineaer.

Then x^* is an efficient solution to (P).

Proof. Although the proof is similar to that of Theorem 3.1), we include for the sake of the completeness. If we consider (i), then proceeding as in Theorem 3.1 (and using the first-order hybrid $(\Phi,\Psi,\rho,\zeta,\theta)-$invexity assumptions instead), we arrive at

$$\begin{aligned}&\Psi\Big(\langle\Sigma_{i=1}^p u_i^*[\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})\bigtriangledown g_i(x^*)],\zeta(x,x^*)\rangle\Big)\\ \geq\ &\rho(x,x^*)\|\theta(x,x^*)\|^2.\end{aligned}\tag{3.14}$$

Since $\rho(x,x^*)\geq 0$, applying the hybrid $(\Phi,\Psi,\rho,\zeta,\theta)-$pseudo-invexity at x^* to (3.14) and assumptions on Φ, we have

$$\Phi\Big(\Sigma_{i=1}^p u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(g^*)})g_i(x)]-\Sigma_{i=1}^p u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]\Big)\geq 0,$$

which implies

$$\begin{aligned}&\Sigma_{i=1}^p u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x)]\\ \geq\ &\Sigma_{i=1}^p u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]])\\ =\ &0.\end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)}) g_i(x)] \geq 0. \tag{3.15}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)}) \leq 0 \ \forall i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)}) < 0 \ \text{ for some } \ j \in \{1, \cdots, p\}.$$

Hence, x^* is an efficient solution to (P).

□

Theorem 3.3. Let $x^* \in Q$, let f_i, g_i for $i \in \{1, \cdots, p\}$ with $\frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be twice continuously differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \mathrm{Re}^p : u > 0, \Sigma_{i=1}^{p} u_i = 1\}$ and $v^* \in \mathrm{Re}_+^m$ such that

$$\Sigma_{i=1}^{p} u_i^* [\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)] + \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*) = 0 \tag{3.16}$$

$$\Big\langle \zeta(x, x^*), \Big[\sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) \Big] z \Big\rangle \geq 0, \tag{3.17}$$

and

$$v_j^* H_j(x^*) = 0, \ j \in \{1, \cdots, m\}. \tag{3.18}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$):

(i) $E_i(.; x^*, u^*)$ $\forall i \in \{1, \cdot \cdot \cdot, p\}$ are hybrid $(\rho, \zeta, \theta)-$pseudo-invex at x^*, and $B_j(., v^*)$ $\forall j \in \{1, \cdots, m\}$ are hybrid $(\rho, \zeta, \theta)-$quasi-invex at x^*.

(ii) $E_i(.; x^*, u^*)$ $\forall i \in \{1, \cdots, p\}$ are prestrictly hybrid $(\rho, \zeta, \theta)-$pseudo-invex at x^*, and $B_j(., v^*)$ $\forall j \in \{1, \cdots, m\}$ are hybrid $(\rho, \zeta, \theta)-$strictly-quasi-invex at x^*.

(iii) $E_i(.; x^*, u^*)$ $\forall i \in \{1, \cdots, p\}$ are strictly hybrid $(\rho, \zeta, \theta)-$pseudo-invex at x^*, and $B_j(., v^*)$ $\forall j \in \{1, \cdots, m\}$ are strictly hybrid $(\rho, \zeta, \theta)-$quasi-invex at x^*.

(iv) For each $i \in \{1, \cdots, p\}$, f_i is hybrid $(\rho_1, \zeta, \theta)-$invex and $-g_i$ is $(\rho_2, \zeta, \theta)-$invex at x^*. $H_j(., v^*)$ $\forall j \in \{1, \cdots, m\}$ is hybrid $(\rho_3, \zeta, \theta)-$quasi-invex at x^*, and $\Sigma_{j=1}^{m} v_j^* \rho_3 + \rho^* \geq 0$ for $\rho^* = \Sigma_{i=1}^{p} u_i^* (\rho_1 + \phi(x^*) \rho_2)$ and for $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)}$.

Then x^* is an efficient solution to (P).

Proof. The proof is similar to that of Theorem 3.1 based on the second order hybrid $(\rho, \zeta, \theta)-$invexity assumptions.

□

We observe that Theorem 3.1 can be further generalized to the case of the $\epsilon-$Efficient conditions based on the hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$invexity frameworks. As a matter of fact, we generalize the $\epsilon-$efficient solvability conditions for problem (P) based on the work of Verma [22], and Kim, Kim and Lee [6], where they have investigated the $\epsilon-$efficiency as well as the weak $\epsilon-$efficiency conditions for multiobjective fractional programming problems under constraint qualifications. To the best of our knowledge, the results established in this communication (Theorem 3.1 and Theorem 3.4) generalize and unify most of the results on the multiobjective fractional programming to the context of the generalized invexities in the literature. We recall some auxiliary concepts (for the hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$invexity) crucial to the problem on hand.

Definition 3.1. A point $x^* \in Q$ is an $\epsilon-$efficient solution to (P) if there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i \, \forall \, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} < \frac{f(_j x^*)}{g_j(x^*)} - \epsilon_j \text{ for some } j \in \{1, \cdots, p\},$$

where $\epsilon_i = (\epsilon_1, \cdots, \epsilon_p)$ is with $\epsilon_i \geq 0$ for $i = 1, \cdots, p$.

For $\epsilon = 0$, Definition 3.1 reduces to the case that $x^* \in Q$ is an efficient solution to (P).

Next, we start with real-valued functions $E_i(., x^*, u^*)$ and $B_j(., v)$, respectively, defined by

$$E_i(x, x^*, u^*) = u_i[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big) g_i(x)], \; i \in \{1, \cdots, p\}$$

and

$$B_j(., \nu) = \nu_j H_j(x), \;\; j = 1, \cdots, m.$$

Theorem 3.4. Let $x^* \in Q$, let f_i, g_i for $i \in \{1, \cdots, p\}$ with $f_i(x^*) \geq \epsilon_i g_i(x^*)$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be twice continuously differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \mathrm{Re}^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$, $\nu^* \in \mathrm{Re}_+^m$ and $z \in \mathrm{Re}^n$ such that

$$\Sigma_{i=1}^p u_i^* [\nabla f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big) \bigtriangledown g_i(x^*)] + \Sigma_{j=1}^m \nu_j^* \bigtriangledown H_j(x^*) = 0, \tag{3.19}$$

$$\Big\langle \zeta(x, x^*), \Big[\sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big) \nabla^2 g_i(x^*)] + \sum_{j=1}^m \nu_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0, \tag{3.20}$$

and

$$\nu_j^* H_j(x^*) = 0,\ j \in \{1, \cdots, m\}. \tag{3.21}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$):

(i) $E_i(.; x^*, u^*)\ \forall i \in \{1, \cdots, p\}$ are hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$pseudo-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(., \nu^*)\ \forall j \in \{1, \cdots, m\}$ are hybrid $(\tilde{\Phi}, \Psi, \bar{\rho}, \zeta, \theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing with $\tilde{\Phi}(0) = 0$, and Ψ sublinear.

(ii) $E_i(.; x^*, u^*)\ \forall i \in \{1, \cdots, p\}$ are prestrictly hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$pseudo-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(., \nu^*)\ \forall j \in \{1, \cdots, m\}$ are strictly hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing with $\bar{\Phi}(0) = 0$, and Ψ sublinear.

(iii) $E_i(.; x^*, u^*)\ \forall i \in \{1, \cdots, p\}$ are strictly hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$pseudo-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(., \nu^*)\ \forall j \in \{1, \cdots, m\}$ are strictly hybrid $(\bar{\Phi}, \Psi, \bar{\rho}, \zeta, \theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing with $\bar{\Phi}(0) = 0$, and Ψ sublinear.

(iv) For each $i \in \{1, \cdots, p\}$, f_i is hybrid $(\Phi, \Psi, \rho_1, \zeta, \theta)-$invex and $-g_i$ is $(\Phi, \Psi, \rho_2, \zeta, \theta)-$invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $H_j(., \nu^*)\ \forall j \in \{1, \cdots, m\}$ is hybrid $(\bar{\Phi}, \Psi, \rho_3, \zeta, \theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing with $\bar{\Phi}(0) = 0$, Ψ sublinear, and $\Sigma_{j=1}^m \nu_j^* \rho_3 + \rho^* \geq 0$ for $\rho^* = \Sigma_{i=1}^p u_i^* (\rho_1 + \phi(x^*) \rho_2)$, where $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i$.

Then x^* is an $\epsilon-$efficient solution to (P).

Proof. If (i) holds, and if $x \in Q$, then it follows using the sublinearity of Ψ from (3.1) and (3.2) that

$$
\begin{aligned}
& \Psi\Big(\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \bigtriangledown g_i(x^*)] \\
& +\frac{1}{2}\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)z], \zeta(x,x^*)\Big\rangle\Big) \\
+ \quad & \Psi\Big(\langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*) + \frac{1}{2}\nabla^2 H_j(x^*)z, \zeta(x,x^*)\Big\rangle\Big) \geq 0. \qquad (3.22)
\end{aligned}
$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

which implies

$$\Sigma_{j=1}^{m} v_j^* H_j(x) - \Sigma_{j=1}^{m} v_j^* H_j(x^*) \leq 0,$$

so in light of the hybrid $(\tilde{\Phi}, \Psi, \bar{\rho}, \zeta, \theta)$—quasi-invexity of $B_j(., v^*)$ at x^*, and assumptions on $\tilde{\Phi}$, it results in

$$\tilde{\Phi}\Big(\Sigma_{j=1}^{m} v_j^* H_j(x) - \Sigma_{j=1}^{m} v_j^* H_j(x^*)\Big) \leq 0,$$

which implies

$$\Psi\Big(\langle \bigtriangledown H_j(x^*), \zeta(x,x^*)\rangle + \frac{1}{2}\langle \zeta(x,x^*), \nabla^2 H_j(x^*)z\rangle\Big) + \bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2 \leq 0. \qquad (3.23)$$

It follows from (3.22) and (3.23) that

$$
\begin{aligned}
& \Psi\Big(\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \bigtriangledown g_i(x^*)], z\rangle \\
+ \quad & \frac{1}{2}\Big\langle \zeta(x,x^*), \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)z]\Big\rangle\Big) \\
\geq \quad & \bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2 \geq -\rho(x,x^*)\|\theta(x,x^*)\|^2. \qquad (3.24)
\end{aligned}
$$

As a result, since $\rho(x,x^*) \geq 0$, applying the hybrid $(\Phi, \Psi, \rho, \zeta, \theta)-$ pseudo-invexity at x^* to (3.24) and assumptions on Φ, we have

$$
\begin{aligned}
& \Phi\Big(\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \\
& -\Sigma_{i=1}^{p} u_i^*[f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)]\Big) \geq 0,
\end{aligned}
$$

which implies

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)]$$

$$\geq \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)]$$

$$\geq \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)]$$

$$-\Sigma_{i=1}^{p} u_i^* \epsilon_i g_i(x^*)$$

$$= 0.$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \geq 0. \tag{3.25}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{\sum_{i=1}^{p} f_i(x)}{\sum_{i=1}^{p} g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \leq 0 \; \forall \, i = 1, \cdots, p,$$

$$\frac{\sum_{j=1}^{p} f_j(x)}{\sum_{j=1}^{p} g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j) < 0 \text{ for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an ϵ−efficient solution to (P).

If (ii) holds, and if $x \in Q$, then it follows from (3.1) and (3.2) that

$$\begin{aligned} &\Psi\Big(\Big\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \bigtriangledown g_i(x^*)] \\ &+\frac{1}{2}\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)z], \zeta(x, x^*)\Big\rangle\Big) \\ + \quad &\Psi\Big(\Big\langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*) + \frac{1}{2}\nabla^2 H_j(x^*)z, \zeta(x, x^*)\Big\rangle\Big) \geq 0. \end{aligned} \tag{3.26}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

or

$$\Sigma_{j=1}^{m} v_j^* H_j(x) - \Sigma_{j=1}^{m} v_j^* H_j(x^*) \leq 0,$$

which implies based on assumptions on $\tilde{\Phi}$ that

$$\tilde{\Phi}\Big(\Sigma_{j=1}^{m}\nu_j^*H_j(x)-\Sigma_{j=1}^{m}\nu_j^*H_j(x^*)\Big)\leq 0.$$

Next, in light of the strict $(\tilde{\Phi},\Psi,\bar{\rho},\zeta,\theta)$−quasi-invexity of $B_j(.,\nu^*)$ at x^* with $\tilde{\Phi}$ increasing and $\tilde{\Phi}(0)=0$, we find

$$\Psi\Big(\langle\bigtriangledown H_j(x^*),\zeta(x,x^*)\rangle+\frac{1}{2}\langle\zeta(x,x^*),\nabla^2H_j(x^*)z\rangle\Big)+\bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2<0. \tag{3.27}$$

It follows from (3.26) and (3.27) that

$$\begin{aligned}
&\Psi\Big(\langle\Sigma_{i=1}^{p}u_i^*[\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\epsilon_i)\bigtriangledown g_i(x^*)],\zeta(x,x^*)\rangle\\
+\quad&\frac{1}{2}\Big\langle\zeta(x,x^*),\sum_{i=1}^{p}u_i^*[\nabla^2f_i(x^*)z-(\frac{f_i(x^*)}{g_i(x^*)}-\epsilon_i)\nabla^2g_i(x^*)z]\Big\rangle\Big)\\
>\quad&\bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2>-\rho(x,x^*)\|\theta(x,x^*)\|^2.
\end{aligned} \tag{3.28}$$

As a result, since $\rho(x,x^*)\geq 0$, applying the prestrictly hybrid $(\Phi,\Psi,\rho,\zeta,\theta)$−pseudo-invexity at x^* to (3.28) and assumptions on Φ, we have

$$\Phi\Big(\Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)}-\epsilon_i)g_i(x)]-\Sigma_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})-\epsilon_i)g_i(x^*)]\Big)\geq 0,$$

which implies

$$\begin{aligned}
\Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)}-\epsilon_i)g_i(x)]&\geq\Sigma_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\epsilon_i)g_i(x^*)]\\
&\geq\Sigma_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\epsilon_i)g_i(x^*)]-\Sigma_{i=1}^{p}u_i^*\epsilon_ig_i(x^*)\\
&=0.
\end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)}-\epsilon_i)g_i(x)]\geq 0. \tag{3.29}$$

Since $u_i^*>0$ for each $i\in\{1,\cdots,p\}$, we conclude that there does not exist an $x\in Q$ such that

$$\frac{f_i(x)}{g_i(x)}-(\frac{f_i(x^*)}{g_i(x^*)}-\epsilon_i)\leq 0\ \forall\, i=1,\cdots,p,$$

$$\frac{f_j(x)}{g_j(x)}-(\frac{f_j(x^*)}{g_j(x^*)}-\epsilon_j)<0\ \text{for some}\ j\in\{1,\cdots,p\}.$$

Hence, x^* is an ϵ−efficient solution to (P).

The proof applying (iii) is similar to that of (ii), and we just need to include the proof using (iv) as follows: since $x \in Q$, it follows that $H_j(x) \leq H_j(x^*)$. Then applying the $(\bar{\Phi}, \Psi, \rho_3, \zeta, \theta)$−quasi-invexity of H_j at x^* and $\nu^* \in R^m_+$, we have

$$\begin{aligned}&\Psi\Big(\Big\langle \Sigma_{j=1}^m \nu_j^* \bigtriangledown H_j(x^*), \zeta(x,x^*)\rangle + \frac{1}{2}\Big\langle \zeta(x,x^*), \Sigma_{j=1}^m \nu_j^* \nabla^2 H_j(x^*)z\Big\rangle\Big)\\ &\leq -\Sigma_{j=1}^m \nu_j^* \rho_3 \|\theta(x,x^*)\|^2.\end{aligned}$$

Since $u^* \geq 0$ and $f_i(x^*) \geq \epsilon_i g_i(x^*)$, it follows from $(\Phi, \Psi, \rho_3, \zeta, \theta)$−invexity assumptions that

$$\begin{aligned}
& \Phi\Big(\Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)]\Big)\\
= & \Phi\Big(\Sigma_{i=1}^p u_i^*\{[f_i(x) - f_i(x^*)] - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)[g_i(x) - g_i(x^*)] + \epsilon_i g_i(x^*)\}\Big)\\
\geq & \Psi\Big(\Sigma_{i=1}^p u_i^*\{\langle \bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \bigtriangledown g_i(x^*), \zeta(x,x^*)\rangle\}\\
+ & \frac{1}{2}\langle \zeta(x,x^*), \Sigma_{i=1}^p u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)z\rangle]\Big)\\
+ & [\rho_1 + \phi(x^*)\rho_2]\|\theta(x,x^*)\|^2 + \Sigma_{i=1}^p u_i^* \epsilon_i g_i(x^*)\\
\geq & -\Psi\Big([\langle \Sigma_{j=1}^m \nu_j^* \bigtriangledown H_j(x^*), \zeta(x,x^*)\rangle + \frac{1}{2}\Big\langle \zeta(x,x^*), \Sigma_{j=1}^m \nu_j^* \nabla^2 H_j(x^*)z\Big\rangle]\Big)\\
+ & \Sigma_{i=1}^p u_i^*[\rho_1 + (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\rho_2]\|\theta(x,x^*)\|^2 + \Sigma_{i=1}^p u_i^* \epsilon_i g_i(x^*)\\
\geq & (\Sigma_{j=1}^m \nu_j^* \rho_3 + \Sigma_{i=1}^p u_i^*[\rho_1 + (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\rho_2])\|\theta(x,x^*)\|^2 + \Sigma_{i=1}^p u_i^* \epsilon_i g_i(x^*)\\
= & (\Sigma_{j=1}^m \nu_j^* \rho_3 + \rho^*)\|\theta(x,x^*)\|^2 + \Sigma_{i=1}^p u_i^* \epsilon_i g_i(x^*)\\
\geq & (\Sigma_{j=1}^m \nu_j^* \rho_3 + \rho^*)\|\theta(x,x^*)\|^2 \geq 0.
\end{aligned}$$

Therefore, we have

$$\Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \geq 0. \tag{3.30}$$

Thus, we conclude that there does not exist an $x \in Q$ such that

$$\frac{\sum_{i=1}^{p} f_i(x)}{\sum_{i=1}^{p} g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \leq 0 \ \forall\, i = 1, \cdots, p,$$

$$\frac{\sum_{j=1}^{p} f_j(x)}{\sum_{j=1}^{p} g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j) < 0 \text{ for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\epsilon-$efficient solution to (P).

□

4 Concluding Remarks

We observe that the obtained results in this communication can be generalized to the case of multiobjective fractional programming with generalized hybrid invex functions of higher orders (including the exponential type generalized invexities), for instance, based on the work of Mishra and Rueda [11], Mishra, Laha and Verma [13], and Zalmai and Zhang [37] to the case of the efficiency as well as to the $\epsilon-$efficiency conditions relating to the minimax fractional programming problems involving generalized invex functions.

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