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Semi Open sets in bispaces

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ABSTRACT

The notions of semi open sets in a topological space were introduced by N.Levine in 1963. Here we study the same using the idea of $\tau_1(\tau_2)$ semi open sets with respect to $\tau_2(\tau_1)$, pairwise semi open sets in a more general structure of a bispace and investigate how far several results as valid in a bitopological space are affected in bispaces.

RESUMEN

Las nociones de conjuntos semiabiertos en un espacio topológico se introdujeron por N. Levine en 1963. Aquí estudiamos lo mismo usando la idea de conjuntos semiabiertos $\tau_1(\tau_2)$ respecto de conjuntos abiertos semiabiertos dos a dos $\tau_2(\tau_1)$, en una estructura más general de biespacio e investigamos cómo varios resultados válidos en un espacio bitopológico cambian en biespacios.

Keywords and Phrases: bispaces, semi open sets, τ_1 semi open sets with respect to τ_2.

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1 Introduction

The notion of a σ space or simply a space was introduced by A.D.Alexandroff [1] in 1940 generalising the idea of a topological space where only countable unions of open sets were taken to be open. In 2001 the idea of space was used by Lahiri and Das [8] to generalise the notion of a bitopological space to a bispace. N.Levine [9] introduced the concept of semi open sets in a topological space in 1963 and this idea was generalised by S.Bose [3] in the setting of a bitopological space $(X, \mathcal{P}, \mathcal{Q})$ using the idea of $\mathcal{P}(\mathcal{Q})$ semi open sets with respect to $\mathcal{Q}(\mathcal{P})$ etc. Later the same was studied in a space by Lahiri and Das [7] and they critically took the matter of generalisation in this setting. Here we have studied the concept of $\tau_1(\tau_2)$ semi open sets with respect to $\tau_2(\tau_1)$ and some other properties in the setting of a bispace and have shown with typical examples how far several results as valid in[3] are affected in bispaces. Also we have given a necessary and sufficient condition for a bispace to be a bitopological space in terms of $\tau_1(\tau_2)$ semi open sets with respect to $\tau_2(\tau_1)$.

2 Preliminaries

Definition 2.1. *[1] A set X is called an Alexandroff space or simply a space if in it is chosen a system of subsets $\mathcal{F}$ satisfying the following axioms:*

(1) *The intersection of a countable number of sets from $\mathcal{F}$ is a set in $\mathcal{F}$.*

(2) *The union of a finite number of sets from $\mathcal{F}$ is a set in $\mathcal{F}$.*

(3) *The void set ϕ is a set in $\mathcal{F}$.*

(4) *The whole set X is a set in $\mathcal{F}$.*

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Sets of $\mathcal{F}$ are called closed sets. Their complementary sets are called open sets. It is clear that instead of closed set in the definition of the space, one may put open sets with subject to the condition of countable summability, finite intersectibility and the condition that X and ϕ should be open. The collection of all such open sets will sometimes be denoted by τ and the space by (X, τ). Note that, in general τ is not a topology as can be easily seen by taking $X = R$, the set of real numbers and τ as the collection of all F_σ sets in R.

Definition 2.2. *[1] To every set M of (X, τ) we correlate its closure $\overline{M}$ = the intersection of all closed sets containing M. Sometimes the closure of a set M will be denoted by τclM or simply clM when there is no confusion about τ.*

Generally the closure of a set in a space is not a closed set.

From the axioms, it easily follows that

1) $\overline{M \cup N} = \overline{M} \cup \overline{N}$; 2) $M \subset \overline{M}$; 3) $\overline{M} = \overline{\overline{M}}$; 4) $\overline{\phi} = \phi$.

Definition 2.3. *[7] The interior of a set* M *in* (X, τ) *is defined as the union of all open sets contained in* M *and is denoted by* $\tau\text{int}M$ *or* $\text{int}M$ *when there is no confusion.*

Definition 2.4. *[6] A non empty set* X *on which are defined two arbitrary topologies* $\mathcal{P}, \mathcal{Q}$ *is called a bitopological space and denoted by* $(X, \mathcal{P}, \mathcal{Q})$.

Definition 2.5. *[8] Let* X *be a nonempty set. If* τ_1 *and* τ_2 *be two collections of subsets of* X *such that* (X, τ_1) *and* (X, τ_2) *are two spaces, then* X *is called a bispace and is denoted by* (X, τ_1, τ_2).

3 Pairwise Semi Open sets

Definition 3.1. *(cf. Definition 1[3]): Let* (X, τ_1, τ_2) *be a bispace. We say that a subset* A *of* X *is* τ_1 *semi open with respect to* τ_2 *(in short* τ_1 *s.o.w.r.to* τ_2 *) if and only if there exists a* τ_1 *open set* O *such that* $O \subset A \subset \tau_2\text{cl}O$.

Similarly $A \subset X$, is τ_2 semi open with respect to τ_1 (in short τ_2 s.o.w.r.to τ_1) if and only if there exists a τ_2 open set O such that $O \subset A \subset \tau_1\text{cl}O$.
We say that A is pairwise semi open if and only if it is both τ_1 s.o.w.r.to τ_2 and τ_2 s.o.w.r.to τ_1.
Note that a $\tau_1(\tau_2)$ open set is $\tau_1(\tau_2)$ s.o.w.r.to $\tau_2(\tau_1)$.

Throughout our discussion, (X, τ_1, τ_2) or simply X stands for a bispace, R stands for the set of real numbers, Q for the set of rational numbers and N stands for the set of natural numbers and sets are always subsets of X unless otherwise stated.

Theorem 3.2. *Let* (X, τ_1, τ_2) *be a bispace. Let* $A \subset X$*, and* A *is* τ_1 *s.o.w.r.to* τ_2 *then* $\tau_2\text{cl}A = \tau_2\text{cl}(\tau_1\text{int}A)$.

Proof. Let A is τ_1 s.o.w.r.to τ_2 then there exists a τ_1 open set O such that $O \subset A \subset \tau_2\text{cl}O$. Also $O \subset \tau_1\text{int}A$. Therefore, $A \subset \tau_2\text{cl}O \subset \tau_2\text{cl}(\tau_1\text{int}A)$ and hence $\tau_2\text{cl}A \subset \tau_2\text{cl}(\tau_2\text{cl}(\tau_1\text{int}A)) = \tau_2\text{cl}(\tau_1\text{int}A)$. Also $\tau_2\text{cl}(\tau_1\text{int}A) \subset \tau_2\text{cl}A$. Therefore, $\tau_2\text{cl}A = \tau_2\text{cl}(\tau_1\text{int}A)$. □

Corollary 3.3. *If* A *is* τ_1 *s.o.w.r.to* τ_2 *and* $A \neq \phi$ *then* $\tau_1\text{int}A \neq \phi$.

Corollary 3.4. *Let* A *is* τ_1 *s.o.w.r.to* τ_2 *and* $A \subset B$ *then* $A \subset \tau_2\text{cl}(\tau_1\text{int}B)$.

If (X, τ_1, τ_2) is a bitopological space then converse part of the theorem 3.2 also holds which is seen in [3] . But in a bispace, this may not be true as shown below:

Example 3.5. *Let* $X = [0, 2]$ *and* $\{G_i\}$ *be the collection of all countable subsets of irrational numbers in* $[0, 1]$. *Let* τ_1 *be the collection of all sets of the form* $G_i \cup \{\sqrt{2}\}$ *together with* X *and* ϕ*, and* τ_2 *be the collection of all sets* G_i *together with* X *and* ϕ. *Then* (X, τ_1, τ_2) *is a bispace. Now consider a subset* $A = [0, 1] \cup \{\sqrt{2}\}$ *then* $\tau_2\text{cl}A = X$ *and* $\tau_1\text{int}A$ *is set of all irrational numbers in* $[0, 1]$ *together with* $\sqrt{2}$ *and hence* $\tau_2\text{cl}(\tau_1\text{int}A) = X$. *Therefore,* $\tau_2\text{cl}A = \tau_2\text{cl}(\tau_1\text{int}A)$. *But for any* τ_1

open set $G(\neq X, \phi)$, $\tau_2 clG = G \cup Q_1 \cup [1, 2]$, *where* Q_1 *is the set of all rational numbers in* $[0, 1]$. *Clearly* $\tau_2 clG$ *does not contain* A. *Therefore, there does not exist any* τ_1 *open set* G *satisfying* $G \subset A \subset \tau_2 clG$. *So* A *is not* τ_1 *s.o.w.r.to* τ_2.

However we observe in the following theorem that the converse part of theorem 3.2 holds under an additional condition.

Theorem 3.6. *In a bispace* (X, τ_1, τ_2), *let* $\tau_2 clA = \tau_2 cl(\tau_1 intA)$. *Then* A *is* τ_1 *s.o.w.r.to* τ_2 *for any subset* A *of* X *if the condition* C_1 *is satisfied.*

C_1 : *Arbitrary union of* τ_1 *open sets is* τ_1 *s.o.w.r.to* τ_2.

Proof. Let $O = \tau_1 intA$. Then by the condition C_1, O is τ_1 s.o.w.r.to τ_2. So there exists a τ_1 open set G such that $G \subset O \subset \tau_2 clG$. Now since $\tau_2 clO = \tau_2 cl(\tau_1 intA) = \tau_2 clA$ and $O \subset \tau_2 clG$, it follows that $\tau_2 clO \subset \tau_2 cl(\tau_2 clG) = \tau_2 clG$ and hence $\tau_2 clA \subset \tau_2 clG$. Therefore, $G \subset O \subset A \subset \tau_2 clA \subset \tau_2 clG$ and so A is τ_1 s.o.w.r.to τ_2. □

Remark 3.7. *We see in the Example 3.8 below that there is a bispace which is not a bitopological space where the condition* C_1 *holds good.*

Example 3.8. *Let* $X = [0, 2]$, τ_1 *be the collection of all sets* G_i *together with* X *and* ϕ *and* τ_2 *be the collection of all sets* F_i *together with* X *and* ϕ *where* $\{G_i\}$ *and* $\{F_i\}$ *are the collection of all countable subsets of irrational numbers in* $[0, 1]$ *and* $[1, 2]$ *respectively. Then* (X, τ_1, τ_2) *is a bispace but not a bitopological space. Now consider all* τ_1 *open sets* $\{G_i\}$. *Then* $\cup G_i$ *is the set of all irrational numbers in* $[0, 1]$ *which is not* τ_1 *open. But since, for any* τ_1 *open set* G_i, $\tau_2 clG_i = [0, 1] \cup Q_2$ *where* Q_2 *is set of all rational numbers in* $[1, 2]$. *It follows that* $G_i \subset \cup G_i \subset \tau_2 clG_i$. *This implies that* $\cup G_i$ *is* τ_1 *s.o.w.r.to* τ_2 *although it is not* τ_1 *open.*

Theorem 3.9. *Countable union of* τ_1 *s.o.sets w.r.to* τ_2 *is* τ_1 *s.o.w.r.to* τ_2.

Proof. Let $\{A_n : n \in N\}$ be a countable collection of τ_1 s.o.sets w.r.to τ_2. Then for each $n \in N$ there exists a τ_1 open set O_n such that $O_n \subset A_n \subset \tau_2 clO_n$. This implies that $\cup\{O_n : n \in N\} \subset \cup\{A_n : n \in N\} \subset \cup\{\tau_2 clO_n : n \in N\} \subset \tau_2 cl(\cup\{O_n : n \in N\})$ i.e, $O \subset \cup\{A_n : n \in N\} \subset \tau_2 clO$, where $O = \cup\{O_n : n \in N\}$, a τ_1 open set. Hence $\cup\{A_n : n \in N\}$ is τ_1 s.o.w.r.to τ_2. □

Remark 3.10. *In [3] it was proved that in a bitopological space arbitrary union of* τ_1 *s.o.sets w.r.to* τ_2 *is* τ_1 *s.o.w.r.to* τ_2. *But this may not be true in a bispace as shown in the Example 3.11 below.*

Example 3.11. *Consider* $X = R$. *Let* τ_1 *open sets are* X, ϕ *and all sets* G_i *where* $\{G_i\}$ *is the collection of all countable subsets of irrational numbers in* R *and* τ_2 *open sets are* X, ϕ *and all* F_σ *sets in* R. *Clearly* τ_2 *closed sets are the* G_δ *sets. So for any subset* G *in* R *we have* $\tau_2 clG = G$. *Therefore, for any set* A *which is* τ_1 *s.o.w.r.to* τ_2 *in* X, *there exists a* τ_1 *open set* G_i *such that* $G_i \subset A \subset \tau_2 clG_i = G_i$. *This implies that* $A = G_i$, *i.e.,* A *is* τ_1 *open set. So* τ_1 *open sets are the only* τ_1 *s.o.sets w.r.to* τ_2. *Since the union of all* τ_1 *open sets* G_i $(G_i \neq X)$ *is precisely the set* *of all irrational numbers in* R *which is not* τ_1 *open, it follows that arbitrary union of* τ_1 *s.o.sets w.r.to* τ_2 *may not be* τ_1 *s.o.w.r.to* τ_2.

However the additional condition C_1 *ensures the result in theorem 3.9 for arbitrary union.*

Theorem 3.12. *Arbitrary union of* τ_1 *s.o.sets w.r.to* τ_2 *is* τ_1 *s.o.w.r.to* τ_2 *if and only if the condition* C_1 *is satisfied.*

Proof. Assume first that the arbitrary union of τ_1 s.o.sets w.r.to τ_2 is τ_1 s.o.w.r.to τ_2. Since every τ_1 open set is τ_1 s.o.w.r.to τ_2, arbitrary union of τ_1 open sets is τ_1 s.o.w.r.to τ_2, i.e., the condition C_1 holds.

Next assume that the condition C_1 holds. Let $\{A_i\}$ be an arbitrary collection of τ_1 s.o.sets w.r.to τ_2 and $A = \cup A_i$. For each i, there exists a τ_1 open set G_i such that $G_i \subset A_i \subset \tau_2 clG_i$. Therefore, $\cup G_i \subset \cup A_i = A \subset \cup\tau_2 clG_i \subset \tau_2 cl(\cup G_i)$. Since $\cup G_i$, by assumption, is τ_1 s.o.w.r.to τ_2, there exists a τ_1 open set G such that $G \subset \cup G_i \subset \tau_2 clG$. Therefore, $G \subset \cup G_i \subset A \subset \cup\tau_2 clG_i \subset \tau_2 cl(\cup G_i) \subset \tau_2 cl\tau_2 clG = \tau_2 clG$. This proves that A is τ_1 s.o.w.r.to τ_2. □

Theorem 3.13. *Let* A *be* τ_1 *s.o.w.r.to* τ_2 *in a bispace* (X, τ_1, τ_2) *and let* $A \subset B \subset \tau_2 clA$. *Then* B *is* τ_1 *s.o.w.r.to* τ_2.

Proof. Since A is τ_1 s.o.w.r.to τ_2, there exists a τ_1 open set O such that $O \subset A \subset \tau_2 clO$. Therefore, $O \subset A \subset B \subset \tau_2 clA \subset \tau_2 cl(\tau_2 clO) = \tau_2 clO$ and hence B is τ_1 s.o.w.r.to τ_2. □

Theorem 3.14. *Let* (X, τ_1, τ_2) *be a bispace and let* $A \subset Y \subset X$. *If* A *is pairwise s.o. in* X*, it is pairwise s.o. in* Y.

Proof. Let A be τ_1 s.o.w.r.to τ_2. Then there exists a τ_1 open set O such that $O \subset A \subset \tau_2 clO$. Let $O_Y = Y \cap O$ which is τ_1 open in Y. So $O_Y = Y \cap O \subset Y \cap A \subset Y \cap \tau_2 clO = \tau_2 clO_Y$ in Y. Interchanging the role of τ_1 and τ_2 we get the result.

We denote the class of all τ_1 s.o.w.r.to τ_2 by $\tau_1 S.O.(X)_{\tau_2}$. □

Theorem 3.15. *Let* $\mathcal{B} = \{B_\alpha\}$ *be a collection of subsets of* X *such that (i)* $\tau_1 \subset \mathcal{B}$ *and (ii)* $B \in \mathcal{B}$ *and* $B \subset D \subset \tau_2 clB$ *imply* $D \in \mathcal{B}$*, then* $\tau_1 S.O.(X)_{\tau_2} \subset \mathcal{B}$.

Proof. Let $A \in \tau_1 S.O.(X)_{\tau_2}$, then there exists a τ_1 open set O such that $O \subset A \subset \tau_2 clO$. Therefore, $O \in \mathcal{B}$ and $O \subset A \subset \tau_2 clO$ imply that $A \in \mathcal{B}$ and hence the result follows.

We denote the set $\{\tau_1 intA : A \in \tau_1 S.O.(X)_{\tau_2}\}$ by $\tau_1 int(\tau_1 S.O.(X)_{\tau_2})$. Interchanging the role of τ_1 and τ_2 we may denote other such classes at our will.

From the construction, it is obvious that $\tau_1 \subset \tau_1 int(\tau_1 S.O.(X)_{\tau_2})$. But in general, τ_1 may not be equal to $\tau_1 int(\tau_1 S.O.(X)_{\tau_2})$ as shown in the following example. □

Example 3.16. *Let* (X, τ_1, τ_2) *be the bispace as in Example 3.8. Now for any* τ_1 *open set* G_i, $\tau_2 clG_i = [0,1] \cup Q_2$, *where* Q_2 *is set of all rational numbers in* $[1,2]$. *Let* $A = [0,1]$. *Then for any* τ_1 *open set* G_i, $G_i \subset A \subset [0,1] \cup Q_2 = \tau_2 clG_i$. *This implies that* A *is* τ_1 *s.o.w.r.to* τ_2, *i.e.,* $A \in \tau_1 S.O.(X)_{\tau_2}$. *But* $\tau_1 intA$ *is the set of all irrational numbers in [0,1] which is not* τ_1 *open.*

However equality $\tau_1 = \tau_1 int(\tau_1 S.O.(X)_{\tau_2})$ *holds if an additional condition holds.*

Theorem 3.17. *In a bispace* (X, τ_1, τ_2), $\tau_1 = \tau_1 int(\tau_1 S.O.(X)_{\tau_2})$ *if and only if the condition* C_2 *is satisfied.*

C_2 *: For any* $A \subset X$ *which is* τ_1 *s.o.w.r.to* τ_2, *there exists a maximal* τ_1 *open set* O *such that* $O \subset A \subset \tau_2 clO$.

Proof. First assume that $\tau_1 = \tau_1 int(\tau_1 S.O.(X)_{\tau_2})$, and let A be any subset of X which is τ_1 s.o.w.r.to τ_2. Then $\tau_1 intA \in \tau_1$. Also by theorem 3.2, $A \subset \tau_2 cl(\tau_1 intA)$. Again if G is any τ_1 open set satisfying $G \subset A \subset \tau_2 clG$, then $G \subset \tau_1 intA$. Hence $\tau_1 intA$ is the maximal τ_1 open set contained in A such that $\tau_1 intA \subset A \subset \tau_2 cl(\tau_1 intA)$. Taking $O = \tau_1 intA$, we get $O \subset A \subset \tau_2 clO$.

Conversely let $A \in \tau_1 S.O.(X)_{\tau_2}$. By the condition, there exists a maximal τ_1 open set O such that $O \subset A \subset \tau_2 clO$......(1).
If possible let $O \neq \tau_1 intA$. Then there exists a τ_1 open set $G \subset A$ such that G is not contained in O . Since $O \cup G$ is τ_1 open set and $O \cup G \subset A \subset \tau_2 clO \subset \tau_2 cl(O \cup G)$, this contradicts that O is maximal satisfying the condition (1). Hence $\tau_1 intA = O$ and so $\tau_1 intA$ is a τ_1 open set, i.e., $\tau_1 intA \in \tau_1$. Therefore $\tau_1 int(\tau_1 S.O.(X)_{\tau_2}) \subset \tau_1$ and consequently $\tau_1 = \tau_1 int(\tau_1 S.O.(X)_{\tau_2})$. □

Remark 3.18. *We see that there is a bispace which is not bitopological space where the condition* C_2 *holds. For, consider the bispace* (X, τ_1, τ_2) *as in Example 3.11 where the* τ_1 *open sets are the only* τ_1 *s.o.sets w.r.to* τ_2. *So for any set* A *which* τ_1 *s.o.w.r.to* τ_2 *in* X, *there exists a maximal* τ_1 *open set* $O(= A)$ *such that* $O \subset A \subset \tau_2 clO$.

We now give a necessary and sufficient condition in terms of semi open sets for a bispace to be a bitopological space.

Theorem 3.19. *A bispace* (X, τ_1, τ_2) *is a bitopological space if and only if following condition holds:*
(i) arbitrary union of $\tau_1(\tau_2)$ *s.o.sets w.r.to* $\tau_2(\tau_1)$ *is* $\tau_1(\tau_2)$ *s.o.w.r.to* $\tau_2(\tau_1)$
(ii) $\tau_1 = \tau_1 int(\tau_1 S.O.(X)_{\tau_2})$ *and* $\tau_2 = \tau_2 int(\tau_2 S.O.(X)_{\tau_1})$.

Proof. If (X, τ_1, τ_2) is a bitopological space then (i) holds[3] . For (ii), let $O \in \tau_1$. Then $O \in \tau_1 S.O.(X)_{\tau_2}$ and since $O = \tau_1 intO$, $O \in \tau_1 int(\tau_1 S.O.(X)_{\tau_2})$. Therefore, $\tau_1 \subset \tau_1 int(\tau_1 S.O.(X)_{\tau_2})$. On the other hand, let $O \in \tau_1 int(\tau_1 S.O.(X)_{\tau_2})$. Then $O = \tau_1 intA$ for some $A \in \tau_1 S.O.(X)_{\tau_2}$ and hence $O \in \tau_1$. Therefore, $\tau_1 int(\tau_1 S.O.(X)_{\tau_2}) \subset \tau_1$. Therefore, $\tau_1 = \tau_1 int(\tau_1 S.O.(X)_{\tau_2})$. Similarly we can prove that $\tau_2 = \tau_2 int(\tau_2 S.O.(X)_{\tau_1})$.

Conversely, it suffices to show that an arbitrary union of $\tau_1(\tau_2)$ open sets is $\tau_1(\tau_2)$ open in

(X, τ_1, τ_2). Let $\{G_i\}$ be an arbitrary collection of τ_1 open sets and $G = \cup G_i$. Each G_i being τ_1 open is τ_1 s.o.w.r.to τ_2. So by (i) G is τ_1 s.o.w.r.to τ_2. Then by (ii) $\tau_1 \mathrm{int} G$ is τ_1 open. So $\tau_1 \mathrm{int} G = G$ and similarly arbitrary union of τ_2 open sets is τ_2 open set and this proves the theorem. □

Definition 3.20. *(cf. [7]): Let (X, τ_1, τ_2) be a bispace. Two non empty subsets A and B are said to be*
(i) pairwise weakly separated if there exist a τ_1 open set U and a τ_2 open set V such that $A \subset U$, $B \subset V$, $A \cap V = \phi$, $B \cap U = \phi$.
(ii) pairwise strongly separated if there exists a τ_1 open set U and a τ_2 open set V such that $A \subset U$, $B \subset V$, $U \cap V = \phi$.

Definition 3.21. *(cf.[7]): A subset A in a bispace (X, τ_1, τ_2) is said to be pairwise connected if it can not be expressed as the unions of two pairwise weakly separated sets.*

Remark 3.22. *In [3] it is proved that in a bitopological space (X, τ_1, τ_2) if $A = O \cup B$ where (i) $O \neq \phi$ is τ_1 open (ii) A is pairwise connected and (iii) B'_{τ_1} , the derived set of B w.r.to τ_1 is empty, then A is τ_1 s.o.w.r.to τ_2. But this is not true in a bispace as shown in the following example.*

Example 3.23. *Let $X = ([0,1] - Q) \cup \{\sqrt{2}\}$ and $\{G_i\}$ be the collection of all countable subsets of $[0,1] - Q$ where Q is the set of all rational numbers. Let τ_1 be the collection of all sets $G_i \cup \{\sqrt{2}\}$ together with X and ϕ, and τ_2 be the collection of all sets G_i together with X and ϕ where $G_i \in \{G_i\}$. Then (X, τ_1, τ_2) is a bispace. Let $A = ([0, \frac{1}{2}] - Q) \cup \{\sqrt{2}\}$. Then A is pairwise connected, because if $A = A_1 \cup A_2$, then at least one of A_1 and A_2 say A_1 is uncountable and X is the only τ_1 open set containing A_1. Let G be a nonempty countable subset of $[0, \frac{1}{2}] - Q$ and $O = G \cup \{\sqrt{2}\}$. Then $A = O \cup (A - O)$ where $O \neq \phi$, is τ_1 open. Also $(A - O)'_{\tau_1}$, the set of all τ_1 limit points of $A - O$ is empty. Indeed if $\alpha \in [0,1] - Q$, then $\{\alpha, \sqrt{2}\}$ is a τ_1 open set containing α satisfying $\{\alpha, \sqrt{2}\} \cap ((A - O) - \{\alpha\}) = \phi$. Again if $\alpha = \sqrt{2}$ then for any $p \in G$, $\{p, \alpha\}$ is a τ_1 open set containing α satisfying $\{p, \alpha\} \cap ((A - O) - \{\alpha\}) = \phi$. Thus no point of X can be a τ_1 limit point of $A - O$.*

So all the conditions stated in the Remark 3.22 above are satisfied, but A is not τ_1 s.o.w.r.to τ_2, because if $G_i \cup \{\sqrt{2}\}$ is any τ_1 open set contained in A, then $\tau_2 \mathrm{cl}(G_i \cup \{\sqrt{2}\}) = G_i \cup \{\sqrt{2}\}$.

However the following theorem is true.

Theorem 3.24. *Let (X, τ_1, τ_2) be a bispace. If $A = O \cup B$ where (i) $O(\neq \phi)$ is τ_1 open (ii) A is pairwise connected (iii) there exists a τ_2 closed set $F_1 \supset O$ such that $B \cap F_1 \subset G \subset \tau_2 \mathrm{cl} O$ for some τ_1 open set G, then A is τ_1 s.o.w.r.to τ_2.*

Proof. We show that $B \subset \tau_2 \mathrm{cl} O$. If $B \not\subset \tau_2 \mathrm{cl} O$ then there exists a τ_2 closed set $F \supset O$ such that $B \not\subset F$. Let $B_1 = B \cap F$, $B_2 = B - B_1$. Then $B_2 \subset X - F$ and $B_2 \neq \phi$. Further, let $B_1^* = B_1 \cap F_1$ and $B_1^{**} = B_1 \cap (X - F_1)$. Then $A = O \cup B = O \cup B_1 \cup B_2 = (O \cup B_1^*) \cup (B_1^{**} \cup B_2)$. Now $B_1^{**} \cup B_2 \subset (X - F_1) \cup (X - F) = X - (F \cap F_1) = G_2$ (say), which is τ_2 open. Since $B_1^* =$

$B_1 \cap F_1 = B \cap F \cap F_1 \subset B \cap F_1 \subset G$, we have $O \cup B_1^* \subset O \cup G = G_1$ (say), which is τ_1 open set. Since $G \subset \tau_2 clO \subset F \cap F_1$ and $G_1 = O \cup G \subset F \cap F_1$, it follows that $G_1 \cap G_2 = \phi$. This implies that $O \cup B_1^*$, $B_1^{**} \cup B_2$ are non empty strongly separated sets, a contradiction. Hence $O \subset A = O \cup B \subset \tau_2 clO$ and so A is τ_1 s.o.w.r.to τ_2. □

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