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Degenerate k-regularized (C_1, C_2)-existence and uniqueness families

Marko Kostić [1]
Faculty of Technical Sciences,
University of Novi Sad,
Trg D. Obradovića 6, 21125 Novi Sad, Serbia.
marco.s@verat.net

ABSTRACT

In this paper, we consider various classes of degenerate k-regularized (C_1, C_2)-existence and uniqueness families. The main purpose of the paper is to report how the techniques established in a joint paper of C.-G. Li, M. Li and the author [32] can be successfully applied in the analysis of a wide class of abstract degenerate multi-term fractional differential equations with Caputo derivatives.

RESUMEN

En este artículo, consideramos varias clases de familias k-regularizadas (C_1, C_2)-de existencia y unicidad. El principal objetivo de este trabajo es mostrar como las técnicas establecidas en un trabajo conjunto de C.-G. Li, M. Li y el autor [27], pueden ser aplicadas satisfactoriamente en el análisis de una clase amplia de ecuaciones fracionarias multi-término degeneradas con derivadas de Caputo.

Keywords and Phrases: Abstract multi-term fractional differential equations, degenerate differential equations, fractional calculus, Mittag-Leffler functions, Caputo time-fractional derivatives.

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1 Introduction and preliminaries

During the past three decades, considerable interest in fractional calculus and fractional differential equations has been stimulated due to their numerous applications in engineering, physics, chemistry, biology and other sciences. Basic information about fractional calculus and non-degenerate fractional differential equations can be obtained by consulting [6], [15], [23]-[25], [42]-[44] and the references cited therein. For the basic source of information on the abstract degenerate differential equations, we refer the reader to [1], [3], [5], [7], [12], [17], [35], [40]-[41], [45]-[49] and [52]-[53].

The theory of abstract degenerate (multi-term) fractional differential equations is at its beginning stage and we can freely say that it is a still-undeveloped subject. The most important qualitative properties of abstract degenerate (multi-term) fractional differential equations have been recently considered in the papers [19]-[20], [22], [27]-[30] and [34]. The existence and uniqueness of solutions of the Cauchy and Showalter problems for a class of degenerate fractional evolution systems have been analyzed by V. E. Fedorov and A. Debbouche in [19], while the necessary and sufficient conditions for the relative p-boundedness of a pair of operators have been obtained by V. E. Fedorov and D. M. Gordievskikh in [20]. In [27]-[28], the author has investigated degenerate Volterra integro-differential equations in locally convex spaces, as well as the generation of degenerate fractional resolvent operator families associated with abstract differential operators and the generation of various classes of exponentially equicontinuous k-regularized C-resolvent propagation families associated with the degenerate multi-term problem (1.1) below. The hypercyclic and topologically mixing properties of degenerate multi-term fractional differential equations with Caputo derivatives have been analyzed in [29]-[30]. Among many other things, in a joint research study with V. E. Fedorov [22], the author has analyzed the existence and uniqueness of regularized solutions for a class of abstract degenerate multi-term fractional differential equations with Caputo derivatives. The abstract degenerate multi-term fractional differential equations with classical Riemann-Liouville fractional derivatives have been recently investigated by the author in [34], following the methods used in [33] and this paper.

The main subject under our consideration is the following degenerate multi-term problem:

$$\begin{aligned} &B\mathbf{D}_t^{\alpha_n}u(t)+\sum_{i=1}^{n-1}A_i\mathbf{D}_t^{\alpha_i}u(t)=A\mathbf{D}_t^{\alpha}u(t)+f(t), \quad t\geq 0;\\ &u^{(j)}(0)=u_j,\ j=0,\cdots,\lceil\alpha_n\rceil-1, \end{aligned} \tag{1.1}$$

where $n\in\mathbb{N}\setminus\{1\}$, A, B and $A_1,\cdots,A_{n-1}$ are closed linear operators on a sequentially complete locally convex space X, $0\leq\alpha_1<\cdots<\alpha_n$, $0\leq\alpha<\alpha_n$, $f(t)$ is an X-valued function, and $\mathbf{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α ([6], [25]). Define $A_n:=B$, $A_0:=A$, $m:=\lceil\alpha\rceil$, $\alpha_0:=\alpha$ and $m_i:=\lceil\alpha_i\rceil$, $i\in\mathbb{N}_n^0$, where $\mathbb{N}_n:=\{1,2,\cdots,n\}$ and $\mathbb{N}_n^0:=\mathbb{N}_n\cup\{0\}$.

As mentioned in the abstract, the main purpose of this paper is to reconsider the various notions of non-degenerate k-regularized (C_1,C_2)-existence and uniqueness families introduced in the paper [32], whose organization is very similar to that of this paper. Without any doubt, this causes the expositority of our paper in a certain sense. On the other hand, we will not be in wrong if we say that our paper proposes an important theoretical novelty method capable of seeking of solutions of some very atypical degenerate differential equations in L^p-spaces. In this place, it is also worth noting that we initiate the analysis of existence of local solutions of abstract degenerate differential equations in this paper; furthermore, we provide generalizations of [36, Theorem 2.3, Theorem 3.1] for degenerate multi-term problems, and successfully apply the obtained theoretical results in the analysis of some very interesting degenerate differential equations.

Before explaining the notation used in the paper, we would like to note that it is quite questionable whether there exists any other significant reference which treats the existence and uniqueness of various types of automorphic solutions to abstract degenerate multi-term fractional differential equations (cf. [2], [8]-[9] and [13]-[14] for some results in the non-degenerate case). Unless specifed otherwise, we assume that X is a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short. If Y is also an SCLCS over the same field of scalars as X, then we denote by $L(Y, X)$ the space consisting of all continuous linear mappings from Y into X; $L(X) \equiv L(X, X)$. By $\circledast_X$ ($\circledast$, if there is no risk for confusion), we denote the fundamental system of seminorms which defines the topology of X. The fundamental system of seminorms which defines the topology on Y is denoted by $\circledast_Y$. The symbol I denotes the identity operator on X. Let $0 < \tau \leq \infty$. A strongly continuous operator family $(W(t))_{t\in[0,\tau)} \subseteq L(Y, X)$ is said to be locally equicontinuous iff, for every $T \in (0, \tau)$ and for every $p \in \circledast_X$, there exist $q_p \in \circledast_Y$ and $c_p > 0$ such that $p(W(t)y) \leq c_p q_p(y)$, $y \in Y$, $t \in [0, T]$; the notions of equicontinuity of $(W(t))_{t\in[0,\tau)}$ and the exponential equicontinuity of $(W(t))_{t\geq 0}$ are defined similarly. Notice that $(W(t))_{t\in[0,\tau)}$ is automatically locally equicontinuous in case that the space Y is barreled ([39]).

By $\mathcal{B}$ we denote the family consisting of all bounded subsets of Y. Define $p_{\mathbb{B}}(T) := \sup_{y\in\mathbb{B}} p(Ty)$, $p \in \circledast_X$, $\mathbb{B} \in \mathcal{B}$, $T \in L(Y, X)$. Then $p_{\mathbb{B}}(\cdot)$ is a seminorm on $L(Y, X)$ and the system $(p_{\mathbb{B}})_{(p,\mathbb{B})\in\circledast_X\times\mathcal{B}}$ induces the Hausdorff locally convex topology on $L(Y, X)$. If X is a Banach space, then we denote by $\|x\|$ the norm of an element $x \in X$. Suppose that A is a closed linear operator acting on X. Then we denote the domain, kernel space and range of A by $D(A)$, $N(A)$ and $R(A)$, respectively. Since no confusion seems likely, we will identify A with its graph. Set $p_A(x) := p(x) + p(Ax)$, $x \in D(A)$, $p \in \circledast$. Then the calibration $(p_A)_{p\in\circledast}$ induces the Hausdorff sequentially complete locally convex topology on $D(A)$; we denote this space simply by $[D(A)]$. If V is a general topological vector space, then a function $f : \Omega \to V$, where Ω is an open non-empty subset of $\mathbb{C}$, is said to be analytic if it is locally expressible in a neighborhood of any point $z \in \Omega$ by a uniformly convergent power series with coefficients in V. We refer the reader to [4], [25, Section 1.1] and references cited there for the basic information about vector-valued analytic functions. In our approach the space X is sequentially complete, so that the analyticity of a mapping $f : \Omega \to X$ is equivalent with its weak analyticity. It is said that a function $f : [0, \infty) \to E$ is locally Hölder continuous with the exponent $r \in (0, 1]$ iff for each $p \in \circledast$ and $T > 0$ there exists $M \geq 1$ such that $p(f(t) - f(s)) \leq M|t - s|^r$, provided $0 \leq t, s \leq T$.

Sometimes we use the following condition on a scalar-valued function $K(\cdot)$:

(P1) $K(\cdot)$ is Laplace transformable, i.e., it is locally integrable on $[0, \infty)$ and there exists $\beta \in \mathbb{R}$ so that
$$\tilde{K}(\lambda) := \mathcal{L}(K)(\lambda) := \lim_{b \to \infty} \int_0^b e^{-\lambda t} K(t)\, dt := \int_0^\infty e^{-\lambda t} K(t)\, dt$$
exists for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \beta$. Put abs$(K)$:=inf$\{\mathrm{Re}\lambda : \tilde{K}(\lambda)$ exists$\}$, and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

We say that a function $h(\cdot)$ belongs to the class $LT-E$ iff there exists a function $f \in C([0,\infty) : E)$ such that for each $p \in \circledast$ there exists $M_p > 0$ satisfying $p(f(t)) \leq M_p e^{at}$, $t \geq 0$ and $h(\lambda) = (\mathcal{L}f)(\lambda), \lambda > a$. The reader may consult [4], [51, Chapter 1] and [25, Section 1.2] for the basic properties of vector-valued Laplace transform.

Given $\theta \in (0, \pi]$ in advance, define $\Sigma_\theta := \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \theta\}$. Further on, $\lceil \beta \rceil := \inf\{n \in \mathbb{Z} : \beta \leq n\}$ $(\beta \in \mathbb{R})$. A scalar-valued function $k \in L^1_{loc}([0, \tau))$ is said to be a kernel on $[0, \tau)$ iff for any scalar-valued continuous function $t \mapsto u(t)$, $t \in [0, \tau)$, the preassumption $\int_0^t k(t-s)u(s)\, ds = 0$, $t \in [0, \tau)$ implies $u(t) = 0$, $t \in [0, \tau)$. If $\tau < \infty$, then the Titchmarsh–Foiaş theorem (see e.g. [24, Theorem 3.4.40]) states that the function $k(t)$ is a kernel on $[0, \tau)$ iff $0 \in \mathrm{supp}(k)$; on the other hand, if $\tau = \infty$ and $k \neq 0$ in $L^1_{loc}([0, \infty))$, then it is well known that the function $k(t)$ is automatically a kernel on $[0, \infty)$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers; the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)\, ds$. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $0^\zeta := 0$ $(\zeta > 0,\ t > 0)$ and $g_0(t) :=$ the Dirac δ-distribution. If $f : [0, \infty) \to X$ is a continuous function, then we set $g_0 * f \equiv f$. The reader may consult [43, Definition 4.5, p. 96] for the notion of a completely positive function on $[0, \infty)$ (cf. also [37, Remark 3.6, (3.3)]). Denote by $S^{\alpha,p}(\mathbb{R}^n)$ the fractional Sobolev space of order α (cf. [38, Definition 12.3.1, p. 297]).

Let $\zeta > 0$. Then the Caputo fractional derivative $\mathbf{D}_t^\zeta u$ ([6], [25]) is defined for those functions $u \in C^{\lceil \zeta \rceil - 1}([0, \infty) : E)$ for which
$g_{\lceil \zeta \rceil - \zeta} * (u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0) g_{j+1}) \in C^{\lceil \zeta \rceil}([0, \infty) : E)$, by
$$\mathbf{D}_t^\zeta u(t) := \frac{d^{\lceil \zeta \rceil}}{dt^{\lceil \zeta \rceil}} \left[g_{\lceil \zeta \rceil - \zeta} * \left(u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0) g_{j+1} \right) \right].$$

If the Caputo fractional derivative $\mathbf{D}_t^\zeta u(t)$ exists, then for each number $\nu \in (0, \zeta)$ the Caputo fractional derivative $\mathbf{D}_t^\nu u(t)$ exists, as well, and the following equality holds:
$$\mathbf{D}_t^\nu u(t) = \big(g_{\zeta-\nu} * \mathbf{D}_t^\zeta u(\cdot)\big)(t) + \sum_{j=\lceil \nu \rceil}^{\lceil \zeta \rceil - 1} u^{(j)}(0) g_{j+1-\nu}(t), \quad t \geq 0. \tag{1.2}$$

The Mittag-Leffler function $E_{\beta,\gamma}(z)$ $(\beta > 0,\ \gamma \in \mathbb{R})$ is defined by
$$E_{\beta,\gamma}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \gamma)}, \quad z \in \mathbb{C}.$$

In this place, we assume that $1/\Gamma(\beta k+\gamma)=0$ if $\beta k+\gamma\in -\mathbb{N}_0$. Set, for short, $E_\beta(z):=E_{\beta,1}(z)$, $z\in\mathbb{C}$. Let $\beta\in(0,1)$. Then the Wright function $\Phi_\beta(\cdot)$ is defined by

$$\Phi_\beta(t):=\mathcal{L}^{-1}\big(E_\beta(-\lambda)\big)(t),\quad t\geq 0.$$

For further information about the Mittag-Leffler and Wright functions, cf. [6], [25] and references cited there.

2 Degenerate k-regularized (C_1, C_2)-existence and uniqueness propagation families for (1.1)

We start this section by recalling that $n\in\mathbb{N}\setminus\{1\}$, $0\leq\alpha_1<\cdots<\alpha_n$, $0\leq\alpha<\alpha_n$, as well as that A, B and $A_1,\cdots,A_{n-1}$ are closed linear operators acting on X. Further on, $A_n=B$, $A_0=A$, $m=\lceil\alpha\rceil$, $\alpha_0=\alpha$ and $m_i=\lceil\alpha_i\rceil$, $i\in\mathbb{N}_n^0$. Set $D_i:=\{j\in\mathbb{N}_{n-1}: m_j-1\geq i\}$ $(i\in\mathbb{N}_{m_n-1}^0)$.

Let $T>0$ and $f\in C([0,T]:E)$. By a strong solution of problem (1.1) on the interval $[0,T]$ we mean any continuous function $t\mapsto u(t)$, $t\in[0,T]$ satisfying that the term $A_i\mathbf{D}_t^{\alpha_i}u(t)$ is well-defined and continuous on $[0,T]$ $(i\in\mathbb{N}_n^0)$, as well as that (1.1) holds identically on $[0,T]$. Convoluting both sides of (1.1) with $g_{\alpha_n}(t)$, we get that:

$$\begin{aligned} &B\Bigg[u(\cdot)-\sum_{k=0}^{m_n-1}u_k g_{k+1}(\cdot)\Bigg]+\sum_{j=1}^{n-1}g_{\alpha_n-\alpha_j}*A_j\Bigg[u(\cdot)-\sum_{k=0}^{m_j-1}u_k g_{k+1}(\cdot)\Bigg]\\ &\quad=g_{\alpha_n-\alpha}*A\Bigg[u(\cdot)-\sum_{k=0}^{m-1}u_k g_{k+1}(\cdot)\Bigg]+\big(g_{\alpha_n}*f\big)(\cdot),\quad t\in[0,T]. \end{aligned}\tag{2.1}$$

By a mild solution of (1.1) on $[0,T]$ we mean any continuous X-valued function $t\mapsto u(t)$, $t\in[0,T]$ satisfying

$$\begin{aligned} &B\Bigg[u(\cdot)-\sum_{k=0}^{m_n-1}u_k g_{k+1}(\cdot)\Bigg]+\sum_{j=1}^{n-1}A_j\Bigg(g_{\alpha_n-\alpha_j}*\Bigg[u(\cdot)-\sum_{k=0}^{m_j-1}u_k g_{k+1}(\cdot)\Bigg]\Bigg)\\ &\quad=A\Bigg(g_{\alpha_n-\alpha}*\Bigg[u(\cdot)-\sum_{k=0}^{m-1}u_k g_{k+1}(\cdot)\Bigg]\Bigg)+\big(g_{\alpha_n}*f\big)(\cdot),\quad t\in[0,T]. \end{aligned}$$

Consider the following inhomogeneous integral equation:

$$Bu(t)+\sum_{j=1}^{n-1}\big(g_{\alpha_n-\alpha_j}*A_j u\big)(t)=f(t)+\big(g_{\alpha_n-\alpha}*Au\big)(t),\quad t\in[0,T].\tag{2.2}$$

Similarly to the above, we say that a function $u\in C([0,T]:E)$ is:

(i) a strong solution of (2.2) iff $A_j u\in C([0,T]:E)$, $j\in\mathbb{N}_{n-1}^0$ and (2.2) holds for every $t\in[0,T]$.

(ii) a mild solution of (2.2) iff $(g_{\alpha_n-\alpha_j} * u)(t) \in D(A_j)$, $t \in [0,T]$, $j \in \mathbb{N}^0_{n-1}$ and

$$Bu(t) + \sum_{j=1}^{n-1} A_j\big(g_{\alpha_n-\alpha_j} * u\big)(t) = f(t) + A\big(g_{\alpha_n-\alpha} * u\big)(t), \quad t \in [0,T].$$

A mild (strong) solution of problem (1.1), resp. (2.2), on $[0,\infty)$ is defined analogously.

We will be interested in the following notions.

Definition 2.1. (cf. [32, Definition 2.2] for the case $B = I$) Suppose $0 < \tau \le \infty$, $k \in C([0,\tau))$, $C,\ C_1,\ C_2 \in L(X)$, C and C_2 are injective.

(i) A sequence $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ of strongly continuous operator families in $L(X,[D(B)])$ is called a (local, if $\tau < \infty$) k-regularized C_1-existence propagation family for (1.1) iff, for every $i = 0, \cdots, m_n - 1$, the following holds:

$$\begin{aligned}
&B\Big[R_i(\cdot)x - \big(k * g_i\big)(\cdot)C_1x\Big] \\
&\quad + \sum_{j\in D_i} A_j\Big[g_{\alpha_n-\alpha_j} * \Big(R_i(\cdot)x - \big(k * g_i\big)(\cdot)C_1x\Big)\Big] \\
&\quad + \sum_{j\in \mathbb{N}_{n-1}\setminus D_i} A_j\big(g_{\alpha_n-\alpha_j} * R_i\big)(\cdot)x \\
&\quad = \begin{cases} A\big(g_{\alpha_n-\alpha} * R_i\big)(\cdot)x, & m-1 < i,\ x \in X, \\ A\Big[g_{\alpha_n-\alpha} * \big(R_i(\cdot)x - \big(k * g_i\big)(\cdot)C_1x\big)\Big](\cdot), & m-1 \ge i,\ x \in X. \end{cases}
\end{aligned}$$

(ii) A sequence $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ of strongly continuous operator families in $L(X)$ is called a (local, if $\tau < \infty$) k-regularized C_2-uniqueness propagation family for (1.1) iff

$$\begin{aligned}
&\Big[R_i(\cdot)Bx - \big(k * g_i\big)(\cdot)C_2Bx\Big] \\
&\quad + \sum_{j\in D_i} g_{\alpha_n-\alpha_j} * \Big[R_i(\cdot)A_jx - \big(k * g_i\big)(\cdot)C_2A_jx\Big] \\
&\quad + \sum_{j\in \mathbb{N}_{n-1}\setminus D_i} \big(g_{\alpha_n-\alpha_j} * R_i(\cdot)A_jx\big)(\cdot) \\
&\quad = \begin{cases} \big(g_{\alpha_n-\alpha} * R_i(\cdot)Ax\big)(\cdot), & m-1 < i, \\ g_{\alpha_n-\alpha} * \Big[R_i(\cdot)Ax - \big(k * g_i\big)(\cdot)C_2Ax\Big](\cdot), & m-1 \ge i, \end{cases}
\end{aligned}$$

for any $x \in \bigcap_{0\le j\le n} D(A_j)$ and $i \in \mathbb{N}^0_{m_n-1}$.

(iii) A sequence $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ of strongly continuous operator families in $L(X)$ is called a (local, if $\tau < \infty$) k-regularized C-resolvent propagation family for (1.1), in short k-regularized C-propagation family for (1.1), iff $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a k-regularized C-uniqueness propagation family for (1.1), and for every $t \in [0,\tau)$, $i \in \mathbb{N}^0_{m_n-1}$ and $j \in \mathbb{N}^0_n$, one has $R_i(t)A_j \subseteq A_jR_i(t)$, $R_i(t)C = CR_i(t)$ and $CA_j \subseteq A_jC$.

In case $k(t) = g_{\zeta+1}(t)$, where $\zeta \geq 0$, it is also said that $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a ζ-times integrated C_1-existence propagation family for (1.1); 0-times integrated C_1-existence propagation family for (1.1) is simply called C_1-existence propagation family for (1.1). For a k-regularized C_1-existence propagation family $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$, it is said that is locally equicontinuous (exponentially equicontinuous) iff each single operator family
$(R_0(t))_{t\in[0,\tau)} \subseteq L(X, [D(B)])$, $\cdots$, $(R_{m_n-1}(t))_{t\in[0,\tau)} \subseteq L(X, [D(B)])$ is;
$((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$ is said to be an exponentially equicontinuous k-regularized C_1-existence propagation family for problem (1.1), of angle $\alpha \in (0, \pi/2]$, iff the following holds:

(a) For every $x \in E$ and $i \in \mathbb{N}^0_{m_n-1}$, the mappings $t \mapsto R_i(t)x$, $t > 0$ and $t \mapsto BR_i(t)x$, $t > 0$ can be analytically extended to the sector Σ_α; since no confusion seems likely, we shall denote these extensions by the same symbols.

(b) For every $x \in E$, $\beta \in (0, \alpha)$ and $i \in \mathbb{N}^0_{m_n-1}$, one has $\lim_{z\to 0, z\in\Sigma_\beta} R_i(z)x = R_i(0)x$ and $\lim_{z\to 0, z\in\Sigma_\beta} BR_i(z)x = BR_i(0)x$.

(c) For every $\beta \in (0, \alpha)$ and $i \in \mathbb{N}^0_{m_n-1}$, there exists $\omega_\beta \geq \max(0, \mathrm{abs}(k))$ ($\omega_\beta = 0$) such that the family $\{e^{-\omega_\beta z} R_i(z) : z \in \Sigma_\beta\} \subseteq L(E, [D(B)])$ is equicontinuous.

The above terminological agreements and abbreviations can be also understood for the classes of k-regularized C_2-uniqueness propagation families for (1.1) and k-regularized C-resolvent propagation families for (1.1).

The reader with a little experience can simply state a few noteworthy facts about the existence and uniqueness of solutions of mild (strong) solutions of problem (2.2) provided that there exists a k-regularized C_1-existence propagation family for problem (1.1) (k-regularized C_2-uniqueness propagation family for problem (1.1)); because of that, the corresponding discussion is omitted. The proof of following extension of [32, Proposition 2.3] is omitted, too.

Proposition 2.2. *Let $i \in \mathbb{N}^0_{m_n-1}$, and let $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ be a locally equicontinuous k-regularized C_1-existence propagation family for (1.1). If $R_i(t)A_j \subseteq A_j R_i(t)$ ($j \in \mathbb{N}^0_n$, $t \in [0, \tau)$), $R_i(t)C_1 = C_1 R_i(t)$ ($t \in [0, \tau)$), C_1 is injective, $k(t)$ is a kernel on $[0, \tau)$ and $C_1 A_j \subseteq A_j C_1$ ($j \in \mathbb{N}^0_n$), then the following holds:*

(i) The equality

$$R_i(t)R_i(s) = R_i(s)R_i(t), \quad 0 \leq t,\ s < \tau \tag{2.3}$$

holds, provided that $m - 1 < i$ and that the condition

($\diamond$) The assumption $Bf(t) + \sum_{j\in D_i} A_j(g_{\alpha_n-\alpha_j} * f)(t) = 0$, $t \in [0, \tau)$ for some $f \in C([0, \tau) : E)$, implies $f(t) = 0$, $t \in [0, \tau)$,

holds.

(ii) The equality (2.3) holds provided that $m - 1 \geq i$, $\mathbb{N}_{n-1} \setminus D_i \neq \emptyset$, and that the condition

($\diamond\diamond$) *If* $\sum_{j\in\mathbb{N}_{n-1}\setminus D_i} A_j(g_{\alpha_n-\alpha_j}*f)(t)=0$, $t\in[0,\tau)$, *for some* $f\in C([0,\tau):E)$, *then* $f(t)=0$, $t\in[0,\tau)$,

holds.

The assertions of [32, Proposition 2.5, Proposition 2.6] can be reformulated for degenerate multi-term problems. This is also the case with the assertion of generalized variation of parameters formula [32, Proposition 2.8]:

Theorem 2.3. *Let* $C_2\in L(X)$ *be injective. Suppose that* $((R_0(t))_{t\in[0,\tau)},\cdots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ *is a locally equicontinuous* k*-regularized* C_2*-uniqueness propagation family for (1.1),* $T\in(0,\tau)$ *and* $f\in C([0,T]:X)$. *Then the following holds:*

(i) If $m-1<i$, *then any strong solution* $u(t)$ *of (2.2) satisfies the equality:*

$$\big(R_i*f\big)(t)=\big(k*g_i*C_2Bu\big)(t)+\sum_{j\in D_i}\big(g_{\alpha_n-\alpha_j+i}*k*C_2A_ju\big)(t),$$

for any $t\in[0,T]$. *Therefore, there is at most one strong (mild) solution for (2.2), provided that* $k(t)$ *is a kernel on* $[0,\tau)$ *and* ($\diamond$) *holds.*

(ii) If $m-1\geq i$, *then any strong solution* $u(t)$ *of (2.2) satisfies the equality:*

$$\big(R_i*f\big)(t)=-\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\big(g_{\alpha_n-\alpha_j+i}*k*C_2A_ju\big)(t),\quad t\in[0,T].$$

Therefore, there is at most one strong (mild) solution for (2.2), provided that $k(t)$ *is a kernel on* $[0,\tau)$, $\mathbb{N}_{n-1}\setminus D_i\neq\emptyset$ *and* ($\diamond\diamond$) *holds.*

As explained in [25, Section 2.10], the notion of a k-regularized C-resolvent propagation family is probably the best theoretical concept for the investigation of integral solutions of non-degenerate abstract time-fractional equation (1.1) with $A_j\in L(E)$, $1\leq j\leq n-1$. If $A_j\notin L(E)$ for some $j\in\mathbb{N}_{n-1}$, then the vector-valued Laplace transform cannot be so easily applied, which certainly implies that there exist some limitations to this class of propagation families. A similar problem appears in the analysis od degenerate multi-term fractional differential equation (1.1) and, because of that, we will leave the problem of restating [32, Theorem 2.9(i), Theorem 2.10-Theorem 2.12] in our new framework to the reader's own exploration. In contrast to the above, it is very simple to reformulate the assertion of [32, Theorem 2.9(ii)] to degenerate equations, without imposing any additional barriers at:

Theorem 2.4. *Suppose* $k(t)$ *satisfies (P1),* $\omega\geq\max(0,abs(k))$, $(R_i(t))_{t\geq0}$ *is strongly continuous, and the family* $\{e^{-\omega t}R_i(t):t\geq0\}\subseteq L(X)$ *is equicontinuous* $(0\leq i\leq m_n-1)$. *Let* $C_2\in L(X)$ *be injective. Then* $((R_0(t))_{t\geq0},\cdots,(R_{m_n-1}(t))_{t\geq0})$ *is a global* k*-regularized* C_2*-uniqueness propagation family for (1.1) iff, for every* $\lambda\in\mathbb{C}$ *with* $\operatorname{Re}\lambda>\omega$, *and for every* $x\in\bigcap_{0\leq j\leq n}D(A_j)$, *the following* *equality holds:*

$$\int_0^\infty e^{-\lambda t}\big[R_i(t)Bx - \big(k * g_i\big)(t)C_2Bx\big]\,dt$$
$$+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha_n}\int_0^\infty e^{-\lambda t}\big[R_i(t)x - \big(k * g_i\big)(t)C_2A_jx\big]\,dt$$
$$+\sum_{j\in \mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j-\alpha_n}\int_0^\infty e^{-\lambda t}R_i(t)A_jx\,dt$$
$$=\begin{cases}\lambda^{\alpha-\alpha_n}\int_0^\infty e^{-\lambda t}R_i(t)Ax\,dt, & m-1<i,\\ \lambda^{\alpha-\alpha_n}\int_0^\infty e^{-\lambda t}\big[R_i(t)Ax - \big(k * g_i\big)(t)C_2Ax\big]\,dt, & m-1\geq i.\end{cases}$$

Now we would like to present an intriguing example of a local k-regularized I-resolvent propagation family for (1.1):

Example 2.5. (cf. [32, Example 5.2] for non-degenerate case) Suppose $1 \leq p \leq \infty$, $E := L^p(\mathbb{R})$, $m : \mathbb{R} \to \mathbb{C}$ is measurable, $a_j \in L^\infty(\mathbb{R})$, $(A_jf)(x) := a_j(x)f(x)$, $x \in \mathbb{R}$, $f \in E$ $(1 \leq j \leq n)$, $(Af)(x) := m(x)f(x)$, $x \in \mathbb{R}$, with maximal domain, and $\alpha = 0$. Assume $s \in (1, 2)$, $\delta = 1/s$, $M_p = p!^s$ and $k_\delta(t) = \mathcal{L}^{-1}(\exp(-\lambda^\delta))(t)$, $t \geq 0$. Denote by $M(t)$ the associated function of sequence (M_p) (cf. [24, Section 1.3] for more details) and put $\Lambda'_{\alpha',\beta',\gamma'} := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq \gamma'^{-1}M(\alpha'\lambda) + \beta'\}$, α', β', $\gamma' > 0$. Clearly, there exists a constant $C_s > 0$ such that $M(\lambda) \leq C_s|\lambda|^{1/s}$, $\lambda \in \mathbb{C}$. Assume that the following condition holds

(CH): For every $\tau > 0$, there exist $\alpha' > 0$, $\beta' > 0$ and $d > 0$ such that $\tau \leq \frac{\cos(\frac{\delta\pi}{2})}{C_s(\alpha')^{1/s}}$ and

$$\left|\sum_{j=1}^{n}\lambda^{\alpha_j-\alpha}a_j(x) - m(x)\right| \geq d, \quad x \in \mathbb{R},\ \lambda \in \Lambda_{\alpha',\beta',1}.$$

Notice that the above condition holds provided $n = 2$, $\alpha_2 = 2$, $\alpha_1 = 1$, $c_1 \in L^\infty(\mathbb{R})$, $|c_1(x)| \geq d_1 > 0$ for a.e. $x \in \mathbb{R}$, $a_2(x) \in L^\infty(\mathbb{R})$, $a_2(x) = 0$, $x \in (-1, 1)$, $a_1(x) = a_2(x)c_1(x)$ and $m(x) = \frac{1}{4}c_1^2(x)a_2(x) - \frac{1}{16}c_1^4(x)a_2(x) - a_2(x)$, $x \in \mathbb{R}$ (cf. [32, (5.7)]), and that the validity of condition (CH) does not imply, in general, the essential boundedness of function $m(\cdot)$ or the injectivity of the operator B. We will prove that there exists a global (not exponentially bounded, in general) k_δ-regularized I-resolvent propagation family $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$ for (1.1). Clearly, it suffices to show that, for every $\tau > 0$, there exists a local k_δ-regularized I-resolvent propagation family for (1.1) on $[0, \tau)$. Suppose $\tau > 0$ is given in advance, and $\alpha' > 0$, $\beta' > 0$ and $d > 0$ satisfy (CH), with this τ. Let Γ denote the upwards oriented boundary of ultra-logarithmic region

$\Lambda_{\alpha',\beta',1}$. Put, for every $t \in [0,\tau)$, $f \in E$ and $x \in \mathbb{R}$,

$$\big(R_i(t)f\big)(x) := \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t-\lambda^{\delta}} \frac{\Big[\lambda^{\alpha_n-\alpha-i}a_n(x)+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}a_j(x)\Big]f(x)}{\lambda^{\alpha_n-\alpha}a_n(x)+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)}\,d\lambda.$$

Then the analysis contained in [32, Example 5.2] shows that $((R_0(t))_{t\in[0,\tau)},\cdot\cdot\cdot,(R_{m_n-1}(t))_{t\in[0,\tau)})$ is a local k_δ-regularized I-resolvent propagation family for (1.1), as well as that, for every compact set $K \subseteq [0,\infty)$, there exists $h_K > 0$ such that

$$\sup_{t\in K,p\in\mathbb{N}_0,i\in\mathbb{N}^0_{m_n-1}} \frac{\Big\|h_K^p\frac{d^p}{dt^p}R_i(t)\Big\|}{p!^s} < \infty.$$

We can similarly consider the existence of local $k_{1/2}$-regularized I-resolvent propagation families for (1.1) which obey slight modifications of the properties stated above with $s = 2$, and with the operators A_j not belonging to the space $L(E)$ for some indexes $j \in \mathbb{N}_n$. Furthermore, we can similarly construct some relevant examples of local k-regularized I-resolvent propagation families for (1.1) in certain classes of Fréchet function spaces.

3 Degenerate k-regularized (C_1, C_2)-existence and uniqueness families for (1.1)

In this section, we investigate the class of degenerate k-regularized (C_1, C_2)-existence and uniqueness families for (1.1). Recall that $D_i = \{j \in \mathbb{N}_{n-1} : m_j - 1 \geq i\}$ ($i \in \mathbb{N}^0_{m_n-1}$), as well as that A, B and $A_1,\cdot\cdot\cdot,A_{n-1}$ are closed linear operators acting on X. By Y we denote another SCLCS over the same field of scalars as X.

In the following definition, we will generalize the notion introduced in our previous joint research with C.-G. Li and M. Li (cf. [32, Definition 3.1], [31], R. deLaubenfels [10]-[11], and T.-J. Xiao-J. Liang [54] for some other known concepts in the case that $B = I$).

Definition 3.1. Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $C_1 \in L(Y,X)$, and $C_2 \in L(X)$ is injective.

(i) A strongly continuous operator family $(E(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_1-existence family for (1.1) iff, for every $y \in Y$, the following holds: $E(\cdot)y \in C^{m_n-1}([0,\tau) : [D(B)])$, $E^{(i)}(0)y = 0$ for every $i \in \mathbb{N}_0$ with $i < m_n - 1$, $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(\cdot)y \in C([0,\tau) : X)$ for $0 \leq j \leq n$, and

$$\begin{aligned} BE^{(m_n-1)}(t)y + \sum_{j=1}^{n-1}A_j\big(g_{\alpha_n-\alpha_j} * E^{(m_n-1)}\big)(t)y \\ - A\big(g_{\alpha_n-\alpha} * E^{(m_n-1)}\big)(t)y = k(t)C_1y, \end{aligned} \tag{3.1}$$

for any $t \in [0,\tau)$.

(ii) A strongly continuous operator family $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_2-uniqueness family for (1.1) iff, for every $\tau \in [0, \tau)$ and $x \in \bigcap_{0\leq j\leq n} D(A_j)$, the following holds:

$$\begin{aligned} U(t)Bx + \sum_{j=1}^{n-1} \bigl(g_{\alpha_n-\alpha_j} * U(\cdot)A_j x\bigr)(t) \\ - \bigl(g_{\alpha_n-\alpha} * U(\cdot)Ax\bigr)(t)y = \bigl(k * g_{m_n-1}\bigr)(t)C_2x. \end{aligned} \tag{3.2}$$

(iii) A strongly continuous family $((E(t))_{t\in[0,\tau)}, (U(t))_{t\in[0,\tau)}) \subseteq L(Y, X) \times L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized (C_1, C_2)-existence and uniqueness family for (1.1) iff $(E(t))_{t\in[0,\tau)}$ is a k-regularized C_1-existence family for (1.1), and $(U(t))_{t\in[0,\tau)}$ is a k-regularized C_2-uniqueness family for (1.1).

(iv) Suppose $Y = X$ and $C = C_1 = C_2$. Then a strongly continuous operator family $(R(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized C-resolvent family for (1.1) iff $(R(t))_{t\in[0,\tau)}$ is a k-regularized C-uniqueness family for (1.1), $R(t)A_j \subseteq A_jR(t)$, for $0 \leq j \leq n$ and $t \in [0, \tau)$, as well as $R(t)C = CR(t)$, $t \in [0, \tau)$, and $CA_j \subseteq A_jC$, for $0 \leq j \leq n$.

If $k(t) = g_{\zeta+1}(t)$, where $\zeta \geq 0$, then it is also said that $(E(t))_{t\in[0,\tau)}$ is a ζ-times integrated C_1-existence family for (1.1); 0-times integrated C_1-existence family for (1.1) is also said to be a C_1-existence family for (1.1). A similar notion can be introduced for all other classes of uniqueness and resolvent families introduced in Definition 3.1.

Albeit the choice of an SCLCS space Y different from X can produce a larger set of initial data for which the abstract Cauchy problem (1.1) has a strong solution (see e.g. [54, Example 2.5]), in our furher work the most important case will be that in which $Y = X$. Keeping in mind that the operators A, B, $A_1, \cdots, A_{n-1}$ are closed, we can integrate the both sides of (3.1) sufficiently many times in order to see that:

$$\begin{aligned} BE^{(l)}(t)y + \sum_{j=1}^{n-1} A_j\bigl(g_{\alpha_n-\alpha_j} * E^{(l)}\bigr)(t)y \\ - A\bigl(g_{\alpha_n-\alpha} * E^{(l)}\bigr)(t)y = \bigl(k * g_{m_n-1-l}\bigr)(t)C_1y, \end{aligned} \tag{3.3}$$

for any $t \in [0, \tau)$, $y \in Y$ and $l \in \mathbb{N}^0_{m_n-1}$.

Proposition 3.2. *Suppose that* $((E(t))_{t\in[0,\tau)}, (U(t))_{t\in[0,\tau)})$ *is a* k*-regularized* (C_1, C_2)*-existence and uniqueness family for (1.1), and let* $(U(t))_{t\in[0,\tau)}$ *be locally equicontinuous. Then* $C_2E(t)y = U(t)C_1y$, $t \in [0, \tau)$, $y \in Y$.

Proof. The proof of proposition is almost the same as the corresponding proof of [32, Proposition 3.2]. Observe only that we can always assume, without loss of generality, that the number α is less than or equal to α_1. □

Definition 3.3. (cf. [32, Definition 3.3]) Suppose $0 \leq i \leq m_n - 1$. Then we define $D'_i := \{j \in \mathbb{N}^0_{n-1} : m_j - 1 \geq i\}$, $D''_i := \mathbb{N}^0_{n-1} \setminus D'_i$ and

$$\mathbf{D}_i := \Big\{u_i \in \bigcap_{j \in D''_i} D(A_j) : A_j u_i \in R(C_1),\ j \in D''_i\Big\}.$$

It is not so predictable that the assertion of [32, Theorem 3.4] continues to hold in degenerate case without any terminological changes, and that the operator B does not appear in the definition of set $\mathbf{D}_i$, for which it is well known that represents, in non-degenerate case, the set which consists of all initial values for which the homogeneous counterpart of abstract Cauchy problem (1.1), with $B = I$ and $u_j = 0$, $j \in \mathbb{N}^0_{m_n-1} \setminus \{i\}$, has a strong solution (provided that there exists a C_1-existence family for (1.1)). It is also worth nothing that we do not use the injectiveness of the operator B in (ii):

Theorem 3.4. *(i) Suppose* $(E(t))_{t\in[0,\tau)}$ *is a* C_1*-existence family for (1.1),* $T \in (0,\tau)$, *and* $u_i \in \mathbf{D}_i$ *for* $0 \leq i \leq m_n - 1$. *Then the function*

$$u(t) = \sum_{i=0}^{m_n-1} u_i g_{i+1}(t) - \sum_{i=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \Big(g_{\alpha_n-\alpha_j} * E^{(m_n-1-i)}\Big)(t)v_{i,j}$$
$$+ \sum_{i=m}^{m_n-1} \Big(g_{\alpha_n-\alpha} * E^{(m_n-1-i)}\Big)(t)v_{i,0}, \quad 0 \leq t \leq T,$$

is a strong solution of the problem (1.1) on $[0,T]$, *with* $f(t) \equiv 0$, *where* $v_{i,j} \in Y$ *satisfy* $A_j u_i = C_1 v_{i,j}$ *for* $0 \leq j \leq n-1$.

(ii) Suppose $(U(t))_{t\in[0,\tau)}$ *is a locally equicontinuous* k*-regularized* C_2*-uniqueness family for (1.1),* $T \in (0,\tau)$ *and* $0 \in$ *supp*(k). *Then there exists at most one strong (mild) solution of (1.1) on* $[0,T]$, *with* $u_i = 0$, $i \in \mathbb{N}^0_{m_n-1}$.

Proof. We will provide all the relevant details for the sake of completeness. Making use of (3.3), it can be easily verified that:

$$
\begin{aligned}
& B\left[u(\cdot)-\sum_{i=0}^{m_n-1} u_i g_{i+1}(\cdot)\right]+\sum_{j=1}^{n-1} A_j\left(g_{\alpha_n-\alpha_j} *\left[u(\cdot)-\sum_{i=0}^{m_j-1} u_i g_{i+1}(\cdot)\right]\right) \\
& =-\sum_{i=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i}\left(g_{\alpha_n-\alpha_j} * BE^{(m_n-1-i)}\right)(\cdot) v_{i,j} \\
& +\sum_{i=m}^{m_n-1}\left(g_{\alpha_n-\alpha} * BE^{(m_n-1-i)}\right)(\cdot) v_{i,0} \\
& +\sum_{j=1}^{n-1} A_j\left(g_{\alpha_n-\alpha_j} *\left\{\sum_{i=m_j}^{m_n-1} g_{i+1}(\cdot) u_i-\sum_{i=0}^{m_n-1} \sum_{l \in \mathbb{N}_{n-1} \setminus D_i}\left(g_{\alpha_n-\alpha_l} * E^{(m_n-1-i)}\right)(\cdot) v_{i,l}\right.\right. \\
& \left.\left.+\sum_{i=m}^{m_n-1}\left(g_{\alpha_n-\alpha} * E^{(m_n-1-i)}\right)(\cdot) v_{i,0}\right\}\right) \\
& =-\sum_{i=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i}\left(g_{\alpha_n-\alpha_j} * BE^{(m_n-1-i)}\right)(\cdot) v_{i,j} \\
& +\sum_{i=m}^{m_n-1}\left(g_{\alpha_n-\alpha} * BE^{(m_n-1-i)}\right)(\cdot) v_{i,0} \\
& +\sum_{j=1}^{n-1} \sum_{i=m_j}^{m_n-1} C_1 v_{i,j} g_{\alpha_n-\alpha_j+i+1}(\cdot)-\sum_{i=0}^{m_n-1} \sum_{l \in \mathbb{N}_{n-1} \setminus D_i} g_{\alpha_n-\alpha_l} *\left[-BE^{(m_n-1-i)}(\cdot) v_{i,l}\right. \\
& \left.+A\left(g_{\alpha_n-\alpha} * E^{(m_n-1-i)}\right)(\cdot) v_{i,l}+g_{i+1}(\cdot) C_1 v_{i,l}\right] \\
& +\sum_{i=m}^{m_n-1} g_{\alpha_n-\alpha} *\left[-BE^{(m_n-1-i)}(\cdot) v_{i,0}\right. \\
& \left.+A\left(g_{\alpha_n-\alpha} * R^{(m_n-1-i)}\right)(\cdot) v_{i,0}+g_{i+1}(\cdot) C_1 v_{i,0}\right] \\
& =g_{\alpha_n-\alpha} * A\left[u(\cdot)-\sum_{i=0}^{m-1} u_i g_{i+1}(\cdot)\right],
\end{aligned}
$$

since

$$
\sum_{j=1}^{n-1} \sum_{i=m_j}^{m_n-1} C_1 v_{i,j} g_{\alpha_n-\alpha_j+i+1}(\cdot)=\sum_{i=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} C_1 v_{i,j} g_{\alpha_n-\alpha_j+i+1}(\cdot).
$$

This implies that $u(t)$ is a mild solution of (1.1) on $[0, T]$. In order to complete the proof of (i), it suffices to show that $\mathbf{D}_t^{\alpha_n} u(t) \in C([0, T] : X)$ and $A_i \mathbf{D}_t^{\alpha_i} u \in C([0, T] : X)$ for all $i \in \mathbb{N}_n^0$. Towards this end, notice that the partial integration implies that, for every $t \in [0, T]$,

$$
\begin{aligned}
& g_{m_n-\alpha_n} *\left[u(\cdot)-\sum_{i=0}^{m_n-1} u_i g_{i+1}(\cdot)\right](t) \\
& =\sum_{i=m}^{m_n-1}\left(g_{m_n-\alpha+i} * E^{(m_n-1)}\right)(t) v_{i,0}-\sum_{i=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i}\left(g_{m_n-\alpha_j+i} * E^{(m_n-1)}\right)(t) v_{i,j}.
\end{aligned}
$$

Therefore, $\mathbf{D}_t^{\alpha_n} u \in C([0,T] : X)$ and, for every $t \in [0,T]$,

$$\mathbf{D}_t^{\alpha_n} u(t) = \frac{d^{m_n}}{dt^{m_n}} \Biggl\{ g_{m_n-\alpha_n} * \Biggl[u(\cdot) - \sum_{i=0}^{m_n-1} u_i g_{i+1}(\cdot) \Biggr](t) \Biggr\}$$

$$= \sum_{i=m}^{m_n-1} \Bigl(g_{i-\alpha} * E^{(m_n-1)} \Bigr)(t) v_{i,0} - \sum_{i=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \Bigl(g_{i-\alpha_j} * E^{(m_n-1)} \Bigr)(t) v_{i,j}, \qquad (3.4)$$

whence we may directly conclude that $B\mathbf{D}_t^{\alpha_n} u \in C([0,T] : X)$. Suppose, for the time being, $i \in \mathbb{N}_{n-1}^0$. Then $A_i u_j \in R(C_1)$ for $j \geq m_i$. Moreover, the inequality $l \geq \alpha_j$ holds provided $0 \leq l \leq m_n - 1$ and $j \in \mathbb{N}_{n-1} \setminus D_l$, and $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(\cdot)y \in C([0,T] : X)$ for $0 \leq j \leq n-1$ and $y \in Y$. Using (1.2) and (3.4), it is not difficult to prove that:

$$A_i \mathbf{D}_t^{\alpha_i} u(\cdot)$$
$$= \sum_{j=m_i}^{m_n-1} g_{j+1-\alpha_i}(\cdot) A_i u_j - \sum_{l=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_l} \Bigl[g_{l-\alpha_j} * A_i \bigl(g_{\alpha_n-\alpha_i} * E^{(m_n-1)} \bigr) \Bigr](\cdot) v_{l,j}$$
$$+ \sum_{l=m}^{m_n-1} \Bigl[g_{l-\alpha} * A_i \bigl(g_{\alpha_n-\alpha_i} * E^{(m_n-1)} \bigr) \Bigr](\cdot) v_{l,0} \in C([0,T] : X),$$

finishing the proof of (i). The second part of theorem can be proved as follows. Suppose $u(t)$ is a strong solution of (1.1) on $[0,T]$, with $u_i = 0$, $i \in \mathbb{N}_{m_n-1}^0$. Making use of (3.2) and the equality

$$\int_0^t \int_0^{t-s} g_{\alpha_n-\alpha_j}(r) U(t-s-r) A_j u(s)\, dr\, ds = \int_0^t \int_0^s g_{\alpha_n-\alpha_j}(r) U(t-s) A_j u(s-r)\, dr\, ds,$$

holding for any $t \in [0,T]$ and $j \in \mathbb{N}_{n-1}^0$, we have that

$$\bigl(UB * u\bigr)(t) = \bigl(k * g_{m_n-1} C_2 * u\bigr)(t)$$
$$+ \int_0^t \int_0^{t-s} \Bigl[g_{\alpha_n-\alpha_j}(r) U(t-s-r) A_j u(s) - g_{\alpha_n-\alpha}(r) U(t-s-r) A u(s) \Bigr] dr\, ds$$
$$= \bigl(k * g_{m_n-1} C_2 * u\bigr)(t) + \bigl(U * Bu\bigr)(t), \quad t \in [0,T].$$

Therefore, $(k * g_{m_n-1} C_2 * u)(t) = 0$, $t \in [0,T]$ and $u(t) = 0$, $t \in [0,T]$. $\square$

The standard proof of following theorem is omitted.

Theorem 3.5. *Suppose* $k(t)$ *satisfies (P1),* $(E(t))_{t\geq 0} \subseteq L(Y,X)$, $(U(t))_{t\geq 0} \subseteq L(X)$, $\omega \geq \max(0,$ *abs*$(k))$, $C_1 \in L(Y,X)$ *and* $C_2 \in L(X)$ *is injective. Set* $\mathbf{P}_\lambda := B + \sum_{j=1}^{n-1} \lambda^{\alpha_j-\alpha_n} A_j - \lambda^{\alpha-\alpha_n} A$, $\operatorname{Re}\lambda > 0$.

(i) *(a) Let* $(E(t))_{t\geq 0}$ *be a* k*-regularized* C_1*-existence family for (1.1), let the family* $\{e^{-\omega t} E(t) : t \geq 0\}$ *be equicontinuous, and let the family* $\{e^{-\omega t} A_j (g_{\alpha_n-\alpha_j} * E)(t) : t \geq 0\}$ *be equicontinuous* ($0 \leq j \leq n$)*. Then the following holds:*

$$\mathbf{P}_\lambda \int_0^\infty e^{-\lambda t} E(t) y\, dt = \tilde{k}(\lambda) \lambda^{1-m_n} C_1 y, \quad y \in Y, \ \operatorname{Re}\lambda > \omega.$$

(b) *Let the operator* $\mathbf{P}_\lambda$ *be injective for every* $\lambda > \omega$ *with* $\tilde{k}(\lambda) \neq 0$. *Suppose, additionally, that there exist strongly continuous operator families* $(W(t))_{t\geq 0} \subseteq L(Y, X)$ *and* $(W_j(t))_{t\geq 0} \subseteq L(Y, X)$ *such that* $\{e^{-\omega t}W(t) : t \geq 0\}$ *and* $\{e^{-\omega t}W_j(t) : t \geq 0\}$ *are equicontinuous* ($0 \leq j \leq n$) *as well as that:*

$$\int_0^\infty e^{-\lambda t} W(t)y\,dt = \tilde{k}(\lambda)\mathbf{P}_\lambda^{-1}C_1 y$$

and

$$\int_0^\infty e^{-\lambda t} W_j(t)y\,dt = \tilde{k}(\lambda)\lambda^{\alpha_j-\alpha_n}A_j\mathbf{P}_\lambda^{-1}C_1 y,$$

for every $\lambda > \omega$ *with* $\tilde{k}(\lambda) \neq 0$, $y \in Y$ *and* $j \in \mathbb{N}_n^0$. *Then there exists a* k*-regularized* C_1*-existence family for (1.1), denoted by* $(E(t))_{t\geq 0}$. *Furthermore,* $E^{(m_n-1)}(t)y = W(t)y$, $t \geq 0$, $y \in Y$ *and* $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(t)y = W_j(t)y$, $t \geq 0$, $y \in Y$, $j \in \mathbb{N}_{n-1}^0$.

(ii) *Suppose* $(U(t))_{t\geq 0}$ *is strongly continuous and the operator family* $\{e^{-\omega t}U(t) : t \geq 0\}$ *is equicontinuous. Then* $(U(t))_{t\geq 0}$ *is a* k*-regularized* C_2*-uniqueness family for (1.1) iff, for every* $x \in \bigcap_{j=0}^n D(A_j)$, *the following holds:*

$$\int_0^\infty e^{-\lambda t} U(t)\mathbf{P}_\lambda x\,dt = \tilde{k}(\lambda)\lambda^{1-m_n}C_2 x, \quad \operatorname{Re}\lambda > \omega.$$

The assertion of [32, Theorem 3.7], concerning the inhomogeneous Cauchy problem (1.1), can be stated for degenerate equations without any terminological changes, as well:

Theorem 3.6. *Suppose* $(E(t))_{t\in[0,\tau)}$ *is a locally equicontinuous* C_1*-existence family for (1.1),* $T \in (0, \tau)$, *and* $u_i \in \mathbf{D}_i$ *for* $0 \leq i \leq m_n - 1$. *Let* $f \in C([0, T] : X)$, *let* $g \in C([0, T] : Y)$ *satisfy* $C_1 g(t) = f(t)$, $t \in [0, T]$, *and let* $G \in C([0, T] : Y)$ *satisfy* $(g_{\alpha_n-m_n+1} * g)(t) = (g_1 * G)(t)$, $t \in [0, T]$. *Then the function*

$$\begin{aligned} u(t) = \sum_{i=0}^{m_n-1} u_i g_{i+1}(t) - \sum_{i=0}^{m_n-1} \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} \Big(g_{\alpha_n-\alpha_j} * E^{(m_n-1-i)}\Big)(t)v_{i,j} \\ + \sum_{i=m}^{m_n-1} \Big(g_{\alpha_n-\alpha} * E^{(m_n-1-i)}\Big)(t)v_{i,0} + \int_0^t E(t-s)G(s)\,ds, \quad 0 \leq t \leq T, \end{aligned} \tag{3.5}$$

is a mild solution of the problem (2.1) on $[0, T]$, *where* $v_{i,j} \in Y$ *satisfy* $A_j u_i = C_1 v_{i,j}$ *for* $0 \leq j \leq n-1$. *If, additionally,* $g \in C^1([0, T] : Y)$ *and* $(E^{(m_n-1)}(t))_{t\in[0,\tau)} \subseteq L(Y, X)$ *is locally equicontinuous, then the solution* $u(t)$, *given by (3.5), is a strong solution of (1.1) on* $[0, T]$.

Contrary to the assertion of [32, Theorem 3.7], the final conclusions of [32, Remark 3.8] cannot be proved for degenerate equations without imposing some additional conditions. Details can be left to the interested reader.

Concerning the action of subordination principles, we can state the following analogue of [32, Theorem 4.1] for degenerate multi-term problems (the final conclusions of [32, Remark 4.2] can be restated in our new setting, as well).

Theorem 3.7. *Suppose* $C_1 \in L(Y, X)$, $C_2 \in L(X)$ *is injective and* $\gamma \in (0, 1)$.

(i) *Let* $\omega \geq \max(0, abs(k))$, *and let the assumptions of Theorem 3.5(i)-(b) hold. Put*

$$W_\gamma(t) := \int_0^\infty t^{-\gamma} \Phi_\gamma(t^{-\gamma}s) W(s) y \, ds, \quad t > 0, \; y \in Y \text{ and } W_\gamma(0) := W(0). \tag{3.6}$$

Define, for every $j \in \mathbb{N}_n^0$ *and* $t \geq 0$, $W_{j,\gamma}(t)$ *by replacing* $W(t)$ *in (3.6) with* $W_j(t)$. *Suppose that there exist a number* $\nu > 0$ *and a continuous kernel* $k_\gamma(t)$ *on* $[0, \infty)$ *satisfying (P1) and* $\widetilde{k_\gamma}(\lambda) = \lambda^{\gamma-1}\tilde{k}(\lambda^\gamma)$, $\lambda > \nu$. *Then there exists an exponentially equicontinuous* k_γ*-regularized* C_1*-existence family* $(E_\gamma(t))_{t\geq 0}$ *for (1.1), with* α_j *replaced by* $\alpha_j\gamma$ *therein* ($0 \leq j \leq n$). *Furthermore, the family* $\{(1 + t^{\lceil \alpha_n \gamma \rceil - 2})^{-1} e^{-\omega^{1/\gamma} t} E_\gamma(t) : t \geq 0\}$ *is equicontinuous.*

(ii) *Suppose* $(U(t))_{t\geq 0}$ *is a* k*-regularized* C_2*-uniqueness family for (1.1), and the family* $\{e^{-\omega t}U(t) : t \geq 0\}$ *is equicontinuous. Define, for every* $t \geq 0$, $U_\gamma(t)$ *by replacing* $W(t)$ *in (3.6) with* $U(t)$. *Suppose that there exist a number* $\nu > 0$ *and a continuous kernel* $k_\gamma(t)$ *on* $[0, \infty)$ *satisfying (P1) and* $\widetilde{k_\gamma}(\lambda) = \lambda^{\gamma(2-m_n)-2+\lceil \alpha_n \gamma \rceil}\tilde{k}(\lambda^\gamma)$, $\lambda > \nu$. *Then there exists a* k_γ*-regularized* C_2*-uniqueness family for (1.1), with* α_j *replaced by* $\alpha_j\gamma$ *therein* ($0 \leq j \leq n$). *Furthermore, the family* $\{e^{-\omega^{1/\gamma} t} U_\gamma(t) : t \geq 0\}$ *is equicontinuous.*

Of importance is the following abstract degenerate Volterra equation:

$$Bu(t) = f(t) + \sum_{j=0}^{n-1} \big(a_j * A_j u\big)(t), \quad t \in [0, \tau), \tag{3.7}$$

where $0 < \tau \leq \infty$, $f \in C([0, \tau) : X)$, $a_0, \cdot \cdot \cdot, a_{n-1} \in L^1_{loc}([0, \tau))$, and $A = A_0, \cdot \cdot \cdot, A_{n-1}, B$ are closed linear operators on X. We define the notion of a mild (strong) solution of problem (3.7) in the same way as it has been done before for the problem (2.2).

The following definition plays a crucial role in our investigation of problem (3.7).

Definition 3.8. (cf. [32, Definition 4.3] for the case $B = I$) Suppose $0 < \tau \leq \infty$, $k \in C([0, \tau))$, $C_1 \in L(Y, X)$, and $C_2 \in L(X)$ is injective.

(i) A strongly continuous operator family $(E(t))_{t\in[0,\tau)} \subseteq L(Y, [D(B)])$ is said to be a (local, if $\tau < \infty$) k-regularized C_1-existence family for (3.7) iff

$$BE(t)y = k(t)C_1 y + \sum_{j=0}^{n-1} A_j\big(a_j * E\big)(t)y, \quad t \in [0, \tau), \; y \in Y.$$

(ii) A strongly continuous operator family $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_2-uniqueness family for (3.7) iff

$$U(t)Bx = k(t)C_2x + \sum_{j=0}^{n-1} \big(a_j * A_jU\big)(t)x, \quad t \in [0,\tau),\ x \in \bigcap_{j=0}^{n} D(A_j).$$

As in non-degenerate case, we have the following:

(i) Suppose $(E(t))_{t\in[0,\tau)}$ is a k-regularized C_1-existence family for (3.7). Then, for every $y \in Y$, the function $u(t) = E(t)y$, $t \in [0,\tau)$ is a mild solution of (3.7) with $f(t) = k(t)C_1y$, $t \in [0,\tau)$.

(ii) Let $(U(t))_{t\in[0,\tau)}$ be a locally equicontinuous k-regularized C_2-uniqueness family for (3.7). Then there exists at most one mild (strong) solution of (3.7).

The most important structural properties of k-regularized C_1-existence families for (3.7) and k-regularized C_2-uniqueness families for (3.7) are stated in the following analogue of Theorem 3.5.

Theorem 3.9. *Suppose that* $k(t)$ *and* $a_0(t), \cdots, a_{n-1}(t)$ *satisfy (P1),* $(E(t))_{t\geq0} \subseteq L(Y,X)$, $(U(t))_{t\geq0} \subseteq L(X)$, $\omega \geq \max(0, abs(k), abs(a_0), \cdots, abs(a_{n-1}))$, $C_1 \in L(Y,X)$ *and* $C_2 \in L(X)$ *is injective. Set* $\mathcal{P}_\lambda := B - \sum_{j=0}^{n-1} \tilde{a_j}(\lambda)A_j$, $\operatorname{Re}\lambda > \omega$.

(i) *(a)* *Let* $(E(t))_{t\geq0}$ *be a* k*-regularized* C_1*-existence family for (3.7), let the family* $\{e^{-\omega t}E(t) : t \geq 0\} \subseteq L(Y, [D(B)])$ *be equicontinuous, and let the family* $\{e^{-\omega t}A_j(a_j * E)(t) : t \geq 0\} \subseteq L(Y,X)$ *be equicontinuous (*$0 \leq j \leq n-1$*). Then the following holds:*

$$\mathcal{P}_\lambda \int_0^\infty e^{-\lambda t}E(t)y\,dt = \tilde{k}(\lambda)C_1y, \quad y \in Y,\ \operatorname{Re}\lambda > \omega.$$

(b) *Let the operator* $\mathcal{P}_\lambda$ *be injective for every* $\lambda > \omega$ *with* $\tilde{k}(\lambda) \neq 0$*. Suppose, additionally, that there exist strongly continuous operator families* $(E(t))_{t\geq0} \subseteq L(Y,X)$, $(E_B(t))_{t\geq0} \subseteq L(Y,X)$, *and* $(E_j(t))_{t\geq0} \subseteq L(Y,X)$ *such that the operator families* $\{e^{-\omega t}E(t) : t \geq 0\}$, $\{e^{-\omega t}E_B(t) : t \geq 0\}$, *and* $\{e^{-\omega t}E_j(t) : t \geq 0\}$ *are equicontinuous (*$0 \leq j \leq n-1$*) as well as that:*

$$\int_0^\infty e^{-\lambda t}E(t)y\,dt = \tilde{k}(\lambda)\mathcal{P}_\lambda^{-1}C_1y, \quad \int_0^\infty e^{-\lambda t}E_B(t)y\,dt = \tilde{k}(\lambda)B\mathcal{P}_\lambda^{-1}C_1y$$

and

$$\int_0^\infty e^{-\lambda t}E_j(t)y\,dt = \tilde{k}(\lambda)\tilde{a_j}(\lambda)A_j\mathcal{P}_\lambda^{-1}C_1y,$$

for every $\lambda > \omega$ *with* $\tilde{k}(\lambda) \neq 0$, $y \in Y$ *and* $j \in \mathbb{N}_{n-1}^0$*. Then* $(E(t))_{t\geq0}$ *is a* k*-regularized* C_1*-existence family for (3.7). Furthermore,* $BE(t)y = E_B(t)y$, $t \geq 0$, $y \in Y$ *and* $A_j(a_j * E)(t)y = E_j(t)y$, $t \geq 0$, $y \in Y$, $j \in \mathbb{N}_{n-1}^0$.

(ii) *Suppose* $(U(t))_{t\geq 0}$ *is strongly continuous and the operator family* $\{e^{-\omega t}U(t) : t \geq 0\} \subseteq L(X)$ *is equicontinuous. Then* $(U(t))_{t\geq 0}$ *is a* k*-regularized* C_2*-uniqueness family for (3.7) iff, for every* $x \in \bigcap_{j=0}^{n} D(A_j)$*, the following holds:*

$$\int_0^\infty e^{-\lambda t}U(t)\mathcal{P}_\lambda x\,dt = \tilde{k}(\lambda)C_2x, \quad \operatorname{Re}\lambda > \omega.$$

Theorem 3.10. *(Subordination principle)*

(i) *Suppose that the requirements of Theorem 3.9(i)-(b) hold. Let* $c(t)$ *be completely positive, let* $c(t)$, $k(t)$, $k_1(t)$, $a_0(t), \cdots, a_{n-1}(t)$ *and* $b_0(t), \cdots, b_{n-1}(t)$ *satisfy (P1), and let* $\omega_0 > 0$ *be such that, for every* $\lambda > \omega_0$ *with* $\tilde{c}(\lambda) \neq 0$ *and* $\tilde{k}(1/\tilde{c}(\lambda)) \neq 0$*, the following holds:*

$$\widetilde{a_j}(1/\tilde{c}(\lambda)) = \widetilde{b_j}(\lambda),\ j \in \mathbb{N}^0_{n-1} \text{ and } \widetilde{k_1}(\lambda) = \frac{1}{\lambda\tilde{c}(\lambda)}\tilde{k}(1/\tilde{c}(\lambda)). \tag{3.8}$$

Then for each $r \in (0,1]$ *there exists a locally Hölder continuous (with exponent* r*), exponentially equicontinuous* $(k_1 * g_r)$*-regularized* C_1*-existence family for*

$$Bu(t) = f(t) + \sum_{j=0}^{n-1}\big(b_j * A_ju\big)(t), \quad t \in [0,\tau). \tag{3.9}$$

(ii) *Suppose that the requirements of Theorem 3.9(ii) hold. Let* $c(t)$ *be completely positive, let* $c(t)$, $k(t)$, $k_1(t)$ $a_0(t), \cdots, a_{n-1}(t)$ *and* $b_0(t), \cdots, b_{n-1}(t)$ *satisfy (P1), and let* $\omega_0 > 0$ *be such that, for every* $\lambda > \omega_0$ *with* $\tilde{c}(\lambda) \neq 0$ *and* $\tilde{k}(1/\tilde{c}(\lambda)) \neq 0$*, (3.8) holds. Then for each* $r \in (0,1]$ *there exists a locally Hölder continuous (with exponent* r*), exponentially equicontinuous* $(k_1 * g_r)$*-regularized* C_2*-uniqueness family for (3.9).*

The interested reader may try to transfer the final conclusions of [36, Theorem 2.1, Theorem 2.2, Remark 2.1, Proposition 2.1] to degenerate multi-term fractional differential equations. In order to do the same with the perturbation result [36, Theorem 2.3], we need to introduce the following notion.

Definition 3.11. A strongly continuous operator family $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) (k, C_2)-uniqueness family for (1.1) iff, for every $t \in [0,\tau)$ and $x \in \bigcap_{0\leq j\leq n} D(A_j)$, the following holds:

$$U(t)Bx + \sum_{j=1}^{n-1}\big(g_{\alpha_n-\alpha_j} * U(\cdot)A_jx\big)(t) - \big(g_{\alpha_n-\alpha} * U(\cdot)Ax\big)(t)x = k(t)C_2x.$$

Then it is clear that for any strongly continuous operator family $(U(t))_{t\in[0,\tau)}$ the following equivalence relation holds: $(U(t))_{t\in[0,\tau)}$ is a (local) $(k * g_{m_n-1}, C_2)$-uniqueness family for (1.1) iff $(U(t))_{t\in[0,\tau)}$ is a (local) k-regularized C_2-uniqueness family for (1.1).

Consider now the perturbed equation:

$$\begin{aligned} &BD_t^{\alpha_n}u(t) + \sum_{i=1}^{n-1}(A_i + F_i)D_t^{\alpha_i}u(t) = (A + F)D_t^{\alpha}u(t) + f(t), \quad t \geq 0; \\ &u^{(j)}(0) = u_j, \ j = 0, \cdot \cdot \cdot, \lceil \alpha_n \rceil - 1, \end{aligned} \tag{3.10}$$

where $F_i \in L(X)$ for $0 \leq i \leq n-1$ and $F_0 \equiv F$. A similar line of reasoning as in the proof of [36, Theorem 2.3] shows that the following result about the C-wellposedness of problem (3.10) holds good (observe that the employed method is based on the arguments contained in the proof of [43, Theorem 6.1], which does not work any longer if we replace the term $BD_t^{\alpha_n}u(t)$, in (3.10), with $(B + F_n)D_t^{\alpha_n}u(t)$):

Theorem 3.12. *(i) Suppose $Y = X$, $(E(t))_{t\in[0,\tau)} \subseteq L(X)$ is a (local) C_1-existence family for (1.1), $E_j \in L(X)$ and $F_j = C_1E_j$ $(j \in \mathbb{N}_{n-1}^0)$. Suppose that the following conditions hold:*

(a) For every $p \in \circledast_X$ and for every $T \in (0, \tau)$, there exists $c_{p,T} > 0$ such that

$$p\bigl(E^{(m_n-1)}(t)x\bigr) \leq c_{p,T}p(x), \quad x \in X, \ t \in [0, T].$$

(b) For every $p \in \circledast_X$, there exists $c_p > 0$ such that

$$p\bigl(E_jx\bigr) \leq c_pp(x), \quad j \in \mathbb{N}_{n-1}^0, \ x \in X.$$

(c) $\alpha_n - \alpha_{n-1} \geq 1$ and $\alpha_n - \alpha \geq 1$.

Then there exists a (local) C_1-existence propagation family $(R(t))_{t\in[0,\tau)}$ for (3.10). If $\tau = \infty$ and if, for every $p \in \circledast_X$, there exist $M \geq 1$ and $\omega \geq 0$ such that

$$p\bigl(E^{(m_n-1)}(t)x\bigr) \leq Me^{\omega t}p(x), \quad t \geq 0, \ x \in X, \tag{3.11}$$

respectively (3.11) and

$$p\bigl(BE^{(m_n-1)}(t)x\bigr) \leq Me^{\omega t}p(x), \quad t \geq 0, \ x \in X, \tag{3.12}$$

then $(R(t))_{t\geq 0}$ is exponentially equicontinuous, and moreover, $(R(t))_{t\geq 0}$ also satisfies the condition (3.11), repectively (3.11) and (3.12), with possibly different numbers $M \geq 1$ and $\omega > 0$.

(ii) Suppose $Y = X$, $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ is a (local) $(1, C_2)$-uniqueness family for (1.1), $E_j \in L(E)$ and $F_j = E_jC_2$ $(j \in \mathbb{N}_{n-1}^0)$. Suppose that (b)-(c) hold, and that (a) holds with $(E^{(m_n-1)}(t))_{t\in[0,\tau)}$ replaced by $(U(t))_{t\in[0,\tau)}$ therein. Then there exists a (local) $(1, C_2)$-uniqueness family $(W(t))_{t\in[0,\tau)}$ for (3.10). If $\tau = \infty$ and if, for every $p \in \circledast_X$, there exist $M \geq 1$ and $\omega \geq 0$ such that (3.11) holds, then $(W(t))_{t\geq 0}$ is exponentially equicontinuous, and moreover, $(W(t))_{t\geq 0}$ also satisfies the condition (3.11), with possibly different numbers $M \geq 1$ and $\omega > 0$.

For some other results concerning perturbation properties of abstract degenerate differential equations, one may refer e.g. to [18], [21] and [50]. Concerning the existence of strong solutions of (1.1), we can prove the following slight extension of [36, Theorem 3.1]; this result can be viewed of some independent interest.

Theorem 3.13. *Suppose* A, B, $A_1, \cdots, A_{n-1}$ *are closed linear operators on* X, $\omega > 0$, $L(X) \ni C$ *is injective and* $u_0, \cdots, u_{m_n-1} \in X$. *Set* $P_\lambda := \lambda^{\alpha_n-\alpha}B + \sum_{j=1}^{n-1} \lambda^{\alpha_j-\alpha}A_j - A$, $\lambda \in \mathbb{C} \setminus \{0\}$. *Let the following conditions hold:*

(i) The operator P_λ *is injective for* $\lambda > \omega$ *and* $D(P_\lambda^{-1}C) = X$, $\lambda > \omega$.

(ii) If $0 \leq j \leq n$, $0 \leq k \leq m_n - 1$, $m - 1 < k$, $1 \leq l \leq n$, $m_l - 1 \geq k$ *and* $\lambda > \omega$, *then* $Cu_k \in D(P_\lambda^{-1}A_l)$,

$$A_j\Bigg\{\lambda^{\alpha_j}\Bigg[\lambda^{\alpha_n-\alpha-k-1}P_\lambda^{-1}BCu_k + \sum_{l\in D_k}\lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k\Bigg] - \sum_{l=0}^{m_j-1}\delta_{kl}\lambda^{\alpha_j-1-l}Cu_k\Bigg\} \in LT - X \tag{3.13}$$

and

$$\lambda^{\alpha_n}\Bigg[\lambda^{\alpha_n-\alpha-k-1}P_\lambda^{-1}BCu_k + \sum_{l\in D_k}\lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k\Bigg] - \lambda^{\alpha_n-1-k}Cu_k \in LT - X. \tag{3.14}$$

(iii) If $0 \leq j \leq n$, $0 \leq k \leq m_n - 1$, $m - 1 \geq k$, $\mathbb{N}_{n-1} \setminus D_k \neq \emptyset$, $s \in \mathbb{N}_{n-1} \setminus D_k$ *and* $\lambda > \omega$, *then* $Cu_k \in D(A_s)$, $\sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\lambda^{\alpha_l-\alpha-k-1}A_lCu_k \in D(P_\lambda^{-1})$,

$$A_j\Bigg\{\lambda^{\alpha_j}\Bigg[\lambda^{-k-1}Cu_k - P_\lambda^{-1}\sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\lambda^{\alpha_l-\alpha-k-1}A_lCu_k\Bigg] - \sum_{l=0}^{m_j-1}\delta_{kl}\lambda^{\alpha_j-1-l}Cu_k\Bigg\} \in LT - X \tag{3.15}$$

and

$$\lambda^{\alpha_n}\Bigg[\lambda^{-k-1}Cu_k - P_\lambda^{-1}\sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\lambda^{\alpha_l-\alpha-k-1}A_lCu_k\Bigg] - \lambda^{\alpha_n-1-k}Cu_k \in LT - X. \tag{3.16}$$

Then the abstract Cauchy problem (1.1) has a strong solution, with u_k *replaced by* Cu_k $(0 \leq k \leq m_n - 1)$.

Remark 3.14. Let $0 \leq k \leq m_n - 1$ and $m - 1 < k$. Then Theorem 3.13 continues to hold if we replace the term

$$\lambda^{\alpha_n-\alpha-k-1} P_\lambda^{-1} BCu_k + \sum_{l \in D_k} \lambda^{\alpha_l-\alpha-k-1} P_\lambda^{-1} A_l Cu_k$$

i.e., the Laplace transform of $u_k(t)$, in (3.13)-(3.14) by

$$\lambda^{-k-1} Cu_k - \sum_{l \in \mathbb{N}_{n-1} \setminus D_k} \lambda^{\alpha_l-\alpha-k-1} P_\lambda^{-1} A_l Cu_k + \lambda^{-k-1} P_\lambda^{-1} ACu_k;$$

in this case, it is necessary to assume that $Cu_k \in D(P_\lambda^{-1} A_l)$, provided $0 \leq l \leq n-1$, $k > m_l - 1$ and $\lambda > \omega$. Let us also observe that a similar modification can be made in the case $0 \leq k \leq m_n - 1$ and $m - 1 \geq k$. Strictly speaking, one can replace the term

$$\lambda^{-k-1} Cu_k - P_\lambda^{-1} \sum_{l \in \mathbb{N}_{n-1} \setminus D_k} \lambda^{\alpha_l-\alpha-k-1} A_l Cu_k$$

i.e., the Laplace transform of $u_k(t)$, in (3.15)-(3.16) by

$$\lambda^{\alpha_n-\alpha-k-1} P_\lambda^{-1} BCu_k + \sum_{l \in D_k} \lambda^{\alpha_l-\alpha-k-1} P_\lambda^{-1} A_l Cu_k - \lambda^{-k-1} P_\lambda^{-1} ACu_k;$$

in this case, one has to assume that $Cu_k \in D(P_\lambda^{-1} A_l)$, provided $0 \leq l \leq n$, $m_l - 1 \geq k$ and $\lambda > \omega$.

Now we would like to illustrate the obtained results with some examples.

Example 3.15. Suppose $1 \leq p < \infty$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is an open bounded domain with smooth boundary, and $X := L^p(\Omega)$. Consider the equation

$$\begin{aligned} &(\alpha - \Delta) u_{tt} = \beta \Delta u_t + \Delta u + \int_0^t g(t-s) \Delta u(s,x)\, ds = 0, \quad t > 0,\ x \in \Omega; \\ &u(0,x) = \phi(x),\ u_t(0,x) = \psi(x), \end{aligned} \tag{3.17}$$

where $g \in L^1_{loc}([0,\infty))$ satisfies (P1), $\alpha > 0$ and $\beta \in \mathbb{R} \setminus \{0\}$. As explained by M. V. Falaleev and S. S. Orlov in [16], the equation (3.17) appears in the models of nonlinear viscoelasticity provided $n = 3$. Integrating (3.17) twice with the respect to the time-variable t, we obtain the associated integral equation

$$\begin{aligned} (\alpha - \Delta) u(t) &= (\alpha + (\beta - 1)\Delta)\phi(x) \\ &\quad + t(\alpha - \Delta)\psi + \beta \Delta\big(g_1 * u\big)(t) + \Delta\big(g_2 * u\big) + \Delta\big(g_2 * g * u\big)(t), \end{aligned} \tag{3.18}$$

which is of the form (3.7) with $B := \alpha - \Delta$, $A_2 := \beta\Delta$, $A_1 = A_0 := \Delta$ (acting with the Dirichlet boundary conditions) and $a_2(t) := g_1(t)$, $a_1(t) := g_2(t)$, $a_0(t) := (g_2 * g)(t)$. Then

$$\mathcal{P}_\lambda = \frac{\lambda^2 + \beta\lambda + \tilde{g}(\lambda) + 1}{\lambda^2} \left[\frac{\alpha\lambda^2}{\lambda^2 + \beta\lambda + \tilde{g}(\lambda) + 1} - \Delta \right].$$

We assume that $g(t)$ is of the following form:

$$g(t) = \sum_{j=0}^{l} c_j g_{\beta_j}(t) + f(t), \quad t > 0,$$

where $l \in \mathbb{N}$, $c_j \in \mathbb{C}$ $(0 \leq j \leq l)$, $0 < \beta_1 < \cdots < \beta_l < 1$ and the function $f(t)$ satisfies the requirements of [26, Theorem 3.4(i)-(a)] with $\alpha = \pi/2$ and $\omega > 0$ sufficiently large. Using the resolvent equation and the fact that the operator Δ generates a bounded analytic C_0-semigroup of angle $\pi/2$, it can be simply verified that

$$\frac{1}{\lambda}\mathcal{P}_\lambda^{-1} \in LT - L(X),\ \frac{1}{\lambda}B\mathcal{P}_\lambda^{-1} \in LT - L(X) \text{ and } \frac{\widetilde{a_j}(\lambda)}{\lambda}\mathcal{P}_\lambda^{-1} \in LT - L(X),\ j = 0, 1, 2.$$

This implies by Theorem 3.9 that there exists an exponentially bounded I-existence family $(E(t))_{t\geq 0}$ for (3.18), satisfying additionally that for each $f \in X$ the mappings $t \mapsto E(t)f$, $t > 0$, $t \mapsto BE(t)f$, $t > 0$ and $t \mapsto A_j(a_j * E)(t)f$, $t > 0$ can be analytically extended to the sector $\Sigma_{\pi/2}$; furthermore, $(E(t))_{t\geq 0}$ is an exponentailly bounded I-uniqueness family for (3.18). This implies that for each $\phi,\ \psi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, there exists a unique strong solution

$$u(t) = E(t)(\alpha + (\beta - 1)\Delta)\phi(x) + \int_0^t E(s)(\alpha - \Delta)\psi\, ds$$

of the integral equation (3.18), and that $u(t)$ can be analytically extended to the sector $\Sigma_{\pi/2}$.

Example 3.16. Suppose $1 < p < \infty$, $X := L^p(\mathbb{R}^n)$, $1/2 < \gamma \leq 1$, $Q \in \mathbb{N}\setminus\{1\}$, $P_1(x) = \sum_{|\eta|\leq Q} a_\eta x^\eta$, $P_2(x) = -1 - |x|^2$ $(x \in \mathbb{R}^n)$, $P_1(x)$ is positive, $\sigma \geq 0$, the estimate

$$\left| D^\eta \left(\frac{P_1(x)}{P_2(x)} \right) \right| \leq c_\eta \big(1 + |x|\big)^{|\eta|(\sigma-1)}, \quad x \in \mathbb{R}^n$$

holds for each multi-index $\eta \in \mathbb{N}_0^n$ with $|\eta| > 0$, $V_2 \geq 0$ and for each $\eta \in \mathbb{N}_0^n$ there exists $M_\eta > 0$ such that $|D^\eta(P_2(x)^{-1})| \leq M_\eta(1 + |x|)^{|\eta|(V_2-1)}$, $x \in \mathbb{R}^n$. Set $A_2 := \Delta - I$, $A_0 f := \sum_{|\eta|\leq Q} a_\eta D^\eta f$ with maximal distributional domain, where $D^\eta \equiv (-i)^{|\eta|} f^\eta$, and $C_1 := (I - \Delta)^{-\frac{n}{2}|\frac{1}{p}-\frac{1}{2}|\max(\sigma, V_2)}$. Let $E_i \in L(X)$ and $F_i = C_1 E_i$ $(i = 0, 1)$. Then we know (cf. [27]-[28]) that $\lambda(\lambda^2 A_2 - A_0)^{-1} C_1 \in LT - L(X)$ and $\lambda A_2(\lambda^2 A_2 - A_0)^{-1} C_1 \in LT - L(X)$, which implies by Theorem 3.9(i)-(b) that there exists an exponentially bounded C_1-existence family $(E(t))_{t\geq 0}$ for the following degenerate second order Cauchy problem:

$$\begin{cases} (\Delta - I)u_{tt}(t,x) = \sum_{|\eta|\leq Q} a_\eta D^\eta u(t,x), \\ u(0,x) = u_0(x) = \phi(x),\ u_t(0,x) = u_1(x) = \psi(x), \end{cases}$$

obeying the properties (3.11)-(3.12) stated in the formulation of Theorem 3.12. Applying Theorem 3.12, we get there exists an exponentially bounded
C_1-existence family $(R(t))_{t\geq 0}$ for the following degenerate second order Cauchy problem:

$$(P): \begin{cases} (\Delta - I)u_{tt}(t,x) + F_1 u_t(t,x) = \Big(\sum_{|\eta|\leq Q} a_\eta D^\eta + F_0\Big)u(t,x), \\ u(0,x) = u_0(x) = \phi(x),\ u_t(0,x) = u_1(x) = \psi(x). \end{cases}$$

Then Theorem 3.4(i) shows that there exists a strong solution of problem (P) provided that

$$\phi,\ \psi \in S^{Q+n|\frac{1}{p}-\frac{1}{2}|\max(\sigma,V_2),p}(\mathbb{R}^n),\ \big(A_0+F\big)\phi \in S^{n|\frac{1}{p}-\frac{1}{2}|\max(\sigma,V_2),p}(\mathbb{R}^n),$$

$$F_1\psi \in S^{n|\frac{1}{p}-\frac{1}{2}|\max(\sigma,V_2),p}(\mathbb{R}^n) \text{ and } \big(A_0+F\big)\psi \in S^{n|\frac{1}{p}-\frac{1}{2}|\max(\sigma,V_2),p}(\mathbb{R}^n).$$

If we denote by $U(t,x)$, resp. $V(t,x)$, the corresponding strong solution of problem (P) with the initial values $\phi(x)$ and $\psi(x)\equiv 0$, resp. $\phi(x)\equiv 0$ and $\psi(x)$, then one can simply verify that the function

$$u(t,x) := \int\limits_0^\infty t^{-\gamma}\Phi_\gamma\big(st^{-\gamma}\big)U(s,x)\,ds + \int\limits_0^t g_{1-\gamma}(t-s)\int\limits_0^\infty s^{-\gamma}\Phi_\gamma\big(rs^{-\gamma}\big)V(r,x)\,dr\,ds,$$

is a strong solution of the following integral equation

$$\begin{aligned} &A_2\big[u(t,x)-\phi(x)-t\psi(x)\big] + F_1\int_0^t g_\gamma(t-s)\big[u(s,x)-\phi(x)\big]\,ds \\ &\quad = \int_0^t g_{2\gamma}(t-s)\big(A_0+F\big)u(s,x)\,ds,\ t\geq 0,\ x\in\mathbb{R}^n; \end{aligned} \tag{3.19}$$

furthermore, the function $t\mapsto u(t,\cdot)\in X$ can be analytically extended to the sector $\Sigma_{(\frac{1}{\gamma}-1)\frac{\pi}{2}}$. In the present situation, we can only prove that there is at most one strong solution of the integral equation (3.19) provided that $p=2$. Speaking-matter-of-factly, suppose that $u(t,x)$ is a strong solution of (3.19) with $\phi(x)\equiv\psi(x)\equiv 0$. Then $A_2^{-1}\in L(X)$, $C_1=I$ and the function $v(t,x):=A_2u(t,x)$ is a strong solution of the following non-degenerate integral equation

$$\begin{aligned} &u(t,x) + \int_0^t g_\gamma(t-s)F_1A_2^{-1}u(s,x)\,ds \\ &\quad = \int_0^t g_{2\gamma}(t-s)\big(A_0A_2^{-1}+FA_2^{-1}\big)u(s,x)\,ds,\ t\geq 0,\ x\in\mathbb{R}^n. \end{aligned} \tag{3.20}$$

Since $\lambda(\lambda^2-A_0A_2^{-1})^{-1} = \lambda A_2(\lambda^2A_2-A_0)^{-1} \in LT-L(X)$, the operator $A_0A_2^{-1}$ generates a cosine operator function and we can apply Theorem 3.12(ii) in order to see that there exists an exponentially bounded $(1,I)$-uniqueness family for (3.20), with the meaning clear. This proves the claimed assertion on the uniqueness of strong solutions of problem (3.19). In general case $p\neq 2$, it is not clear how we can prove that there is at most one strong solution of the integral equation (3.19) without assuming that F_1 and F take some specific forms.

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