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Reproducing inversion formulas for the Dunkl-Wigner transforms

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ABSTRACT

We define and study the Fourier-Wigner transform associated with the Dunkl operators, and we prove for this transform a reproducing inversion formulas and a Plancherel formula. Next, we introduce and study the extremal functions associated to the Dunkl-Wigner transform.

RESUMEN

Definimos y estudiamos la transformada de Fourier-Wigner asociada a los operadores de Dunkl, y probamos una fórmula de inversion y una formula de Plancherel para esta transformada. Luego introducimos y estudiamos las funciones extramales asociadas a la transformada de Dunkl-Wigner.

Keywords and Phrases: Dunkl transform; Dunkl-Wigner transform; inversion formulas; extremal functions.

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1 Introduction

In this paper, we consider $\mathbb{R}^d$ with the Euclidean inner product $\langle .,. \rangle$ and norm $|y| := \sqrt{\langle y, y\rangle}$. For $\alpha \in \mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α:

$$\sigma_\alpha y := y - \frac{2\langle \alpha, y\rangle}{|\alpha|^2}\alpha.$$

A finite set $\mathrm{Re} \subset \mathbb{R}^d\backslash\{0\}$ is called a root system, if $\mathrm{Re}\cap\mathbb{R}.\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha \mathrm{Re} = \mathrm{Re}$ for all $\alpha \in \mathrm{Re}$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in \mathrm{Re}$. For a root system Re, the reflections σ_α, $\alpha \in \mathrm{Re}$, generate a finite group G. The Coxeter group G is a subgroup of the orthogonal group $O(d)$. All reflections in G, correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d\backslash \bigcup_{\alpha\in\mathrm{Re}} H_\alpha$, we fix the positive subsystem $\mathrm{Re}_+ := \{\alpha \in \mathrm{Re} : \langle \alpha, \beta\rangle > 0\}$. Then for each $\alpha \in \mathrm{Re}$ either $\alpha \in \mathrm{Re}_+$ or $-\alpha \in \mathrm{Re}_+$.

Let $k : \mathrm{Re} \to \mathbb{C}$ be a multiplicity function on Re (a function which is constant on the orbits under the action of G). As an abbreviation, we introduce the index $\gamma = \gamma_k := \sum_{\alpha\in\mathrm{Re}_+} k(\alpha)$.

Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in \mathrm{Re}$. Moreover, let w_k denote the weight function $w_k(y) := \prod_{\alpha\in\mathrm{Re}_+} |\langle \alpha, y\rangle|^{2k(\alpha)}$, for all $y \in \mathbb{R}^d$, which is G-invariant and homogeneous of degree 2γ.

Let c_k be the Mehta-type constant given by $c_k := (\int_{\mathbb{R}^d} e^{-|y|^2/2} w_k(y)dy)^{-1}$. We denote by μ_k the measure on $\mathbb{R}^d$ given by $d\mu_k(y) := c_k w_k(y)dy$; and by $L^p(\mu_k)$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L^p(\mu_k)} := \Big(\int_{\mathbb{R}^d} |f(y)|^p d\mu_k(y)\Big)^{1/p} < \infty, \quad 1 \leq p < \infty,$$
$$\|f\|_{L^\infty(\mu_k)} := \operatorname*{ess\,sup}_{y\in\mathbb{R}^d} |f(y)| < \infty,$$

and by $L^p_{rad}(\mu_k)$ the subspace of $L^p(\mu_k)$ consisting of radial functions.

For $f \in L^1(\mu_k)$ the Dunkl transform of f is defined (see [3]) by

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y)f(y)d\mu_k(y), \quad x \in \mathbb{R}^d,$$

where $E_k(-ix, y)$ denotes the Dunkl kernel. (For more details see the next section.)

The Dunkl translation operators τ_x, $x \in \mathbb{R}^d$, [18] are defined on $L^2(\mu_k)$ by

$$\mathcal{F}_k(\tau_x f)(y) = E_k(ix, y)\mathcal{F}_k(f)(y), \quad y \in \mathbb{R}^d.$$

Let $g \in L^2_{rad}(\mu_k)$. The Dunkl-Wigner transform V_g is the mapping defined for $f \in L^2(\mu_k)$ by

$$V_g(f)(x, y) := \int_{\mathbb{R}^d} f(t)\overline{\tau_x g_{k,y}(-t)}d\mu_k(t),$$

where

$$g_{k,y}(z) := \mathcal{F}_k\Big(\sqrt{\tau_y|\mathcal{F}_k(g)|^2}\Big)(z).$$

We study some of its properties, and we prove reproducing inversion formulas for this transform. Next, Building on the ideas of Matsuura et al. [6], Saitoh [11, 13] and Yamada et al. [20], and using the theory of reproducing kernels [10], we give best approximation of the mapping V_g on the Sobolev-Dunkl spaces $H^s(\mu_k)$. More precisely, for all $\lambda > 0$, $h \in L^2(\mu_k \otimes \mu_k)$, the infimum

$$\inf_{f\in H^s(\mu_k)}\Big\{\lambda\|f\|^2_{H^s(\mu_k)} + \|h - V_g(f)\|^2_{L^2(\mu_k\otimes\mu_k)}\Big\},$$

is attained at one function $f^*_{\lambda,h}$, called the extremal function, and given by

$$f^*_{\lambda,h}(y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{E_k(iy,z)\sqrt{\tau_t|\mathcal{F}_k(g)|^2(z)}\mathcal{F}_k(h(.,t))(z)}{\lambda(1+|z|^2)^s + \|g\|^2_{L^2_{rad}(\mu_k)}}d\mu_k(t)d\mu_k(z).$$

In the Dunkl setting, the extremal functions are studied in several directions [14, 15, 16, 17].

In the classical case, the Fourier-Wigner transforms are studied by Weyl [21] and Wong [22]. In the Bessel-Kingman hypergroups, these operators are studied by Dachraoui [1].

This paper is organized as follows. In Section 2, we recall some properties of harmonic analysis for the Dunkl operators. Next, we define the Fourier-Wigner transform V_g in the Dunkl setting, and we have established for it a reproducing inversion formulas. In Section 3, we introduce and study the extremal functions associated to the Dunkl-Wigner transform V_g.

2 The Dunkl-Wigner transform

The Dunkl operators $\mathcal{D}_j$; $j = 1, ..., d$, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given, for a function f of class C^1 on $\mathbb{R}^d$, by

$$\mathcal{D}_j f(y) := \frac{\partial}{\partial y_j}f(y) + \sum_{\alpha\in\mathrm{Re}_+} k(\alpha)\alpha_j\frac{f(y) - f(\sigma_\alpha y)}{\langle\alpha, y\rangle}.$$

For $y \in \mathbb{R}^d$, the initial problem $\mathcal{D}_j u(., y)(x) = y_j u(x, y)$, $j = 1, ..., d$, with $u(0, y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x, y)$ and called Dunkl kernel [2, 4]. This kernel has a unique analytic extension to $\mathbb{C}^d \times \mathbb{C}^d$ (see [7]). The Dunkl kernel has the Laplace-type representation [8]

$$E_k(x, y) = \int_{\mathbb{R}^d} e^{\langle y,z\rangle}d\Gamma_x(z), \quad x \in \mathbb{R}^d, y \in \mathbb{C}^d, \tag{2.1}$$

where $\langle y, z\rangle := \sum_{i=1}^d y_i z_i$ and Γ_x is a probability measure on $\mathbb{R}^d$, such that $\mathrm{supp}(\Gamma_x) \subset \{z \in \mathbb{R}^d : |z| \le |x|\}$. In our case,

$$|E_k(ix, y)| \le 1, \quad x, y \in \mathbb{R}^d. \tag{2.2}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}^d$, and was introduced by Dunkl in [3], where already many basic properties were established. Dunkl's results were completed and extended later by De Jeu [4]. The Dunkl transform of a function f in $L^1(\mu_k)$, is defined by

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y) f(y) d\mu_k(y), \quad x \in \mathbb{R}^d.$$

We notice that $\mathcal{F}_0$ agrees with the Fourier transform $\mathcal{F}$ that is given by

$$\mathcal{F}(f)(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle x,y\rangle} f(y) dy, \quad x \in \mathbb{R}^d.$$

Some of the properties of Dunkl transform $\mathcal{F}_k$ are collected bellow (see [3, 4]).

Theorem 2.1. (i) $L^1 - L^\infty$-boundedness. *For all* $f \in L^1(\mu_k)$, $\mathcal{F}_k(f) \in L^\infty(\mu_k)$, *and*

$$\|\mathcal{F}_k(f)\|_{L^\infty(\mu_k)} \le \|f\|_{L^1(\mu_k)}.$$

(ii) Inversion theorem. *Let* $f \in L^1(\mu_k)$, *such that* $\mathcal{F}_k(f) \in L^1(\mu_k)$. *Then*

$$f(x) = \mathcal{F}(\mathcal{F}_k(f))(-x), \quad a.e. \ x \in \mathbb{R}^d.$$

(iii) Plancherel theorem. *The Dunkl transform* $\mathcal{F}_k$ *extends uniquely to an isometric isomorphism of* $L^2(\mu_k)$ *onto itself. In particular, we have*

$$\|f\|_{L^2(\mu_k)} = \|\mathcal{F}_k(f)\|_{L^2(\mu_k)}.$$

(iv) Parseval theorem. *For* $f, g \in L^2(\mu_k)$, *we have*

$$\langle f, g\rangle_{L^2(\mu_k)} = \langle \mathcal{F}_k(f), \mathcal{F}_k(g)\rangle_{L^2(\mu_k)}.$$

The Dunkl transform $\mathcal{F}_k$ allows us to define a generalized translation operators on $L^2(\mu_k)$ by setting

$$\mathcal{F}_k(\tau_x f)(y) = E_k(ix, y)\mathcal{F}_k(f)(y), \quad y \in \mathbb{R}^d. \tag{2.3}$$

It is the definition of Thangavelu and Xu given in [18]. It plays the role of the ordinary translation $\tau_x f = f(x + .)$ in $\mathbb{R}^d$, since the Euclidean Fourier transform satisfies $\mathcal{F}(\tau_x f)(y) = e^{ixy}\mathcal{F}(f)(y)$. Note that from (2.2) and Theorem 2.1 (iii), the definition (2.3) makes sense, and

$$\|\tau_x f\|_{L^2(\mu_k)} \le \|f\|_{L^2(\mu_k)}, \quad f \in L^2(\mu_k). \tag{2.4}$$

Rösler [9] introduced the Dunkl translation operators for radial functions. If f are radial functions, $f(x) = F(|x|)$, then

$$\tau_x f(y) = \int_{\mathbb{R}^d} F\Big(\sqrt{|x|^2 + |y|^2 + 2\langle y, z\rangle}\Big) d\Gamma_x(z); \quad x, y \in \mathbb{R}^d,$$

where Γ_x is the representing measure given by (2.1).

This formula allows us to establish the following results [18, 19].

Proposition 2.2. (i) *For all* $p \in [1, 2]$ *and for all* $x \in \mathbb{R}^d$*, the Dunkl translation* $\tau_x : L^p_{rad}(\mu_k) \to L^p(\mu_k)$ *is a bounded operator, and for* $f \in L^p_{rad}(\mu_k)$*, we have*

$$\|\tau_x f\|_{L^p(\mu_k)} \leq \|f\|_{L^p_{rad}(\mu_k)}.$$

(ii) *Let* $f \in L^1_{rad}(\mu_k)$*. Then, for all* $x \in \mathbb{R}^d$*, we have*

$$\int_{\mathbb{R}^d} \tau_x f(y) d\mu_k(y) = \int_{\mathbb{R}^d} f(y) d\mu_k(y).$$

The Dunkl convolution product $*_k$ of two functions f and g in $L^2(\mu_k)$ is defined by

$$f *_k g(x) := \int_{\mathbb{R}^d} \tau_x f(-y) g(y) d\mu_k(y), \quad x \in \mathbb{R}^d. \tag{2.5}$$

We notice that $*_k$ generalizes the convolution $*$ that is given by

$$f * g(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x - y) g(y) dy, \quad x \in \mathbb{R}^d.$$

The Proposition 2.2 allows us to establish the following properties for the Dunkl convolution on $\mathbb{R}^d$ (see [18]).

Proposition 2.3. (i) *Assume that* $p \in [1, 2]$ *and* $q, r \in [1, \infty]$ *such that* $1/p + 1/q = 1 + 1/r$*. Then the map* $(f, g) \to f *_k g$ *extends to a continuous map from* $L^p_{rad}(\mu_k) \times L^q(\mu_k)$ *to* $L^r(\mu_k)$*, and*

$$\|f *_k g\|_{L^r(\mu_k)} \leq \|f\|_{L^p_{rad}(\mu_k)} \|g\|_{L^q(\mu_k)}.$$

(ii) *For all* $f \in L^1_{rad}(\mu_k)$ *and* $g \in L^2(\mu_k)$*, we have*

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\, \mathcal{F}_k(g).$$

(iii) *Let* $f \in L^2_{rad}(\mu_k)$ *and* $g \in L^2(\mu_k)$*. Then* $f *_k g$ *belongs to* $L^2(\mu_k)$ *if and only if* $\mathcal{F}_k(f)\mathcal{F}_k(g)$ *belongs to* $L^2(\mu_k)$*, and*

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g), \quad \text{in the } L^2(\mu_k) - \text{case}.$$

(iv) *Let* $f \in L^2_{rad}(\mu_k)$ *and* $g \in L^2(\mu_k)$. *Then*

$$\int_{\mathbb{R}^d} |f * g(x)|^2 d\mu_k(x) = \int_{\mathbb{R}^d} |\mathcal{F}_k(f)(z)|^2 |\mathcal{F}_k(g)(z)|^2 d\mu_k(z),$$

where both sides are finite or infinite.

Let $g \in L^2_{rad}(\mu_k)$ and $y \in \mathbb{R}^d$. The modulation of g by y is the function $g_{k,y}$ defined by

$$g_{k,y}(z) := \mathcal{F}_k\left(\sqrt{\tau_y |\mathcal{F}_k(g)|^2}\right)(z), \quad z \in \mathbb{R}^d.$$

Thus,

$$\|g_{k,y}\|_{L^2(\mu_k)} = \|g\|_{L^2_{rad}(\mu_k)}. \tag{2.6}$$

Let $g \in L^2_{rad}(\mu_k)$. The Fourier-Wigner transform associated to the Dunkl operators, is the mapping V_g defined for $f \in L^2(\mu_k)$ by

$$V_g(f)(x, y) := \int_{\mathbb{R}^d} f(t) \overline{\tau_x g_{k,y}(-t)} d\mu_k(t), \quad x, y \in \mathbb{R}^d. \tag{2.7}$$

Proposition 2.4. *Let* $(f, g) \in L^2(\mu_k) \times L^2_{rad}(\mu_k)$.

(i) $V_g(f)(x, y) = \overline{g_{k,y}} *_k f(x)$.

(ii) $V_g(f)(x, y) = \int_{\mathbb{R}^d} E_k(ix, z) \mathcal{F}_k(f)(z) \sqrt{\tau_y |\mathcal{F}_k(g)|^2(z)} d\mu_k(z)$.

(iii) *The function* $V_g(f)$ *belongs to* $L^\infty(\mu_k \otimes \mu_k)$, *and*

$$\|V_g(f)\|_{L^\infty(\mu_k \otimes \mu_k)} \leq \|f\|_{L^2(\mu_k)} \|g\|_{L^2_{rad}(\mu_k)}.$$

Proof. (i) follows from (2.5), (2.7) and the fact that $\overline{\tau_x g_{k,y}(-t)} = \tau_x \overline{g_{k,y}}(-t)$.

(ii) By Theorem 2.1 (iv) and (2.3) we have

$$V_g(f)(x, y) = \int_{\mathbb{R}^d} E_k(ix, z) \mathcal{F}_k(f)(z) \overline{\mathcal{F}_k(g_{k,y})(-z)} d\mu_k(z).$$

We obtain the result from the fact that

$$\overline{\mathcal{F}_k(g_{k,y})(-z)} = \mathcal{F}_k(\overline{g_{k,y}})(z) = \sqrt{\tau_y |\mathcal{F}_k(g)|^2(z)}.$$

(iii) follows from (2.7), by using Hölder's inequality, (2.4) and (2.6). □

Theorem 2.5. *Let* $g \in L^2_{rad}(\mu_k)$.

(i) Plancherel formula: *For every* $f \in L^2(\mu_k)$, *we have*

$$\|V_g(f)\|_{L^2(\mu_k \otimes \mu_k)} = \|g\|_{L^2_{rad}(\mu_k)} \|f\|_{L^2(\mu_k)}.$$

(ii) Parseval formula: *For every* $f, h \in L^2(\mu_k)$*, we have*

$$\langle V_g(f), V_g(h)\rangle_{L^2(\mu_k\otimes\mu_k)} = \|g\|^2_{L^2_{rad}(\mu_k)}\langle f, h\rangle_{L^2(\mu_k)}.$$

(iii) Inversion formula: *For all* $f \in L^1 \cap L^2(\mu_k)$ *such that* $\mathcal{F}_k(f) \in L^1(\mu_k)$*, we have*

$$f(z) = \frac{1}{\|g\|^2_{L^2_{rad}(\mu_k)}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_g(f)(x,y)\overline{\tau_z g_{k,y}(-x)}d\mu_k(x)d\mu_k(y).$$

Proof. (i) From Theorem 2.1 (iii), Proposition 2.2 (ii), Proposition 2.3 (iv) and Proposition 2.4 (i), we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|V_g(f)(x,y)|^2 d\mu_k(x)d\mu_k(y) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\overline{g_{k,y}} *_k f(x)|^2 d\mu_k(x)d\mu_k(y)\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\mathcal{F}_k(\overline{g_{k,y}})(z)|^2|\mathcal{F}_k(f)(z)|^2 d\mu_k(z)d\mu_k(y)\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\tau_y|\mathcal{F}_k(g)|^2(z)|\mathcal{F}_k(f)(z)|^2 d\mu_k(z)d\mu_k(y)\\ &= \|g\|^2_{L^2_{rad}(\mu_k)}\int_{\mathbb{R}^d}|\mathcal{F}_k(f)(z)|^2 d\mu_k(z).\end{aligned}$$

(ii) follows from (i) by polarization.

(iii) From Theorem 2.1 (iv), Proposition 2.3 (ii) and (iii), we have

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_g(f)(x,y)\overline{\tau_z g_{k,y}(-x)}d\mu_k(x)d\mu_k(y)\\ &\qquad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\tau_y|\mathcal{F}_k(g)|^2(t)\mathcal{F}_k(f)(t)E_k(iz,t)d\mu_k(t)d\mu_k(y).\end{aligned}$$

Then, by Fubini's theorem, Theorem 2.1 (ii) and Proposition 2.2 (ii) we deduce that

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_g(f)(x,y)\overline{\tau_z g_{k,y}(-x)}d\mu_k(x)d\mu_k(y) &= \|g\|^2_{L^2_{rad}(\mu_k)}\int_{\mathbb{R}^d}\mathcal{F}_k(f)(t)E_k(iz,t)d\mu_k(t)\\ &= \|g\|^2_{L^2_{rad}(\mu_k)}f(z).\end{aligned}$$

□

In the following we establish reproducing inversion formula of Calderón's type for the Dunkl-Wigner transform on $\mathbb{R}^d$.

Theorem 2.6. *Let* $\Delta = \prod_{j=1}^d [a_j, b_j]$*,* $-\infty < a_j < b_j < \infty$*; and let* $g \in L^2_{rad}(\mu_k)$ *such that* $\mathcal{F}_k(g) \in L^\infty(\mu_k)$*. Then, for* $f \in L^2(\mu_k)$*, the function* f_Δ *given by*

$$f_\Delta(z) = \frac{1}{\|g\|_{L^2_{rad}(\mu_k)}}\int_\Delta\int_{\mathbb{R}^d} V_g(f)(x,y)\overline{\tau_z g_{k,y}(-x)}d\mu_k(x)d\mu_k(y),$$

belongs to $L^2(\mu_k)$ *and satisfies*

$$\lim_{\substack{a_j \to -\infty \\ b_j \to +\infty}} \|f_\Delta - f\|_{L^2(\mu_k)} = 0. \tag{2.8}$$

Proof. From Theorem 2.1 (iii), Proposition 2.3 (iv) and Proposition 2.4 (i), we have

$$f_\Delta(z) = \frac{1}{\|g\|^2_{L^2_{rad}(\mu_k)}} \int_\Delta \int_{\mathbb{R}^d} \tau_y |\mathcal{F}_k(g)|^2(t) \mathcal{F}_k(f)(t) E_k(iz, t) d\mu_k(t) d\mu_k(y).$$

By Fubini's theorem we get

$$f_\Delta(z) = \int_{\mathbb{R}^d} K_\Delta(t) \mathcal{F}_k(f)(t) E_k(iz, t) d\mu_k(t). \tag{2.9}$$

where

$$K_\Delta(t) = \frac{1}{\|g\|^2_{L^2_{rad}(\mu_k)}} \int_\Delta \tau_y |\mathcal{F}_k(g)|^2(t) d\mu_k(y).$$

It is easily to see that $\|K_\Delta\|_{L^\infty(\mu_k)} \leq 1$. On the other hand, by Hölder's inequality, we deduce that

$$|K_\Delta(t)|^2 \leq \frac{\mu_k(\Delta)}{\|g\|^4_{L^2_{rad}(\mu_k)}} \int_\Delta |\tau_y |\mathcal{F}_k(g)|^2(t)|^2 d\mu_k(y).$$

Hence, by (2.4) we find

$$\|K_\Delta\|^2_{L^2(\mu_k)} \leq \frac{(\mu_k(\Delta))^2}{\|g\|^4_{L^2_{rad}(\mu_k)}} \int_{\mathbb{R}^d} |\mathcal{F}_k(g)(t)|^4 d\mu_k(t) \leq \frac{(\mu_k(\Delta))^2 \|\mathcal{F}_k(g)\|^2_{L^\infty(\mu_k)}}{\|g\|^2_{L^2_{rad}(\mu_k)}}.$$

Thus $K_\Delta \in L^\infty \cap L^2(\mu_k)$. Therefore and by (2.9) we obtain

$$\mathcal{F}_k(f_\Delta)(t) = K_\Delta(t) \mathcal{F}_k(f)(t).$$

From this relation and Theorem 2.1 (iii), it follows that $f_\Delta \in L^2(\mu_k)$ and

$$\|f_\Delta - f\|^2_{L^2(\mu_k)} = \int_{\mathbb{R}^d} |\mathcal{F}_k(f)(t)|^2 (1 - K_\Delta(t))^2 d\mu_k(t).$$

But by Proposition 2.2 (ii) we have

$$\lim_{\substack{a_j \to -\infty \\ b_j \to +\infty}} K_\Delta(t) = 1, \quad \text{for all } t \in \mathbb{R}^d,$$

and

$$|\mathcal{F}_k(f)(t)|^2 (1 - K_\Delta(t))^2 \leq |\mathcal{F}_k(f)(t)|^2, \quad \text{for all } t \in \mathbb{R}^d.$$

So, the relation (2.8) follows from the dominated convergence theorem. □

3 Extremal functions for the mapping V_g

Let $s \geq 0$. We define the Sobolev-Dunkl space of order s, that will be denoted $H^s(\mu_k)$, as the set of all $f \in L^2(\mu_k)$ such that $(1+|z|^2)^{s/2}\mathcal{F}_k(f) \in L^2(\mu_k)$. The space $H^s(\mu_k)$ provided with the inner product

$$\langle f, g\rangle_{H^s(\mu_k)} = \int_{\mathbb{R}^d} (1+|z|^2)^s \mathcal{F}_k(f)(z)\overline{\mathcal{F}_k(g)(z)}d\mu_k(z),$$

and the norm

$$\|f\|_{H^s(\mu_k)} = \left[\int_{\mathbb{R}^d} (1+|z|^2)^s |\mathcal{F}_k(f)(z)|^2 d\mu_k(z)\right]^{1/2}.$$

The space $H^s(\mu_k)$ satisfies the following properties.

(a) $H^0(\mu_k) = L^2(\mu_k)$.

(b) For all $s > 0$, the space $H^s(\mu_k)$ is continuously contained in $L^2(\mu_k)$ and $\|f\|_{L^2(\mu_k)} \leq \|f\|_{H^s(\mu_k)}$.

(c) For all $s, t > 0$, such that $t > s$, the space $H^t(\mu_k)$ is continuously contained in $H^s(\mu_k)$ and $\|f\|_{H^s(\mu_k)} \leq \|f\|_{H^t(\mu_k)}$.

(d) The space $H^s(\mu_k)$, $s \geq 0$ provided with the inner product $\langle ., .\rangle_{H^s(\mu_k)}$ is a Hilbert space.

Remark 3.1. For $s > \gamma + d/2$, the function $y \to (1+|z|^2)^{-s/2}$ belongs to $L^2(\mu_k)$. Hence for all $f \in H^s(\mu_k)$, we have $\|\mathcal{F}_k(f)\|_{L^2(\mu_k)} \leq \|f\|_{H^s(\mu_k)}$, and by Hölder's inequality

$$\|\mathcal{F}_k(f)\|_{L^1(\mu_k)} \leq \left[\int_{\mathbb{R}^d} \frac{d\mu_k(z)}{(1+|z|^2)^s}\right]^{1/2} \|f\|_{H^s(\mu_k)}.$$

Then the function $\mathcal{F}_k(f)$ belongs to $L^1 \cap L^2(\mu_k)$, and therefore

$$f(x) = \int_{\mathbb{R}^d} E_k(ix, z)\mathcal{F}_k(f)(z)d\mu_k(z), \quad \text{a.e. } x \in \mathbb{R}^d.$$

Let $\lambda > 0$. We denote by $\langle ., .\rangle_{\lambda,H^s(\mu_k)}$ the inner product defined on the space $H^s(\mu_k)$ by

$$\langle f, h\rangle_{\lambda,H^s(\mu_k)} := \lambda\langle f, h\rangle_{H^s(\mu_k)} + \langle V_g(f), V_g(h)\rangle_{L^2(\mu_k\otimes\mu_k)},$$

and the norm $\|f\|_{\lambda,H^s(\mu_k)} := \sqrt{\langle f, f\rangle_{\lambda,H^s(\mu_k)}}$.

In the next we suppose that $g \in L^2_{rad}(\mu_k)$. By Theorem 2.5 (ii), the inner product $\langle ., .\rangle_{\lambda,H^s(\mu_k)}$ can be written

$$\langle f, h\rangle_{\lambda,H^s(\mu_k)} = \lambda\langle f, h\rangle_{H^s(\mu_k)} + \|g\|^2_{L^2_{rad}(\mu_k)}\langle f, h\rangle_{L^2(\mu_k)}. \tag{3.1}$$

Theorem 3.2. *Let* $\lambda > 0$ *and* $s > \gamma + d/2$ *and let* $g \in L^2_{rad}(\mu_k)$. *The space* $(H^s(\mu_k), \langle ., .\rangle_{\lambda,H^s(\mu_k)})$ *has the reproducing kernel*

$$K_s(x, y) = \int_{\mathbb{R}^d} \frac{E_k(ix, z)E_k(-iy, z)}{\lambda(1+|z|^2)^s + \|g\|^2_{L^2_{rad}(\mu_k)}} d\mu_k(z), \tag{3.2}$$

that is

(i) *For all* $y \in \mathbb{R}^d$, *the function* $x \to K_s(x,y)$ *belongs to* $H^s(\mu_k)$.

(ii) *The reproducing property: for all* $f \in H^s(\mu_k)$ *and* $y \in \mathbb{R}^d$,

$$\langle f, K_s(.,y)\rangle_{\lambda,H^s(\mu_k)} = f(y).$$

Proof. (i) Let $y \in \mathbb{R}^d$. From (2.2), the function $\Phi_y : z \to \frac{E_k(-iy,z)}{\lambda(1+|z|^2)^s+\|g\|^2_{L^2_{rad}(\mu_k)}}$ belongs to $L^1 \cap L^2(\mu_k)$. Then, the function K_s is well defined and by Theorem 2.1 (ii), we have

$$K_s(x,y) = \mathcal{F}_k^{-1}(\Phi_y)(x), \quad x \in \mathbb{R}^d.$$

From Theorem 2.1 (iii), it follows that $K_s(.,y)$ belongs to $L^2(\mu_k)$, and we have

$$\mathcal{F}_k(K_s(.,y))(z) = \frac{E_k(-iy,z)}{\lambda(1+|z|^2)^s + \|g\|^2_{L^2_{rad}(\mu_k)}}, \quad z \in \mathbb{R}^d. \tag{3.3}$$

Then by (2.2), we obtain

$$|\mathcal{F}_k(K_s(.,y))(z)| \leq \frac{1}{\lambda(1+|z|^2)^s},$$

and

$$\|K_s(.,y)\|^2_{H^s(\mu_k)} \leq \frac{1}{\lambda^2}\int_{\mathbb{R}^d} \frac{d\mu_k(z)}{(1+|z|^2)^s} < \infty.$$

This proves that for all $y \in \mathbb{R}^d$ the function $K_s(.,y)$ belongs to $H^s(\mu_k)$.

(ii) Let $f \in H^s(\mu_k)$ and $y \in \mathbb{R}^d$. From (3.1) and (3.3), we have

$$\langle f, K_s(.,y)\rangle_{\lambda,H^s(\mu_k)} = \int_{\mathbb{R}^d} E_k(iy,z)\mathcal{F}_k(f)(z)d\mu_k(z),$$

and from Remark 3.1, we obtain the reproducing property:

$$\langle f, K_s(.,y)\rangle_{\lambda,H^s(\mu_k)} = f(y).$$

This completes the proof of the theorem. □

The main result of this subsection can then be stated as follows.

Theorem 3.3. *Let* $s > \gamma + d/2$ *and* $g \in L^2_{rad}(\mu_k)$. *For any* $h \in L^2(\mu_k \otimes \mu_k)$ *and for any* $\lambda > 0$, *there exists a unique function* $f^*_{\lambda,g}$, *where the infimum*

$$\inf_{f \in H^s(\mu_k)} \left\{\lambda\|f\|^2_{H^s(\mu_k)} + \|h - V_g(f)\|^2_{L^2(\mu_k\otimes\mu_k)}\right\} \tag{3.4}$$

is attained. Moreover, the extremal function $f^*_{\lambda,h}$ *is given by*

$$f^*_{\lambda,h}(y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(x,t)Q_s(x,y,t)d\mu_k(t)d\mu_k(x),$$

where

$$Q_s(x,y,t)=\int_{\mathbb{R}^d}\frac{E_k(-ix,z)E_k(iy,z)\sqrt{\tau_t|\mathcal{F}_k(g)|^2(z)}}{\lambda(1+|z|^2)^s+\|g\|^2_{L^2_{rad}(\mu_k)}}d\mu_k(z).$$

Proof. The existence and unicity of the extremal function $f^*_{\lambda,h}$ satisfying (3.4) is given by Kimeldorf and Wahba [5], Matsuura et al. [6] and Saitoh [12]. Especially, $f^*_{\lambda,h}$ is given by the reproducing kernel of $H^s(\mu_k)$ with $\|.\|_{\lambda,H^s(\mu_k)}$ norm as

$$f^*_{\lambda,h}(y)=\langle h, V_g(K_s(.,y))\rangle_{L^2(\mu_k\otimes\mu_k)}, \tag{3.5}$$

where K_s is the kernel given by (3.2).
But by Proposition 2.4 (ii) and (3.3), we have

$$\begin{aligned}V_g(K_s(.,y))(x,t) &= \int_{\mathbb{R}^d}E_k(ix,z)\mathcal{F}_k(K_s(.,y))(z)\sqrt{\tau_t|\mathcal{F}_k(g)|^2(z)}d\mu_k(z)\\ &= \int_{\mathbb{R}^d}\frac{E_k(ix,z)E_k(-iy,z)\sqrt{\tau_t|\mathcal{F}_k(g)|^2(z)}}{\lambda(1+|z|^2)^s+\|g\|^2_{L^2_{rad}(\mu_k)}}d\mu_k(z).\end{aligned}$$

This clearly yields the result. □

Theorem 3.4. *Let* $s>\gamma+d/2$ *and* $g\in L^2_{rad}(\mu_k)$. *For any* $h\in L^2(\mu_k\otimes\mu_k)$ *and for any* $\lambda>0$, *we have*

(i) $|f^*_{\lambda,h}(y)|\leq\frac{\|h\|_{L^2(\mu_k\otimes\mu_k)}}{2\sqrt{\lambda}}\left[\int_{\mathbb{R}^d}\frac{d\mu_k(z)}{(1+|z|^2)^s}\right]^{1/2}$.

(ii) $\|f^*_{\lambda,h}\|^2_{L^2(\mu_k)}\leq\frac{1}{4\lambda}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|h(x,t)|^2e^{(|x|^2+|t|^2)/2}d\mu_k(t)d\mu_k(x)$.

Proof. (i) From (3.5) and Theorem 2.5 (i), we have

$$\begin{aligned}|f^*_{\lambda,h}(y)| &\leq \|h\|_{L^2(\mu_k\otimes\mu_k)}\|V_g(K_s(.,y))\|_{L^2(\mu_k\otimes\mu_k)}\\ &\leq \|h\|_{L^2(\mu_k\otimes\mu_k)}\|g\|_{L^2_{rad}(\mu_k)}\|K_s(.,y)\|_{L^2(\mu_k)}.\end{aligned}$$

Then, by Theorem 2.1 (iii) and (3.3), we deduce that

$$\begin{aligned}|f^*_{\lambda,g}(y)| &\leq \|h\|_{L^2(\mu_k\otimes\mu_k)}\|g\|_{L^2_{rad}(\mu_k)}\|\mathcal{F}_k(K_s(.,y))\|_{L^2(\mu_k)}\\ &\leq \|h\|_{L^2(\mu_k\otimes\mu_k)}\|g\|_{L^2_{rad}(\mu_k)}\left[\int_{\mathbb{R}^d}\frac{d\mu_k(z)}{[\lambda(1+|z|^2)^s+\|g\|^2_{L^2_{rad}(\mu_k)}]^2}\right]^{1/2}.\end{aligned}$$

Using the fact that $\left[\lambda(1+|z|^2)^s+\|g\|^2_{L^2_{rad}(\mu_k)}\right]^2\geq 4\lambda(1+|z|^2)^s\|g\|^2_{L^2_{rad}(\mu_k)}$, we obtain the result.

(ii) We write

$$f^*_{\lambda,h}(y)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{-(|x|^2+|t|^2)/4}e^{(|x|^2+|t|^2)/4}h(x,t)Q_s(x,y,t)d\mu_k(t)d\mu_k(x).$$

Applying Hölder's inequality, we obtain

$$|f^*_{\lambda,h}(y)|^2\leq\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|h(x,t)|^2e^{(|x|^2+|t|^2)/2}|Q_s(x,y,t)|^2d\mu_k(t)d\mu_k(x).$$

Thus and from Fubini-Tonnelli's theorem, we get

$$\|f^*_{\lambda,h}\|^2_{L^2(\mu_k)} \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |h(x,t)|^2 e^{(|x|^2+|t|^2)/2}\|Q_s(x,.,t)\|^2_{L^2(\mu_k)} d\mu_k(t)d\mu_k(x).$$

The function $z \to \frac{E_k(-ix,z)\sqrt{\tau_t|\mathcal{F}_k(g)|^2(z)}}{\lambda(1+|z|^2)^s+\|g\|^2_{L^2_{rad}(\mu_k)}}$ belongs to $L^1 \cap L^2(\mu_k)$, then by Theorem 2.1 (ii), we get

$$Q_s(x,y,t) = \mathcal{F}_k^{-1}\Big(\frac{E_k(-ix,z)\sqrt{\tau_t|\mathcal{F}_k(g)|^2(z)}}{\lambda(1+|z|^2)^s+\|g\|^2_{L^2_{rad}(\mu_k)}}\Big)(y).$$

Thus, by Theorem 2.1 (iii) we deduce that

$$\begin{aligned}\|Q_s(x,.,t)\|^2_{L^2(\mu_k)} &= \int_{\mathbb{R}^d} |\mathcal{F}_k(Q_s(x,.,t))(z)|^2 d\mu_k(z)\\ &\leq \int_{\mathbb{R}^d} \frac{\tau_t|\mathcal{F}_k(g)|^2(z)d\mu_k(z)}{[\lambda(1+|z|^2)^s+\|g\|^2_{L^2_{rad}(\mu_k)}]^2}.\end{aligned}$$

Then

$$\|Q(x,.,t)\|^2_{L^2(\mu_k)} \leq \frac{1}{4\lambda\|g\|^2_{L^2_{rad}(\mu_k)}}\int_{\mathbb{R}^d} \tau_t|\mathcal{F}_k(g)|^2(z)d\mu_k(z) \leq \frac{1}{4\lambda}.$$

From this inequality we deduce the result. □

Theorem 3.5. *Let* $s > \gamma + d/2$ *and* $g \in L^2_{rad}(\mu_k)$*. For any* $h \in L^2(\mu_k \otimes \mu_k)$ *and for any* $\lambda > 0$*, we have*

(i) $f^*_{\lambda,h}(y) = \displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{E_k(iy,z)\sqrt{\tau_t|\mathcal{F}_k(g)|^2(z)}\mathcal{F}_k(h(.,t))(z)}{\lambda(1+|z|^2)^s+\|g\|^2_{L^2_{rad}(\mu_k)}} d\mu_k(t)d\mu_k(z).$

(ii) $\mathcal{F}_k(f^*_{\lambda,h})(z) = \dfrac{\int_{\mathbb{R}^d}\sqrt{\tau_t|\mathcal{F}_k(g)|^2(z)}\mathcal{F}_k(h(.,t))(z)d\mu_k(t)}{\lambda(1+|z|^2)^s+\|g\|^2_{L^2_{rad}(\mu_k)}}.$

(iii) $\|f^*_{\lambda,h}\|_{H^s(\mu_k)} \leq \dfrac{1}{2\sqrt{\lambda}}\|h\|_{L^2(\mu_k\otimes\mu_k)}.$

Proof. (i) From Theorem 3.3 and Fubini's theorem, we have

$$\begin{aligned}f^*_{\lambda,h}(y) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{E_k(iy,z)\sqrt{\tau_t|\mathcal{F}_k(g)|^2(z)}}{\lambda(1+|z|^2)^s+\|g\|^2_{L^2_{rad}(\mu_k)}}\Big[\int_{\mathbb{R}^d} h(x,t)E_k(-ix,z)d\mu_k(x)\Big] d\mu_k(t)d\mu_k(z)\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{E_k(iy,z)\sqrt{\tau_t|\mathcal{F}_k(g)|^2(z)}\mathcal{F}_k(h(.,t))(z)}{\lambda(1+|z|^2)^s+\|g\|^2_{L^2_{rad}(\mu_k)}} d\mu_k(t)d\mu_k(z).\end{aligned}$$

(ii) The function $z \to \frac{\int_{\mathbb{R}^d}\sqrt{\tau_t|\mathcal{F}_k(g)|^2(z)}\mathcal{F}_k(h(.,t))(z)d\mu_k(t)}{\lambda(1+|z|^2)^s+\|g\|^2_{L^2_{rad}(\mu_k)}}$ belongs to $L^1 \cap L^2(\mu_k)$. Then by Theorem 2.1 (ii) and (iii), it follows that $f^*_{\lambda,h}$ belongs to $L^2(\mu_k)$, and

$$\mathcal{F}_k(f^*_{\lambda,h})(z) = \frac{\int_{\mathbb{R}^d} \sqrt{\tau_t|\mathcal{F}_k(g)|^2(z)}\mathcal{F}_k(h(.,t))(z)d\mu_k(t)}{\lambda(1+|z|^2)^s + \|g\|^2_{L^2_{rad}(\mu_k)}}.$$

(iii) From (ii), Hölder's inequality and (2.6) we have

$$|\mathcal{F}_k(f^*_{\lambda,h})(z)|^2 \le \frac{\|g\|^2_{L^2_{rad}(\mu_k)}}{[\lambda(1+|z|^2)^s + \|g\|^2_{L^2_{rad}(\mu_k)}]^2} \int_{\mathbb{R}^d} |\mathcal{F}_k(h(.,t))(z)|^2 d\mu_k(t).$$

Thus,

$$\begin{aligned} \|f^*_{\lambda,h}\|^2_{H^s(\mu_k)} &\le \int_{\mathbb{R}^d} \frac{(1+|z|^2)^s\|g\|^2_{L^2_{rad}(\mu_k)}}{[\lambda(1+|z|^2)^s + \|g\|^2_{L^2_{rad}(\mu_k)}]^2} \left[\int_{\mathbb{R}^d} |\mathcal{F}_k(h(.,t))(z)|^2 d\mu_k(t)\right] d\mu_k(z) \\ &\le \frac{1}{4\lambda}\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} |\mathcal{F}_k(h(.,t))(z)|^2 d\mu_k(t)\right] d\mu_k(z) = \frac{1}{4\lambda}\|h\|^2_{L^2(\mu_k\otimes\mu_k)}, \end{aligned}$$

which ends the proof. □

Theorem 3.6. *Let* $s > \gamma + d/2$ *and* $g \in L^2_{rad}(\mu_k)$. *For any* $h \in L^2(\mu_k \otimes \mu_k)$ *and for any* $\lambda > 0$, *we have*

$$V_g(f^*_{\lambda,h})(x,y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{E_k(ix,z)\sqrt{\tau_t|\mathcal{F}_k(g)|^2(z)\tau_y|\mathcal{F}_k(g)|^2(z)}\mathcal{F}_k(h(.,t))(z)}{\lambda(1+|z|^2)^s + \|g\|^2_{L^2_{rad}(\mu_k)}} d\mu_k(t)d\mu_k(z).$$

Proof. From Proposition 2.4 (ii), we have

$$V_g(f^*_{\lambda,h})(x,y) = \int_{\mathbb{R}^d} E_k(ix,z)\mathcal{F}_k(f^*_{\lambda,h})(z)\sqrt{\tau_y|\mathcal{F}_k(g)|^2(z)}d\mu_k(z).$$

Then by Theorem 3.5 (ii), we obtain the result. □

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