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On a Type of Volterra Integral Equation in the Space of Continuous Functions with Bounded Variation valued in Banach Spaces

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ABSTRACT

In this paper we prove existence and uniqueness of the solutions for a kind of Volterra equation, with an initial condition, in the space of the continuous functions with bounded variation which take values in an arbitrary Banach space. Then we give a parameters variation formula for the solutions of certain kind of linear integral equation. Finally, we prove exact controllability of a particular integral equation using that formula. Moreover, under certain condition, we find a formula for a control steering of a type of system which is studied in the current work, from an initial state to a final one in a prescribed time.

RESUMEN

En este trabajo probamos existencia y unicidad de las soluciones para una ecuación de Volterra, con condición inicial, en el espacio de funciones continuas con variación acotada y valores en un espacio de Banach arbitrario. Damos una formula de variación de parámetros para las soluciones de cierta clase de ecuación lineal integral. Finalmente probamos la controlabilidad exacta de una ecuación integral particular usando esa formula. Más aún, bajo cierta condición, encontramos una formula para una dirección de control de un tipo de Sistema que se estudia en el presente trabajo, desde un estado inicial a uno final en un tiempo prescrito.

Keywords and Phrases: Existence and uniqueness of solutions of integral equations in Banach spaces; continuous functions; bounded variation norm; parameters variation formula; controllability.

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1 Introduction

In the course of the last decades, many authors have studied integral equations in the Banach spaces. Some examples of this kind of work are [1], [10], [13], [17] and [22]. Simultaneously, such as it has been done in [7], [9] and [21], the study of the solutions of the integral equations has considered various spaces of bounded variation functions. Even more, there are works that combine these two trends in the field of the integral equations, such as it is done in [2], [8], [11] and [15]. The present paper follows this tendency, since in section 3 we prove the existence and uniqueness of the solution of the nonlinear problem with initial condition

$$\begin{cases} x(t) - x(a) = A(g)(t) + \lambda \int_a^t K(t,s) f(x(s)) ds, \\ x(a) = x_0 \in \mathbb{X}, \end{cases}$$

in the space of continuous functions of bounded variation defined on the interval $[a, b]$ with values in a normed space $\mathbb{X}$, endowed with the norm $|x|_1 := |x(a)| + \bigvee(x)$, where $\bigvee(x)$ is the bounded variation of the function x, A is a convenient function defined on such functions space and the symbol of integral is referred to the Riemann integral on any Banach space $\mathbb{X}$. Furthermore, $K(t,s)$ is a continuous linear operator from $\mathbb{X}$ in $\mathbb{X}$ for each (t,s) belonging to a convenient subset in $[a, b] \times [a, b]$, $f : \mathbb{X} \longrightarrow \mathbb{X}$ and where the expression $K(t,s)f(x(s))$ in the integral part means that the linear operator $K(t,s)$ is evaluated in $f(x(s))$.

An important part of our considerations is the representation of the solutions. There are known many formulas for the solutions of several kind of integral equations in the space of the real continuous functions defined on set of the real numbers, such as we can find in [20]. In the case of abstract Banach spaces, we find in [16] a parameters variation formula for the Volterra-Stieltjest integral equations in the space of regulated functions. In the same way, in section 4, we give a formula of this type for the solution of linear problem

$$\begin{cases} x(t) - x(a) = \int_a^t B(g)(s) ds + \lambda \int_a^t K(t,s) x(s) ds, \\ x(a) = x_0 \in \mathbb{X}, \end{cases}$$

where $B : \mathcal{F} \longrightarrow \mathcal{F}$ is an adequate function, $\mathcal{F}$ denotes the space of continuous functions of bounded variation defined from the interval $[a, b]$ into any fixed normed space $\mathbb{X}$ and the expression $K(t,s)x(s)$ in the integral part means that the continuous linear operator $K(t,s) : \mathbb{X} \longrightarrow \mathbb{X}$ is evaluated in $x(s)$.

Another interesting question is referred to the controllability of such equations. In papers like [3], [4] and [5] is studied the controllability of some types of integral equations, and most recently in [2], [13] and [19], is studied the controllability of Hammerstein or Volterra integral equation on Banach or Hilbert spaces. In this setting, in section 5, we prove the existence of an exact control of the linear system

$$\begin{cases} x(t) - x(a) = \int_a^t B(g)(s) ds + \lambda \int_a^t K(t,s) x(s) ds, \\ x(a) = x_0 \in \mathbb{X}, \end{cases}$$

and then, in section 6, we verify the controllability of nonlinear system

$$\begin{cases} x(t) - x(a) = \displaystyle\int_a^t B(g)(s)ds \ + \ \lambda \int_a^t T(s)f(x(s))ds, \\ x(a) = x_0 \in \mathbb{X}, \end{cases}$$

under assumption that f is a globally Lipschitz function and such that $T(s) : \mathbb{X} \longrightarrow \mathbb{X}$ is a bounded linear operator for each $s \in [a, b]$. The study of the above two systems was motivated by the work [13], since they are particular forms of the integral equation which is studied in it.

In section 2, we have gathered the definitions and properties of Riemann integral and functions of bounded variations in normed spaces which are used in this work.

2 Preliminares

In this paper, we use the symbol $\mathbb{X}$ to denote any Banach space endowed with a norm $|\cdot|$. Now, let us recall the following definition.

Definition 2.1. [18] Let us fix an interval $I := [a, b]$ and consider a function $y : I \longrightarrow \mathbb{X}$. Let $\Pi := \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of interval I. We define $\bigvee(y, \Pi)$ by

$$\bigvee(y, \Pi) = \sum_{i=1}^{n} \left| \ y(t_i) - y(t_{t_{i-1}}) \ \right|.$$

We call the least upper bound of the set of all possible sums $\bigvee(y, \Pi)$ **the total variation of the function** $y(t)$ on the interval $[a, b]$ and we denote it by $\bigvee(y)$. If $\bigvee(y) < \infty$, then we say that the **function** $y(t)$ is **of bounded variation** on $[a, b]$.

Also, we recall two basic properties of functions of bounded variation in the subsequent proposition.

Proposition 2.1. *If* x and y *are of bounded variation, then*

1. $\bigvee(x + y) \ \leq \ \bigvee(x) \ + \ \bigvee(y)$ *and*
2. $\bigvee(\alpha y) \ = | \ \alpha \ | \bigvee(y)$ *for each real number* $\alpha \in \mathbb{R}$.

Notation 2.1. We denote the vector space of functions of bounded variation on $[a, b]$ by $BV[a, b]$.

The following proposition will be useful later.

Proposition 2.2. *If* $y \in BV[a, b]$, *then* $y(t)$ *is bounded. Furthermore,*

$$\sup_{s \in I} | \ y(s) \ | \leq | \ y(a) \ | \ + \ \bigvee(y).$$

Definition 2.2. We define the set of functions $\mathcal{F}$ by

$$\mathcal{F} := C[a,b] \bigcap BV[a,b],$$

where $C[a,b] := \Big\{ \ x : I \longrightarrow \mathbb{X} \ : \ x \textit{ is continuous} \Big\}$.

Proposition 2.3. *The vector space $\mathcal{F}$ endowed with the norm*

$$|x|_1 := \ |x(a)| + \bigvee(x)$$

is a Banach space.

Now, let us recall the definitions of the Riemann integral for the functions of one and two real variables with values in a normed space and some of their properties, which we shall use in the following sections.

Definition 2.3. [18] Let us consider a function $x : [a,b] \longrightarrow \mathbb{X}$. We denote a partition $\Pi := \{a = t_0 < t_1 < \cdots < t_n = b\}$, together with the set of real numbers $\tau_i \in [t_{i-1}, t_i]$ for $i = 1, \ldots, n$, by P and put $|P| := \max\{|t_i - t_{i-1}| : i = 1, \ldots, n\}$. We define the Riemann sum S_P by

$$S_P := \sum_{i=1}^{n} (\ t_i - t_{i-1})\ x(\tau_i).$$

Moreover, we say that the Riemann integral is $\mathcal{I} \in \mathbb{X}$ if for each real number $\epsilon > 0$ there exists $\delta > 0$ such that when $|P| < \delta$, then $|\ \mathcal{I} - S_P\ | < \epsilon$. In this case the element $\mathcal{I} \in \mathbb{X}$ is called **the Riemann integral of the function** $x(t)$ and is denoted by

$$\int_a^b x(t)dt.$$

Proposition 2.4. [18] *Using the definition of Riemann integral one can easily verify the following properties:*

1.
$$\int_a^b x(t)dt = -\int_b^a x(t)dt,$$

provided that one of integrals exists.

2.
$$\int_a^b x(t)dt = \int_a^c x(t)dt \ + \int_c^b x(t)dt \ , \quad a < c < b$$

provided that the integral on the left member exists.

3. If $x(t) = x_0 \in \mathbb{X}$ for all $t \in [a, b]$, *then*

$$\int_a^b x_0 = (b - a)x_0.$$

4. *If* $x : [a, b] \longrightarrow \mathbb{X}$ *is continuous, then the Riemann integral* $\int_a^b x(t)dt$ *exists.*

5. If $x : [a, b] \longrightarrow \mathbb{X}$ *is continuous, then*

$$\left| \int_a^b x(t)dt \right| \leq \int_a^b | x(t) | \, dt.$$

Definition 2.4. Let us consider a function $F : [a, b] \times [a, b] \longrightarrow \mathbb{X}$. We denote two partitions $\Pi_1 := \{a = \sigma_0 < \sigma_1 < \cdots < \sigma_n = b\}$, together with the set of real numbers $\alpha_i \in [\sigma_{i-1}, \sigma_i]$ for $i = 1, \ldots, n$ and $\Pi_2 := \{a = s_0 < s_1 < \cdots < s_m = b\}$, together with the set of real numbers $\beta_j \in [s_{j-1}, s_j]$ for $j = 1, \ldots, m$, by P put and $|P| := \max\{|\sigma_i - \sigma_{i-1}| : i = 1, \ldots, n\} + \max\{|s_j - s_{j-1}| : j = 1, \ldots, m\}$. We define the Riemann sum S_P by the expression

$$S_P := \sum_{i=1}^{n} \sum_{j=1}^{m} (\sigma_i - \sigma_{i-1}) (s_j - s_{j-1}) F(\alpha_i, \beta_j) .$$

Moreover, we say that the Riemann integral is $\mathcal{I} \in \mathbb{X}$ if for each real number $\epsilon > 0$ there exists $\delta > 0$ such that when $|P| < \delta$, then $| \mathcal{I} - S_P | < \epsilon$. In this case the element $\mathcal{I} \in \mathbb{X}$ is called **the Riemann integral of the function** F and is denoted by

$$\int\int F \, d\sigma ds.$$

Theorem 2.1. *Let us assume that there exist the integrals* $\int_a^b F(\sigma, s)ds$ *for each* $\sigma \in [a, b]$ *and* $\int_a^b \left[\int_a^b F(\sigma, s)ds\right] d\sigma$. *If there exists* $\int\int F \, d\sigma ds$, *then*

$$\int\int F \, d\sigma ds = \int_a^b \left[\int_a^b F(\sigma, s)ds \right] d\sigma.$$

3 Existence and uniqueness of the solutions of the nonlinear problem

Let us assume that $f : \mathbb{X} \longrightarrow \mathbb{X}$ is globally Lispchitz with Lipschitz constant $L \geq 0$. We denote by the letter $\boldsymbol{\mathcal{L}}$ the set of continuous linear operator acting from $\mathbb{X}$ in $\mathbb{X}$ and given $(t, s) \in \Delta :=$

$\{(t,s) \in \mathbb{R}^2 : a \leq s \leq t \leq b\}$, we shall suppose that $K(t,s) \in \boldsymbol{\mathcal{L}}$. In this section we prove the existence and uniqueness of the solution for the problem

$$\begin{cases} x(t) - x(a) = A(g)(t) + \lambda \int_a^t K(t,s) f(x(s)) ds, \\ x(a) = x_0 \in \mathbb{X}, \end{cases}$$

in the Banach space $\mathcal{F}$, where A is a function from $\mathcal{F}$ in $\mathcal{F}$, such that $A(g)(a)$ is equal to the null vector of $\mathbb{X}$ for each $g \in \mathcal{F}$, the integral symbol is referred to the Riemann integral in $\mathbb{X}$ and the expression $K(t,s)f(x(s))$ in the integral part means that the linear operator $K(t,s)$ is evaluated in $f(x(s))$.

Example 3.1. A pair of examples of the function $f : \mathbb{X} \longrightarrow \mathbb{X}$ are $f(x) := Lx + x_0$ and $f(x) := L \sin \|x\| x_0$, where L is a real number and x_0 is any fixed element belonging to $\mathbb{X}$.

Example 3.2. Two examples of the above function $A : \mathcal{F} \longrightarrow \mathcal{F}$ are $A(g)(t) := g(t) - g(a)$ and $A(g)(t) := \int_a^t B(g)(s) ds$, where B is any function from $\mathcal{F}$ into $\mathcal{F}$.

Definition 3.1. A **function** $K : \Delta \longrightarrow \mathcal{L}$ is **uniformly Lipschitz in the first variable**, if there exists $\widehat{L} > 0$ such that

$$\| K(t,s) - K(\tau,s) \| \leq \widehat{L} \ | t - \tau |$$

for all pairs (t,s) and (τ,s) belonging to the set Δ, where Δ is defined by

$$\Delta := \{ (t,s) \in \mathbb{R}^2 : a \leq s \leq t \leq b \}$$

and the symbol $\| \cdot \|$ is referred to the usual *operator norm* in the space $\mathcal{L}$.

Assumption 3.1. We suppose that:

1. the function $K : \Delta \longrightarrow \mathcal{L}$ is uniformly Lipschitz in the first variable,
2. $K(t,\cdot) : [a,t] \longrightarrow \mathcal{L}$ is continuous for each $t \in (a,b]$ and
3. $\sup_{s \in [a,b]} \|K(s,s)\| < \infty$.

Example 3.3. Let $T : \mathbb{X} \longrightarrow \mathbb{X}$ be any fixed bounded linear operator different than the null operator in $\mathcal{L}$. An example of the function $K : \Delta \longrightarrow \mathcal{L}$ which is mentioned in the above assumption is $K(t,s) := Q(t,s)T$, where $Q : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ is a continuous function such that $\frac{\partial Q}{\partial t}$ is continuous.

The following proposition is a straightforward consequence from above Assumption 3.1.

Proposition 3.1. *The function* K *is bounded.*

Definition 3.2. Given $x \in \mathcal{F}$, we define $F(x) : [a, b] \longrightarrow \mathbb{X}$ by

$$F(x)(t) := \int_a^t K(t, s) f(x(s)) ds.$$

Now, we shall prove a pair of propositions about the function F.

Proposition 3.2. *If* $x \in \mathcal{F}$, *then* $F(x) \in \mathcal{F}$.

Proof. Let us note that

$$\begin{aligned}
&\left|F(x)(t+h) - F(x)(t)\right| = \left| \int_a^{t+h} K(t+h, s) f(x(s)) ds - \int_a^t K(t, s) f(x(s)) ds \right| \\
&\le \int_a^t \left\|K(t+h, s) - K(t, s)\right\| \cdot \left|f(x(s))\right| ds + \int_t^{t+h} \left\|K(t+h, s)\right\| \cdot \left|f(x(s))\right| ds \\
&\le \widehat{L} \cdot |h| \cdot \max_{s\in[a,b]} \left|f(x(s))\right| \int_a^t ds + \sup_{(t,s)\in\Delta} \left\|K(t, s)\right\| \cdot \max_{s\in[a,b]} \left|f(x(s))\right| \int_t^{t+h} ds \\
&\le \left\{ \widehat{L} \cdot (b-a) \cdot \max_{s\in[a,b]} \left|f(x(s))\right| ds + \sup_{(t,s)\in\Delta} \left\|K(t, s)\right\| \cdot \max_{s\in[a,b]} \left|f(x(s))\right| \right\} |h| .
\end{aligned}$$

From the above inequality we can deduce that the function F is continuous. Now we shall convince ourselves that if $x \in \mathcal{F}$, then $F(x) \in BV$. Let $\Pi := \{ a = t_0 < t_1 < \cdots < t_n = b \}$ be a partition of interval $[a, b]$ and observe that

$$\begin{aligned}
&\sum_{k=1}^n | F(x)(t_k) - F(x)(t_{k-1}) | \\
&= \sum_{k=1}^n \left| \int_a^{t_k} K(t_k, s) f(x(s)) ds - \int_a^{t_{k-1}} K(t_{k-1}, s) f(x(s)) ds \right| \\
&= \sum_{k=1}^n \left| \int_a^{t_{k-1}} K(t_k, s) f(x(s)) ds - \int_a^{t_{k-1}} K(t_{k-1}, s) f(x(s)) ds + \int_{t_{k-1}}^{t_k} K(t_k, s) f(x(s)) ds \right| \\
&\le \sum_{k=1}^n \int_a^{t_{k-1}} \|K(t_k, s) - K(t_{k-1}, s)\| \, | f(x(s)) | \, ds + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|K(t_k, s)\| \, | f(x(s)) | \, ds \\
&\le \max_{s\in[a,b]} | f(x(s)) | \left\{ \widehat{L}(b-a) \sum_{k=1}^n (t_k - t_{k-1}) + \sup_{(t,s)\in\Delta} \|K(t, s)\| \sum_{k=1}^n (t_k - t_{k-1}) \right\} \\
&= \max_{s\in[a,b]} | f(x(s)) | \left\{ \widehat{L}(b-a) + \sup_{(t,s)\in\Delta} \|K(t, s)\| \right\} (b-a) .
\end{aligned}$$

□

Proposition 3.3. *There exists a constant* $C > 0$ *such that*

$$\left|F(x) - F(y)\right|_1 \;\leq\; C\left|x - y\right|_1,$$

for each pair of functions $x, y \in \mathcal{F}$.

Proof. First, let us note that

$$|F(x) - F(y)|_1 \;=\; \bigvee (F(x) - F(y)).$$

Now, if $\prod := \{a = t_0 < t_1 < \cdots < t_n = b\}$ is any partition of interval $[a, b]$, then we obtain that

$$\sum_{i=1}^{n} |(F(x) - F(y))(t_i) - (F(x) - F(y))(t_{i-1})|$$

$$= \sum_{i=1}^{n} \left| \int_a^{t_i} K(t_i, s)\left[f(x(s)) - f(y(s))\right] ds - \int_a^{t_{i-1}} K(t_{i-1}, s)\left[f(x(s)) - f(y(s))\right] ds \right|$$

$$\leq \sum_{i=1}^{n} \int_a^{t_{i-1}} \|K(t_i, s) - K(t_{i-1}, s)\| \cdot |\, f(x(s)) - f(y(s)) \,|\, ds$$

$$+ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|K(t_i, s)\| \cdot |\, f(x(s)) - f(y(s)) \,|\, ds$$

$$\leq L|x - y|_1 \left[\sum_{i=1}^{n} \int_a^{t_{i-1}} \widehat{L}(t_i - t_{i-1}) ds \;+\; \sup_{(t,s)\in\Delta} \|K(t, s)\| \sum_{i=1}^{n} (t_i - t_{i-1}) \right]$$

$$\leq L\ (b - a) \left[\widehat{L}(b - a) \;+\; \sup_{(t,s)\in\Delta} |\, K(t, s) \,| \right] \left|x - y\right|_1.$$

This proof ends when we realize that one adequate constant is

$$C := L\ (b - a) \left[\widehat{L}(b - a) \;+\; \sup_{(t,s)\in\Delta} |\, K(t, s) \,| \right].$$

□

Theorem 3.1. *Let* x_0 *be any element belonging to* $\mathbb{X}$. *Given* $g \in \mathcal{F}$, *there is a real number* $\rho > 0$ *such that for each fixed real number* λ with $|\lambda| \leq \rho$, *there exists a unique solution* $x_g \in \mathcal{F}$ *for the initial value problem*

$$\begin{cases} x(t) - x(a) \;=\; A(g)(t) \;+\; \lambda \displaystyle\int_a^t K(t, s) f(x(s)) ds, \\ x(a) = x_0. \end{cases}$$

Proof. Given $x, g \in \mathcal{F}$, $x_0 \in \mathbb{X}$ and $\lambda \in \mathbb{R}$, we define

$$G(x) : [a, b] \longrightarrow \mathbb{X}\ ;$$

$$G(x)(t) := x_0 + A(g)(t) + \lambda F(x)(t).$$

Observe that if $x \in \mathcal{F}$, then $G(x) \in \mathcal{F}$. Moreover, note that

$$|G(x) - G(y)|_1 = |\lambda| \, |F(x) - F(y)|_1.$$

From this equality, together with Proposition 3.3, we ensure the existence of a real number $\rho > 0$ such that the function $G_{g,\lambda,x_0} : \mathcal{F} \longrightarrow \mathcal{F}$; $G_{g,\lambda,x_0}(x) := G(x)$ is a contraction for each $\lambda \in \mathbb{R}$ with $|\lambda| \leq \rho$. The Banach's fixed point theorem implies the existence of an unique function $x_g \in \mathcal{F}$, which is the fixed point of the function $G_{g,\lambda,x_0} : \mathcal{F} \longrightarrow \mathcal{F}$. But this indicates that the function x_g is a solution of the above initial value problem.

□

Remark 3.1. The number $\rho > 0$ in Theorem 3.1 does not depend on the function $g \in \mathcal{F}$.

4 A formula for the solution of the linear problem

In this section we find a parameters variation formula for the solutions of the linear problem

$$\begin{cases} x(t) - x(a) = \int_a^t B(g)(s)ds + \lambda \int_a^t K(t,s)x(s)ds, \\ x(a) = x_0 \in \mathbb{X}, \end{cases} \tag{1}$$

in the Banach space $\mathcal{F}$, where $B : \mathcal{F} \longrightarrow \mathcal{F}$. Let us begin this part with two definitions.

Definition 4.1. We say that the function $U : \Delta \longrightarrow \mathcal{L}$ belongs to the set $\mathbb{H}$, if U is continuous and uniformly Lispchitz in the first variable.

Remark 4.1. If $U \in \mathbb{H}$, then U holds each one of the properties in Assumption 3.1.

Definition 4.2. We define the function $\| \cdot \|_1 : \mathbb{H} \longrightarrow \mathbb{R}$ by

$$\| U \|_1 := \| U \|_{11} + \| U \|_{12} , \quad \text{where}$$

$$\| U \|_{11} := \max_{(t,s)\in\Delta} \|U(t,s)\| \quad \text{and} \quad \| U \|_{12} := \sup_{s\in[a,b),\ t\neq\tau} \frac{\|U(t,s) - U(\tau,s)\|}{|t-\tau|},$$

such that $(t,s), (\tau,s) \in \Delta$.

Theorem 4.1. *The real vectorial space* $\mathbb{H}$ *endowed with the above function* $\|\cdot\|_1$ *is a Banach space.*

Now, we define an integral operator on the Banach space $\mathbb{H}$.

Definition 4.3. Let K be a fixed element belonging to $\mathbb{H}$. Given $U \in \mathbb{H}$, we define $F(U) : \Delta \longrightarrow \mathcal{L}$ by

$$F(U)(t,s) := \int_s^t K(t,\sigma)U(\sigma,s)d\sigma,$$

where $K(t,\sigma)U(\sigma,s)$ is referred to the composition of the linear operators $U(\sigma,s)$ and $K(t,\sigma)$.

Remark 4.2. The operator F in above Definition 4.3. is different than function F in Definition 3.2. Moreover, the integral symbol in above Definition 4.3. is referred to the Riemann integral in the space of the linear continuous operator $\mathcal{L} := \mathcal{L}(\mathbb{X},\mathbb{X})$.

The following theorem will play an important role in this section.

Theorem 4.2. *The above operator* F *is a well defined bounded linear operator from* $\mathbb{H}$ into $\mathbb{H}$.

Proof. The proof will be given by claims.

Claim 1: The operator F is well defined, since the integral part

$$[s,t] \ni \sigma \mapsto K(t,\sigma)U(\sigma,s) \in \mathcal{L}$$

is a continuous function and the Riemann integral is unique.

Claim 2: If $U \in \mathbb{H}$, then $F(U)$ is continuous.

Claim 3: If $U \in \mathbb{H}$, then the function $F(U)$ is uniformly Lispchitz in the first variable. To prove this claim, let us suppose that $s \in [a,b)$. Without lost of generality, we assume that $a \le s \le \tau < t < b$ and observe that

$$\begin{aligned}
\frac{\left\|F(U)(t,s) - F(U)(\tau,s)\right\|}{|\,t-\tau\,|} &= \frac{\left\|\int_s^t K(t,\sigma)U(\sigma,s)d\sigma - \int_s^\tau K(\tau,\sigma)U(\sigma,s)d\sigma\right\|}{|\,t-\tau\,|} \\
&\le \max_{(t,s)\in\Delta}\|U(t,s)\| \cdot \sup_{s\in[a,b)} \frac{\left\|K(t,s)-K(\tau,s)\right\|}{|\,t-\tau\,|}\int_s^\tau d\sigma \\
&+ \frac{1}{|\,t-\tau\,|}\max_{(t,s)\in\Delta}\|U(t,s)\| \cdot \max_{(t,s)\in\Delta}\|K(t,s)\|\int_\tau^t d\sigma \\
&\le \| U \|_{11} \,(\| K \|_{11} + (b-a)\,\| K \|_{12}).
\end{aligned}$$

Claim 4: The function F is a bounded linear transformation from $\mathbb{H}$ into $\mathbb{H}$. Since it is easy to verify that the function F is a linear transformation from $\mathbb{H}$ into $\mathbb{H}$, then only we prove that such operator is bounded. Firstly, note that

$$\max_{(t,s)\in\Delta}\left\|F(U)(t,s)\right\| \le (b-a)\max_{(t,\sigma)\in\Delta}\left\|K(t,\sigma)\right\| \cdot \max_{(\sigma,s)\in\Delta}\left\|U(\sigma,s)\right\|.$$

From here, together with the inequality in Claim 3, we can conclude that the linear operator F is bounded.

□

Now, we are ready to present and prove one important result of this section.

Theorem 4.3. *Let* K *be any fixed function belonging to* $\mathbb{H}$. *There is a real number* $\rho > 0$ *such that if* $|\lambda| < \rho$, *then there exists a unique function* $R \in \mathbb{H}$, *which satisfies the equality*

$$R(t,s) = I + \lambda \int_s^t K(t,\sigma)R(\sigma,s)d\sigma$$

for each pair of real numbers t and s *such that* $a \leq s \leq t \leq b$, *where the symbol* I *denote the identity operator* $I : \mathbb{X} \longrightarrow \mathbb{X}$.

Proof. The proof follows from the fact that $F : \mathbb{H} \longrightarrow \mathbb{H}$ is a bounded linear operator and applying the Banach's fixed point theorem to the operator $\mathcal{T}_\lambda : \mathbb{H} \longrightarrow \mathbb{H}$, which is given by

$$\mathcal{T}_\lambda(U) := I + \lambda F(U).$$

□

Remark 4.3. In above Theorem 4.3, the function R could depend on the real number λ and $R(t,t) = I$ for each $t \in [a,b]$.

The function R in Theorems 4.3 will allow us to find a representation of the solution of the linear problem (1) by means of two functions u and v, which are defined below.

Definition 4.4. Given $x_0 \in \mathbb{X}$, we define $u : [a,b] \longrightarrow \mathbb{X}$ by $u(t) := R(t,a)x_0$.

Lemma 4.1. *The above function* $u(t)$ *belongs to the space* $\mathcal{F}$.

Lemma 4.2. *Let us fix* $\lambda \in \mathbb{R}$ *such that* $|\lambda| < \rho$. *Given any* $x_0 \in \mathbb{X}$, *we have that*

$$u(t) = x_0 + \lambda \int_a^t K(t,\sigma)u(\sigma)d\sigma,$$

for each real number $t \in [a,b]$.

Proof. Evaluate both of the members of the equality of Theorem 4.3 in x_0 with $s = a$.

□

Now we define another function v.

Definition 4.5. We define $\nu : [a, b] \longrightarrow \mathbb{X}$ by

$$\nu(t) := \int_a^t R(t,s)B(g)(s)ds.$$

Lemma 4.3. *The function* ν *in Definition* 4.5 *belongs to* $\mathcal{F}$.

We need the following technical lemma in order to prove below Lemma 4.5.

Lemma 4.4. *Let us fix* $t \in [a, b]$. *If* $K \in \mathbb{H}$, *then*

$$\int_a^t \left[\int_s^t K(t,\sigma)R(\sigma,s)d\sigma \right] B(g)(s)ds = \int_a^t \left[K(t,\sigma) \int_a^\sigma R(\sigma,s)B(g)(s)ds \right] d\sigma.$$

Proof. Lemma 4.4 can be deduced from Theorem 2.1.

□

Lemma 4.5. *If* $K \in \mathbb{H}$, *then*

$$\nu(t) = \int_a^t B(g)(s)ds + \lambda \int_a^t K(t,\sigma)\nu(\sigma)d\sigma.$$

Proof. From Lemma 4.4 and Theorem 4.3, we have that

$$\nu(t) - \lambda \int_a^t K(t,\sigma)\nu(\sigma)d\sigma$$

$$= \int_a^t R(t,s)B(g)(s)ds - \lambda \int_a^t K(t,\sigma) \left[\int_a^\sigma R(\sigma,s)B(g)(s)ds \right] d\sigma$$

$$= \int_a^t R(t,s)B(g)(s)ds - \lambda \int_a^t \left[\int_s^t K(t,\sigma)R(\sigma,s)d\sigma \right] B(g)(s)ds$$

$$= \int_a^t R(t,s)B(g)(s)ds - \int_a^t [\, R(t,s) - I \,] B(g)(s)ds = \int_a^t B(g)(s)ds.$$

□

As a consequence of Lemmas 4.2 and 4.5, we obtain the following important result of this work.

Theorem 4.4. Let us suppose that $|\lambda| < \rho$, $K \in \mathbb{H}$ and $y_0 \in \mathbb{X}$. If B is any fixed function from $\mathcal{F}$ into $\mathcal{F}$ and $g \in \mathcal{F}$, then a solution $y_g \in \mathcal{F}$ for the problem with initial condition

$$\begin{cases} y(t) - y(a) = \int_a^t B(g)(s)ds + \lambda \int_a^t K(t,s)y(s)ds, \\ y(a) = y_0 \end{cases}$$

can be expressed by

$$y_g(t) = R(t,a)y_0 + \int_a^t R(t,s)B(g)(s)ds,$$

where R is the function which was found in Theorem 4.3.

Remark 4.4. The number $\rho > 0$ in above Theorem 4.4 does not depend on the functions B or g. Moreover, if this real number ρ is small enough, then the mentioned solution y_g is unique.

5 Controllability of the linear integral equation

Anew, we consider the linear problem which was studied in section 4

$$\begin{cases} y(t) - y(a) = \int_a^t B(g)(s)ds + \lambda \int_a^t K(t,s)y(s)ds, \\ y(a) = y_0, \end{cases} \tag{2}$$

where $K \in \mathbb{H}$, $B : \mathcal{F} \longrightarrow \mathcal{F}$ and $g \in \mathcal{F}$.

Definition 5.1. We say that the system (2) is **exactly controllable on the interval** $[a, b]$, if for each pair of elements y_0 and y_1 belonging to $\mathbb{X}$, there exists a function $g \in \mathcal{F}$ such that the corresponding solution $y \in \mathcal{F}$ of the problem (2) verify that $y(a) = y_0$ and $y(b) = y_1$.

Assumption 5.1. Let us consider a function $B : \mathcal{F} \longrightarrow \mathcal{F}$, which is not necessarily a linear operator. From now on, we assume that the function B is surjective and λ is such as in Theorem 4.4 statement.

Example 5.1. An example of a function $B : \mathcal{F} \longrightarrow \mathcal{F}$, such as in the above assumption, is defined by $B(g) := \gamma g + g_0$, where γ is a real number different than zero and g_0 is a fixed function belonging to $\mathcal{F}$.

Definition 5.2. We define the **controller map** $G : \mathcal{F} \longrightarrow \mathbb{X}$ by

$$G(g) = \int_a^b R(b,s)B(g)(s)ds.$$

As a consequence of Theorem 4.4, we have the following proposition.

Proposition 5.1. *The system* (2) *is exactly controllable on* $[a, b]$ *if, and only if*, $\mathrm{Rang}(G) = \mathbb{X}$.

Theorem 5.1. *If the function* B *is surjective, then the system*

$$\begin{cases} x(t) - x(a) \ = \ \int_a^t B(g)(s) \ + \ \lambda \int_a^t K(t,s)x(s)ds, \\ x(a) = x_0 \end{cases}$$

is exactly controllable on the interval $[a, b]$.

Proof. From the foregoing Proposition 5.1, it is enough to prove that Rang$(G) = \mathbb{X}$. In order to show this, let us recall that the function $R(b,\cdot) : [a,b] \longrightarrow \mathcal{L}$ is continuous by Theorem 4.3 and Definition 4.1. From this and Remark 4.1., there is a real number $\delta > 0$ such that $\| I - R(b,s) \| < \frac{1}{4}$ for all $s \in I = (b-\delta, b] \subset [a,b]$. Moreover, there exists a continuous function $\alpha : [a,b] \longrightarrow \mathbb{R}$ such that:

1. $\alpha(s) = 0$ *if* $s \in [a\ ,\ b-\delta]$,
2. $0 < \alpha(s) \ \leq \ \frac{1}{\delta}$ *if* $s \ \in (\ b-\delta\ ,\ b]$,
3. $\int_{b-\delta}^{b} (\frac{1}{\delta} - \alpha(s))ds \ < \ \frac{1}{4}$,
4. $0 < \int_{b-\delta}^{b} \alpha(s)ds$,
5. $\bigvee(\alpha) \ < \infty$.

Now, we define the function $\mathcal{H} : \mathbb{X} \longrightarrow \mathcal{F}$ by

$$\mathcal{H}(x)(s) := \alpha(s)x.$$

Furthermore, we consider the linear operator $T : \mathbb{X} \longrightarrow \mathbb{X}$ which is defined by

$$T(x) := \int_a^b R(b,s)\mathcal{H}(x)(s)\ ds \ = \ \int_a^b R(b,s)\alpha(s)x\ ds.$$

Let us prove that $\| \ I - T \ \| < 1$. To this end, observe that

$$\begin{aligned}
\left| \ I(x) - T(x) \ \right| &= \left| \ x - \int_a^b R(b,s)\big(\alpha(s)x\big)ds \ \right| = \left| \ \frac{1}{\delta}\int_{b-\delta}^{b} xds - \int_{b-\delta}^{b} \alpha(s)R(b,s)(x)ds \ \right| \\
&\leq \left| \ \frac{1}{\delta}\int_{b-\delta}^{b} xds \ - \int_{b-\delta}^{b} \alpha(s)xds \ \right| + \left| \ \int_{b-\delta}^{b} \alpha(s)xds \ - \int_{b-\delta}^{b} \alpha(s)R(b,s)(x)ds \ \right| \\
&= \left| \ \int_{b-\delta}^{b} \Big(\frac{1}{\delta} \ - \ \alpha(s)\Big)xds \ \right| + \left| \ \int_{b-\delta}^{b} \alpha(s)\Big(I \ - \ R(b,s)\Big)(x)ds \ \right| \\
&\leq \int_{b-\delta}^{b} \Big(\frac{1}{\delta} \ - \ \alpha(s)\Big) \ |x|ds \ + \ \int_{b-\delta}^{b} |\alpha(s)|\big\|I \ - \ R(b,s)\big\| \ |x|ds \\
&= \left[\int_{b-\delta}^{b} \left(\frac{1}{\delta} \ - \ \alpha(s)\right)ds \ \ + \ \int_{b-\delta}^{b} |\alpha(s)|\big\|I \ - \ R(b,s)\big\| \ ds\right] \ |x|
\end{aligned}$$

$$\leq \left[\frac{1}{4} + \int_{b-\delta}^{b} \frac{1}{\delta}\frac{1}{4}\, ds\right] |x| = \left[\frac{1}{4} + \frac{1}{4}\right] |x| = \frac{1}{2}\,|x|.$$

We have just proved that $(T - I)$ is bounded. Now, let us recall the following theorem.

Theorem.[14] *Let* $V : \mathbb{X} \longrightarrow \mathbb{X}$ *be a bounded linear operator on a Banach space* $\mathbb{X}$ *such that* $\| V \| < 1$. *Then the inverse* $(I - V)^{-1}$ *exists on* $\mathbb{X}$ *and is bounded.*

From above two paragraphs, we can infer that $T^{-1} : \mathbb{X} \longrightarrow \mathbb{X}$ exists and is bounded. Since we have assumed that B is surjective, then for a given $x \in \mathbb{X}$ there exists a function $g_x \in \mathcal{F}$ such that $B(g_x) = \mathcal{H}(T^{-1}(x))$. Now observe that

$$G(g_x) = \int_a^b R(b,s)B(g_x)(s)ds = \int_a^b R(b,s)\mathcal{H}(T^{-1}(x))ds$$

$$= \int_a^b R(b,s)\alpha(s)T^{-1}(x)ds = T(T^{-1}(x)) = x.$$

From here we conclude that $\mathrm{Rang}(G) = \mathbb{X}$.

□

Theorem 5.2. *The function* $B : \mathcal{F} \longrightarrow \mathcal{F}$ *admits a right inverse* $\Lambda : \mathcal{F} \longrightarrow \mathcal{F}$, *i.e.* $B \circ \Lambda = I$ *and there exists a control* g_{y_0,y_1} *steering the system* (2) *from the initial state* y_0 *to a final state* y_1 *which is given by*

$$g_{y_0,y_1} := \Lambda \circ \mathcal{H} \circ T^{-1}\big(y_1 - R(b,a)y_0\big) \in \mathcal{F}.$$

Proof.

Recall that any surjective function admits a right inverse. Now observe that

$$G(g_{y_0,y_1}) = \int_a^b R(b,s)B(g_{y_0,y_1})(s)ds$$

$$= \int_a^b R(b,s)B \circ \Lambda\Big(\mathcal{H} \circ T^{-1}\big(y_1 - R(b,a)y_0\big)\Big)(s)ds$$

$$= \int_a^b R(b,s)\Big(\mathcal{H} \circ T^{-1}\big(y_1 - R(b,a)y_0\big)\Big)(s)ds = y_1 - R(b,a)y_0.$$

Therefore,

$$y_1 = R(b,a)y_0 + \int_a^b R(b,s)B(g_{y_0,y_1})(s)ds.$$

□

Remark 5.1.[6] If $B : \mathcal{F} \longrightarrow \mathcal{F}$ is a surjective continuous linear operator and $\mathsf{Ker}(B)$ admits a complement, then B has a right inverse $\Lambda : \mathcal{F} \longrightarrow \mathcal{F}$, that is to say, $B \circ \Lambda = I$, such that Λ is a continuous linear operator.

Definition 5.3. Let $\Lambda : \mathcal{F} \longrightarrow \mathcal{F}$ be a right inverse of B, i.e. $B \circ \Lambda = I$. We define $\Gamma : \mathbb{X} \longrightarrow \mathcal{F}$ by $\Gamma(x) := \Lambda \circ \mathcal{H} \circ T^{-1}\big(x - R(b,a)x_0\big)$, where T^{-1} and $\mathcal{H}$ are defined in the proof of above Theorem 5.1.

Remark 5.2. In above Theorem 5.2, we proved that $\big(G \circ \Gamma\big)(x) = x - R(b,a)x_0$ for each $x \in \mathbb{X}$

6 Controllability of a type of nonlinear integral equation

In this section we prove the controllability of nonlinear system

$$\begin{cases} x(t) - x(a) = \displaystyle\int_a^t B(g)(s)ds + \lambda \int_a^t T(s)f(x(s))ds, \\ x(a) = x_0, \end{cases} \tag{3}$$

but before to do this, we shall prove that the following semilinear system is controllable,

$$\begin{cases} x(t) - x(a) = \displaystyle\int_a^t B(g)(s)ds + \lambda \int_a^t T(s)x(s)ds + \lambda \int_a^t T(s)f(x(s))ds, \\ x(a) = x_0. \end{cases} \tag{4}$$

To this end, we shall consider a particular integral equation under the following assumption.

Assumption 6.1. The function $T : [a,b] \longrightarrow \mathcal{L}$ is Lipschitz on $[a,b]$.

Remark 6.1. The function $K : \Delta \longrightarrow \mathcal{L}$, such that $K(t,s) := T(s)$, belongs to the Banach space $\mathbb{H}$ which was defined at the beginning in section 4.

Theorem 6.1. *There exists $\rho > 0$ such that for any λ with $|\lambda| \le \rho$ and $g \in \mathcal{F}$, the problem* (4) *admits only one solution $x_g \in \mathcal{F}$, which simultaneously is a solution for the integral equation*

$$x_g(t) = R(t,a)x_0 + \int_a^t R(t,s)B(g)(s)ds + \lambda \int_a^t R(t,s)T(s)f(x_g(s))ds, \tag{5}$$

where R is the function mentioned in Theorem 4.4.

Proof. The existence of x_g and ρ is a straightforward consequence of Theorem 3.1 with $A(g)(t) := \displaystyle\int_a^t B(g)(s)ds$, $K(t,s) := T(s)$ and the function $\Big(I + f\Big)$ instead of f. Now, in order to prove the above equality (5), given a function $g \in \mathcal{F}$, we define the function $B_g : [a,b] \longrightarrow \mathbb{X}$ by

$$B_g(s) := B(g)(s) + \lambda T(s)f(x_g(s)),$$

where $|\lambda| < \rho$ and x_g is the function just mentioned. Observe that the function $x_g \in \mathcal{F}$ is a solution in $\mathcal{F}$ of the problem with initial value

$$\begin{cases} y(t) - y(a) \;=\; \int_a^t B_g(s)ds + \;\lambda \int_a^t K(t,s)y(s)ds, \\ y(a) = x_0 \;\in \mathbb{X}, \end{cases}$$

where $K(t,s) := T(s)$. From Theorem 4.4, we get

$$x_g(t) = R(t,a)x_0 \;+\; \int_a^t R(t,s)B_g(s)ds,$$

which indicates that

$$x_g(t) \;=\; R(t,a)x_0 \;+\; \int_a^t R(t,s)B(g)(s)ds \;+\; \lambda \int_a^t R(t,s)T(s)f(x_g(s))ds.$$

□

We continue this section with a pair of definitions, and then we prove a set of technical propositions, which we shall use in order to verify that above semilinear problem (4) is controllable.

Definition 6.1. We define **the controller map** $G_f : \mathcal{F} \longrightarrow \mathbb{X}$ for the system (4) by

$$\begin{aligned} G_f(g) \;&=\; R(b,a)x_0 \;+\; \int_a^b R(b,s)B(g)(s)ds \;+\; \lambda \int_a^b R(b,s)T(s)f(x_g(s))ds. \\ &=\; G_R(g) \;+\; H(g)\,, \end{aligned}$$

such that $G_R : \mathcal{F} \longrightarrow \mathbb{X}$ is defined by

$$G_R(g) = \; R(b,a)x_0 \;+\; \int_a^b R(b,s)B(g)(s)ds \;=\; R(b,a)x_0 \;+\; G(g),$$

where G is the controller map in Definition 5.2. and $H : \mathcal{F} \longrightarrow \mathbb{X}$ is defined by

$$H(g) = \;\lambda \int_a^b R(b,s)T(s)f(x_g(s))ds.$$

Now, let us give the definition of controllability for the nonlinear system (4).

Definition 6.2. We say that the **system** (4) **is exactly controllable on the interval** $[a,b]$, if for each pair of elements x_0 and x_1 belonging to $\mathbb{X}$, there exists a function $g \in \mathcal{F}$ and a solution $x_g \in \mathcal{F}$ of the system (4) which could depend on g, such that $x_g(a) = x_0$ and $x_g(b) = x_1$.

Proposition 6.1. *The system* (4) *is exactly controllable on* $[a,b]$ *if, and only if,* $\mathrm{Rang}(G_f) = \mathbb{X}$.

Assumption 6.2. In this section we assume that there exists $C \geq 0$ such that $|B(u)-B(v)|_1 \leq C\ |u-v|_1$ for all pair $u, v \in \mathcal{F}$.

Proposition 6.2. *Let us fix any two functions* $u, v \in \mathcal{F}$. *If we define the function*

$$t \ \mapsto \ \int_a^t \Big[\ B(u)(s) - B(v)(s)\ \Big] ds,$$

then there is a constant $C > 0$ *such that*

$$\left| \int_a^t \Big[\ B(u)(s) - B(v)(s)\ \Big] ds \right|_1 \ \leq \ C\ (b-a)\ \big|\ u - v\ \big|_1$$

for each pair of elements u, v *belonging to* $\mathcal{F}$.

Proof. This proposition is a consequence of Proposition 2.4 and Assumption 6.2.

□

Proposition 6.3. *There exists a constant* $C > 0$ *such that*

$$|H(u) - H(v)| \ \leq \ |\lambda|\ C\ |x_u - x_v|_1$$

for each pair of elements u, v *belonging to* $\mathcal{F}$.

Proposition 6.4. *Let us fix any two functions* $u, v \in \mathcal{F}$. *If we define the function*

$$t \ \mapsto \ \int_a^t T(s) \big[\ f(x_u(s)) - f(x_v(s))\ \big] ds,$$

then there is a constant $C > 0$ *such that*

$$\left| \int_a^t T(s) \Big[\ f(x_u(s)) - f(x_v(s))\ \Big] ds \right|_1 \ \leq \ C \big|\ x_u - x_v\ \big|_1$$

for each pair of elements u, v *belonging to* $\mathcal{F}$.

Proof. Observe that for any partition $\big\{\ a = t_0 < t_1 < \cdots < t_n = b\ \big\}$ of interval $[a, b]$, we have that

$$\sum_{i=1}^{n} \left| \int_a^{t_i} T(s) \big[f(x_u(s)) - f(x_v(s))\big] \ - \ \int_a^{t_{i-1}} T(s) \big[f(x_u(s)) - f(x_v(s))\big]\ ds \right|$$

$$= \ \sum_{i=1}^{n} \left| \int_{t_{i-1}}^{t_i} T(s) \big[f(x_u(s)) - f(x_v(s))\big]\ ds \right|$$

$$
\begin{aligned}
&\le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \big\|T(s)\big\| \cdot \big|f(x_u(s)) - f(x_v(s))\big| \, ds \\
&\le \sum_{i=1}^{n} L \max_{s\in[a,b]} \big\|T(s)\big\| \cdot \max_{s\in[a,b]} \big|x_u(s) - x_v(s)\big| \int_{t_{i-1}}^{t_i} ds \\
&= L \max_{s\in[a,b]} \big\|T(s)\big\| \cdot \max_{s\in[a,b]} \big|x_u(s) - x_v(s)\big| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} ds \\
&= L \max_{s\in[a,b]} \big\|T(s)\big\| \cdot \max_{s\in[a,b]} \big|x_u(s) - x_v(s)\big| \, (b-a) \\
&\le L(b-a) \max_{s\in[a,b]} \big\|T(s)\big\| \, \big|x_u - x_v\big|_1 \, .
\end{aligned}
$$

□

Proposition 6.5. *There exist two constants* $C_1 > 0$ *and* $C_2 > 0$, *such that if* $|\lambda| < \frac{1}{C_2}$, *then*

$$\big|x_u - x_v\big|_1 \le \frac{C_1}{1 - |\lambda| C_2} \big|u - v\big|_1$$

for each pairs of functions $u, v \in \mathcal{F}$.

Proof. Since x_u and x_v denote the solutions for the system (4) for u and v respectively, we have that

$$
\begin{aligned}
x_u(t) - x_v(t) &= \int_a^t \Big(B(u)(s) - B(v)(s)\Big) ds \\
&+ \lambda \int_a^t T(s)\big[x_u(s) - x_v(s)\big] ds \\
&+ \lambda \int_a^t T(s)\big[f(x_u(s)) - f(x_v(s))\big] ds.
\end{aligned}
$$

Now observe that

$$\big|x_u - x_v\big|_1 \le \Big| \int_a^t \Big(B(u)(s) - B(v)(s)\Big) ds \Big|_1$$

$$+ |\lambda| \left[\Big| \int_a^t T(s)\big[x_u(s) - x_v(s)\big] ds \Big|_1 + \Big| \int_a^t T(s)\big[f(x_u(s)) - f(x_v(s))\big] ds \Big|_1 \right].$$

If we use once Proposition 6.2 and twice Proposition 6.4, we obtain

$$\big|x_u - x_v\big|_1 \le C(b-a)\big|u - v\big|_1 + |\lambda|\big(\tilde{C} + \widehat{C}\big)\big| x_u - x_v \big|_1 \, .$$

Therefore

$$\Big(1 - |\lambda|\big(\tilde{C} + \widehat{C}\,\big)\Big) \big|x_u - x_v\big|_1 \le C(b-a)\big|u - v\big|_1 \, .$$

□

Proposition 6.6. *There exists a constant* $C > 0$ *such that*

$$|H(u) - H(v)| \leq |\lambda| C |u - v|_1$$

for each pair of functions $u, v \in \mathcal{F}$.

Proof. This is a consequence of Propositions 6.3 and 6.5.

□

Let us consider a further result before we can prove the controllability of system (4).

Theorem 6.2. [12] *Let* Z *be a Banach space and* $S : Z \longrightarrow Z$ *a Lipschitz function with a Lipschitz constant* $l < 1$ *and consider* $\widehat{G}(z) := z + S(z)$. *Then* $\widehat{G}$ *is a homeomorphism whose inverse is a Lipschitz function with a Lipschitz constant* $(1 - l)^{-1}$.

Now we shall prove the exact controllability of the system (4).

Theorem 6.3. *Let us assume that* $\Lambda : \mathcal{F} \longrightarrow \mathcal{F}$ *is a right inverse of* B, *i.e.* $B \circ \Lambda = I$, *such that* Λ *is a Lipschitz function. There exists a real number* $\rho > 0$ *such that if* $|\lambda| \leq \rho$, *then the system* (4) *is exactly controllable on* $[a, b]$.

Proof. In view of Proposition 6.1 it is enough to prove that Rang$(G_f) = \mathbb{X}$. To this end, we consider the operator $\mathcal{G} : \mathbb{X} \to \mathbb{X}$, which is defined by $\mathcal{G} := G_f \circ \Gamma$, where Γ is the function in Definition 5.3. Now, from the Remark 5.2. is obtained that $G_R \circ \Gamma(x) = I(x) = x$. Then, $\mathcal{G}(x) = x + \mathcal{S}(x)$, where $\mathcal{S} := H \circ \Gamma$. Moreover, due to the Proposition 6.6 and the Definition 5.3., we can assure that

$$\Big| \mathcal{S}(x) - \mathcal{S}(y) \Big| \leq |\lambda| \cdot C \cdot \Big| x - y \Big|$$

for some fixed real number $C \geq 0$ and all pair of elements $x, y \in \mathbb{X}$. If the real number λ is small enough, then $\mathcal{S}$ is a Lipschitz function with a Lipschitz constant $\kappa < 1$. Then, by the Theorem 6.2, the function $\mathcal{G} : \mathbb{X} \longrightarrow \mathbb{X}$ is a homeomorphisms. Hence we obtain that the function $\mathcal{G}$ is surjective , which implies that Rang$(G_f) = \mathbb{X}$. Therefore, the system (4) is exactly controllable on the interval $[a, b]$.

□

Under the assumptions made in the statement of above Theorem 6.3, we obtain the following two theorems.

Theorem 6.4. *Let* $I : \mathbb{X} \longrightarrow \mathbb{X}$ *be the identity function* $I(x) := x$. *The operator* $\hat{\Gamma} : \mathbb{X} \longrightarrow \mathcal{F}$ *defined by* $\hat{\Gamma}(x) = \Big(\Gamma \circ (I + \mathcal{S})^{-1}\Big)(x)$ *is a right inverse of the nonlinear operator* G_f, i.e. ,

$G_f \circ \hat{\Gamma} = I$. *Moreover, a control* $g \in \mathcal{F}$ *steering the system* (4) *from an initial state* x_0 *to a final state* x_1 *on* $[a, b]$ *is given by*

$$g(t) = \hat{\Gamma}(x_1)(t) = \Big(\Gamma \circ (I + S)^{-1}\Big)(x_1)(t).$$

Proof. This theorem is a consequence from the proof of the above Theorem 6.3.

□

Before to prove the controllability of the system (3), let us consider one more definition.

Definition 6.3. We say that the **system** (3) **is exactly controllable on the interval** $[a, b]$, if for each pair of real numbers x_0 and x_1 there exists a function $g \in \mathcal{F}$ and a solution $x \in \mathcal{F}$ of the system (3), which could depend on g, such that $x(a) = x_0$ and $x(b) = x_1$.

Now we are ready to prove the exact controllability of the nonlinear system (3).

Theorem 6.5. *There exists a real number* $\rho > 0$ *such that if* $|\lambda| \leq \rho$, *then the system* (3) *is exactly controllable on the interval* $[a, b]$.

Proof. Putting $f(x) = x + (f(x) - x) = x + \hat{f}(x)$, we obtain that $\hat{f} := f(x) - x$ has the same properties as f. Therefore, the system (3) can be written as follows

$$\begin{cases} x(t) - x(a) = g(t) - g(a) + \lambda \int_a^t T(s)x(s)ds + \lambda \int_a^t T(s)\hat{f}(x(s))ds, \\ x(a) = x_0. \end{cases}$$

Then, applying Theorem 6.3 to this system, we obtain the result which we were looking for.

□

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