On the supersingular loci of quaternionic Siegel space

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ABSTRACT

The paper studies the supersingular locus of the characteristic p moduli space of principally polarized abelian 8-folds that are equipped with an action of a maximal order in a quaternion algebra, that is non-split at ∞ but split at p. The main result is that its irreducible components are Fermat surfaces of degree $p + 1$.

RESUMEN

El artículo estudia el lugar supersingular del espacio módulo característico p de abeliano polarizado 8-veces principal que son equipados con una acción de un orden maximal en un álgebra quaterniona, que es no-divisible en ∞, pero se divide en p. El principal resultados es que los componentes irreducibles son superficies de Fermat de grado $p + 1$.

Keywords and Phrases: supersingular abelian varieties, Shimura varieties, orthogonal groups.

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1   Introduction

Let p be a prime number. In [9] Oort and Li give a description of the supersingular locus $\mathcal{S}_{g,1}$ of $\mathcal{A}_{g,1} \times \mathbb{F}_p$, the fibre over p of the Siegel modular variety of principally polarized abelian g-folds. Among their results are that $\mathcal{S}_{g,1}$ has $H_g(p,1)$ irreducible components if g is odd and $H_g(1,p)$ if g is even, and all of these components have dimension $[g^2/4]$.

In this paper we study the supersingular locus of certain PEL-moduli spaces S_K of type $D_4^{\mathbb{H}}$, see body of text for a more precise explanation. These moduli spaces are associated to groups G that are twists of $GO(8)$. In the complex analytic context there exist uniformisations by quaternionic Siegel half-spaces, these are tube domains of the shape

$$\mathfrak{h} = \{X + iY | X, Y \in \operatorname{Mat}_2(\mathbb{H}), X^{t,\iota} = X, Y^{t,\iota} = Y > 0\}, \tag{1}$$

where $\mathbb{H}$ is the non-split quaternion algebra over $\mathbb{R}$, and ι is the standard involution.

In the algebraic context S_K is a 6-dimensional variety parameterizing abelian 8-folds with a particular kind of additional structure, and on a mild assumption on the level structure this variety is smooth. For every prime of good reduction we introduce the usual integral model for this Shimura variety, and we move on to exhibit the geometry of each individual irreducible component of the supersingular locus of the mod p-reduction. Our main result says that these components are Fermat surfaces. This comes as a surprise, because for a more general Shimura variety, the structure of the supersingular locus is usually quite complicated and might not even be smooth, for example this happens in the case of $\mathcal{S}_{3,1}$ the 2-dimensional space of supersingular principally polarized abelian 3-folds, cf. [12, Paragraph(4)], [14, Proposition 2.4], [9, Example(9.4)] or [15] for a very precise exposition. We round off the discussion by turning to the non-supersingular points also, we prove that their p-divisible groups do not have parameters, which is somewhat the exact opposite to their behaviour on the supersingular locus. This too seems unusual, as one sees from the non-supersingular principally polarized abelian 3-folds. These results were already applied in [2] to obtain an Eichler-Shimura congruence relation for S_K and for its Shimura divisors.

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The paper is organized as follows: Section 2 focuses on local aspects, section 3 on global ones. In subsections 2.1/ 3.1 we sum up definitions and conventions. In subsections 2.2 we explain techniques needed to understand supersingular Dieudonné modules. We apply these techniques to supersingular Dieudonné modules with the particular additional structure under consideration in the subsections 2.3- 2.4. In subsection 2.5 a result on the non-supersingular locus is obtained (Corollary 2.7).

2 Structure Theorems on Dieudonné Modules with Pairing

2.1 Notions and Notations

We continue to fix a prime p. If k is a perfect field of characteristic p, then one denotes by $W(k)$ and $K(k)$ the Witt ring and fraction field thereof. Unless otherwise said, k will be assumed to be algebraically closed. The absolute Frobenius $x \mapsto x^p$ induces automorphisms $x \mapsto {}^F x$ on $W(k)$ and $K(k)$ which again will be referred to as absolute Frobenii. Recall that a Dieudonné module is a finitely generated, torsion free $W(k)$-module together with a ^{F}-linear endomorphism F and a ${}^{F^{-1}}$-linear endomorphism V that satisfies $FV = VF = p$. If one tensorizes with $\mathbb{Q}$ one obtains the isocrystal of M, this is a finite dimensional $K(k)$-vector space together with a ^{F}-linear bijection F. Dieudonné modules are called isogenous if they give rise to isomorphic isocrystals.

By a pairing on a Dieudonné module M one understands a $W(k)$-bilinear map $\varphi : M \times M \to W(k)$ which satisfies $\varphi(x, Fy) = {}^F \varphi(Vx, y)$. When thinking of M as the co-variant Dieudonné module of a p-divisible group A over k, this means that φ gives rise to a morphism from A to the Serre-dual of A. Dieudonné modules with pairings (M, φ) and (M', φ') are called isometric if there exists an isomorphism from M to M' taking φ to φ'. The dimension, $\dim_k(M/VM)$ of a Dieudonné module with non-degenerate pairing is equal to the codimension $\dim_k(M/FM)$, the rank of M is necessarily even. A pairing is called antisymmetric if $\varphi(x, y) = -\varphi(y, x)$ and symmetric if $\varphi(x, y) = \varphi(y, x)$. In either of these cases we denote $\{x \in M \otimes \mathbb{Q} | \varphi(M, x) \subset W(k)\}$ by M^t, if $M = M^t$ we say that φ is perfect.

In [13, Definition(3.1)] the crucial notion of crystalline discriminant of a non-degenerate symmetric pairing is introduced: Say the underlying Dieudonné module has rank $2n$, and choose a $K(k)$-basis $x_1, \ldots, x_{2n}$ with the additional property $Fx_1 \wedge \cdots \wedge Fx_{2n} = p^n x_1 \wedge \cdots \wedge x_{2n}$. The determinant $\det(\varphi(x_i, x_j))$, regarded as an element in $K(\mathbb{F}_p)^\times / (K(\mathbb{F}_p)^\times)^2$, can be checked to be independent of the choice of basis and is called the crystalline discriminant $\operatorname{crisdisc}(M, \varphi)$. It only depends on the isogeny class of M allowing one to also write $\operatorname{crisdisc}(M \otimes \mathbb{Q}, \varphi)$ for $\operatorname{crisdisc}(M, \varphi)$. When fixing once and for all an element $t \in W(\mathbb{F}_{p^2})^\times$ with $t^\sigma = -t$, the target group $K(\mathbb{F}_p)^\times / (K(\mathbb{F}_p)^\times)^2$ can be identified with $\{1, t^2, p, pt^2\}$ if p is odd and with $\{\pm 1, \pm t^2, \pm p, \pm pt^2\}$ if $p = 2$, notice that the kernel of the forgetful map from $K(\mathbb{F}_p)^\times / (K(\mathbb{F}_p)^\times)^2$ to $K(k)^\times / (K(k)^\times)^2$ consists of $\{1, t^2\}$. Notice also that the image of $\operatorname{crisdisc}(M, \varphi)$ in the group $K(k)^\times / (K(k)^\times)^2$ is the discriminant of a symmetric pairing in the usual sense of linear algebra, hence is independent of the structure of M as a Dieudonné module. The following characterization of crystalline discriminants within $\{1, t^2\}$ will be useful, see [13, Corollary(3.5)] for a proof:

Fact 1 (Ogus)**.** *Assume that φ is a non-degenerate symmetric pairing on a Dieudonné module M of rank $2n$. Assume also that there exists a φ-isotropic $K(k)$-subspace $A \subset M \otimes \mathbb{Q}$ of dimension n. Then*

$$\operatorname{crisdisc}(M, \varphi) = (-1)^n t^{2 \dim_{K(k)}(A + FA/A)},$$

in particular, if A is an isocrystal, then $\operatorname{crisdisc}(M, \varphi) = (-1)^n$.

Recall that the Grassmannian of $\mathfrak{n}$-dimensional isotropic subspaces of the $K(k)$-vector space $M \otimes \mathbb{Q}$ has two connected components. Two such spaces A_1 and A_2 lie in the same component if and only if the integer $\dim_{K(k)}(A_1 + A_2/A_1)$ is even, we will say that A_1 and A_2 have the same parity if this is the case. Thus by the above fact $(-1)^{\mathfrak{n}} \operatorname{crisdisc}(M, \phi)$ is the trivial element of $K(\mathbb{F}_\mathfrak{p})^\times/(K(\mathbb{F}_\mathfrak{p})^\times)^2$ if and only if the bijection F does not change the parity of the maximal isotropic subspaces. If $\mathfrak{p}$ is odd and if ϕ is a perfect form on M, then one can deduce a further formulation: Pick a maximal isotropic k-subspace $\overline{A} \subset \overline{M} = M/\mathfrak{p}M$ complementary to $V\overline{M}$. Lift it to $W(k)$ to obtain a maximal isotropic $W(k)$ submodule A of M (the Grassmannian is smooth). Observe that $F\overline{A} = F\overline{M}$ to conclude that (mod 2):

$$
\begin{aligned}
& \dim_k(\overline{A} + F\overline{A}/\overline{A}) \\
\equiv\ & \dim_k(\overline{A} + V\overline{M}/\overline{A}) + \dim_k(V\overline{M} + F\overline{M}/V\overline{M}) \\
\equiv\ & \dim_k(\overline{M}/V\overline{M} + F\overline{M}) \quad (\mathrm{mod}\ 2).
\end{aligned}
$$

The integer $\dim_k(M/VM + FM)$ is called the Oort invariant of M and denoted $\mathfrak{a}(M)$. Thus, we have derived the consequence, implicitly stated in [11, Section 5.3]:

Fact 2 (Moonen)**.** *Assume that ϕ is a perfect symmetric pairing on the Dieudonné module M of rank $2\mathfrak{n}$. If $\mathfrak{p}$ is odd, then* $\operatorname{crisdisc}(M, \phi) = (-1)^{\mathfrak{n}} t^{2\mathfrak{a}(M)}$.

The Dieudonné module M is called superspecial if it satisfies $FM = VM$, i.e. if $\operatorname{rank}_{W(k)}(M) = 2\mathfrak{a}(M)$. Superspecial Dieudonné modules may conveniently be described in terms of their skeletons, these are the $W(\mathbb{F}_{\mathfrak{p}^2})$-submodules defined by $\tilde{M} = \{x \in M | Fx = Vx\}$. We write $\mathcal{O}_\mathbb{B}$ for the ring extension of $W(\mathbb{F}_{\mathfrak{p}^2})$, obtained by adjoining an indeterminate σ subject to the relations $\sigma^2 = \mathfrak{p}$ and $\sigma\mathfrak{a} = {}^F\mathfrak{a}\sigma$, it operates in a self-explanatory way on $\tilde{M}$. As remarked in [8] the assignment $M \mapsto \tilde{M}$ sets up an equivalence of the category of superspecial Dieudonné modules with the category of finitely generated torsion free $\mathcal{O}_\mathbb{B}$-modules. We also write $\mathbb{B}$ for $\mathcal{O}_\mathbb{B} \otimes \mathbb{Q}$, it is the unique non-split quaternion algebra over $K(\mathbb{F}_\mathfrak{p})$. Observe that $\mathcal{O}_\mathbb{B}$ is the maximal order of $\mathbb{B}$. We let $\mathfrak{m}_\mathbb{B}$ be the maximal ideal of $\mathcal{O}_\mathbb{B}$, one has $\mathcal{O}_\mathbb{B}/\mathfrak{m}_\mathbb{B} \cong \mathbb{F}_{\mathfrak{p}^2}$.

We need to put pairings into the picture as follows. If ϕ is a pairing on a superspecial Dieudonné module M, then one considers a $\mathcal{O}_\mathbb{B}$-valued pairing on $\tilde{M}$ defined by:

$$
\Phi(x, y) = \phi(x, \sigma t y) - \phi(x, y)\sigma t \tag{2}
$$

This is $\mathcal{O}_\mathbb{B}$-sesquilinear, i.e. satisfies $\Phi(ux, vy) = u\Phi(x, y)v^\iota$, for $u, v \in \mathcal{O}_\mathbb{B}$. The involution ι is the standard one, mapping $\mathfrak{a} + \mathfrak{b}\sigma$ to ${}^F\mathfrak{a} - \mathfrak{b}\sigma$. Conversely any $\mathcal{O}_\mathbb{B}$-sesquilinear form arises from a pairing on M in the way described, ϕ is non-degenerate/perfect if and only if Φ is.
Unless otherwise said we assume from now on that ϕ is symmetric, in terms of Φ this means

$\Phi(y,x)^{\iota} = -\Phi(x,y)$ for all $x, y \in \tilde{M}$. The $\mathcal{O}_{\mathbb{B}}$-module $\tilde{M}$ with form Φ is called hyperbolic if on a suitable $\mathcal{O}_{\mathbb{B}}$-basis $e_1, \ldots, e_{n/2}, f_1, \ldots, f_{n/2}$ of $\tilde{M}$ one has

$$\Phi(e_i, e_j) = \Phi(f_i, f_j) = 0 \quad , \quad \Phi(e_i, f_j) = w\delta_{i,j} \tag{3}$$

for some non-zero $w \in \mathcal{O}_{\mathbb{B}}$, uniquely determined only up to multiplication by $\mathcal{O}_{\mathbb{B}}^{\times}$. It turns out that $w = -\sigma^r t$ is a very convenient choice as the values of the corresponding form φ will then read:

$$\varphi(e_i, f_j) = \begin{cases} 0 & r \equiv 0 \pmod 2 \\ p^{r-1/2}\delta_{i,j} & r \equiv 1 \pmod 2 \end{cases}$$

and

$$\varphi(e_i, Ff_j) = \begin{cases} p^{r/2}\delta_{i,j} & r \equiv 0 \pmod 2 \\ 0 & r \equiv 1 \pmod 2 \end{cases}$$

and $\varphi(e_i, e_j) = \varphi(f_i, f_j) = 0$. Equivalently, (M, φ) is hyperbolic if and only if the Dieudonné module M allows a decomposition into a direct sum of Dieudonné modules A and B with $\varphi(A, A) = \varphi(B, B) = 0$, and $M^t = F^{-r}M$, so that φ identifies the dual of A with $F^{-r}B$.

2.2 Results of Oort and Li

A Dieudonné module is called supersingular if it is isogenous to a superspecial one, or equivalently if all its Newton slopes are equal to $1/2$. This section is primarily concerned with supersingular Dieudonné modules, so recall some of the techniques which are usually applied to them: If M is supersingular it has a biggest superspecial sub-module $S_0(M)$ which one can construct as $S_0(M) = \tilde{M} \otimes_{W(\mathbb{F}_{p^2})} W(k)$. Dually there is $S^0(M)$, the smallest superspecial module containing M, see [10, Chapter III.2] for proofs of this.

The following facts on the relation of the lattices $S_0(M) \subset M \subset S^0(M)$ are basic to the study of supersingular Dieudonné modules. The first of them can be found in [8, Corollary(1.7)], along with more information on the functors S_0 and S^0. For the other two facts we refer the reader to [8, Lemma(1.5/1.6)] (or [9, Fact(5.8)]) and [8, 1.10(i)] (or [9, Chapter(12.2)]):

Fact 3 (Li)**.** *Let M be a supersingular Dieudonné module of rank $2g$ over $W(k)$. Then one has $F^{g-1}S^0(M) = \sum_{i+j=g-1} F^i V^j M$. It follows that $F^{g-1}S^0(M) \subset S_0(M)$, in particular the length of the $W(k)$-module $S^0(M)/S_0(M)$ is bounded by $g(g-1)$ and equality is acquired if and only if $a(M) = 1$.*

Fact 4 (Li)**.** *Let N be a superspecial Dieudonné module of rank $2g$ over $W(k)$. Let x be an element of N. Then one has $S^0(W(k)[F, V]x) = N$ if and only if the elements*

$$F^{g-1}x, F^{g-2}Vx, \ldots, FV^{g-2}x, V^{g-1}x$$

form a basis of the k-vector space $F^{g-1}N/F^g N$. Moreover, an element with this property exists.

Fact 5 (Li)**.** *Let M be a supersingular Dieudonné module of rank $2g$ over $W(k)$. For a non-negative integer i let*

$$s_i = \dim_k(M \cap F^i S^0(M)/M \cap F^{i+1} S^0(M)).$$

Then one has $s_i \leq s_{i+1}$ and equality holds if and only if $s_i = g$.

    The work [9] studies supersingular Dieudonné modules which are equipped with a perfect anti-symmetric form ψ. Following their method we notice that we have to incorporate additional structure which by the Morita-equivalence of subsection 3.2 leads to Dieudonné modules M of rank 8 equipped with a symmetric form ϕ.

Analogous to [9, Proposition(6.1)] we need to analyze the restriction of ϕ to $N = S_0(M)$, or more generally, a classification of non-degenerate symmetric forms on superspecial Dieudonné modules:

Theorem 2.1. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let N be a superspecial Dieudonné module of rank $2n$ over $W(k)$, which is equipped with a non-degenerate symmetric pairing ϕ. Then N contains Dieudonné modules N_i of rank 2, with $\phi(N_i, N_j) = 0$, for $i \neq j$, and $N = \bigoplus_{i=1}^{n} N_i$. Moreover, each N_i has a $W(k)$-basis consisting of elements $x_i, Fx_i = Vx_i = y_i$ such that one of the two cases:*

   (i) $\phi(x_i, x_i) = \phi(y_i, y_i) = 0$, *and* $\phi(x_i, y_i) = p^{n_i}$,

   (ii) $\phi(x_i, y_i) = 0$, $\phi(x_i, x_i) = \epsilon_i p^{n_i}$, *and* $\phi(y_i, y_i) = \epsilon_i^{\sigma} p^{n_i+1}$,

holds for some integers n_i and some elements $\epsilon_i \in W(\mathbb{F}_{p^2})^{\times}$ which are unique up to multiplication by elements in $(W(\mathbb{F}_{p^2})^{\times})^2$. Moreover, the cristalline discriminant can be computed from this decomposition as

$$\operatorname{crisdisc}(N_i, \phi) = \begin{cases} -t^2 & (N_i, \phi) \text{ of type (i)} \\ pt^2 \epsilon_i \epsilon_i^{\sigma} & (N_i, \phi) \text{ is of type (ii)} \end{cases},$$

and $\operatorname{crisdisc}(N, \phi) = \prod_{i=1}^{n} \operatorname{crisdisc}(N_i, \phi)$.

Proof. The skeleton construction descends N to a $W(\mathbb{F}_{p^2})$-Dieudonné module $\tilde{N}$ which at the same time is a $\mathcal{O}_{\mathbb{B}}$-module. As in (2) we consider the $\mathcal{O}_{\mathbb{B}}$-valued sesquilinear form Φ and diagonalize it as follows: Let $x_0 \in \tilde{N}$ be an element with $\Phi(x_0, x_0)$ of $\mathfrak{m}_{\mathbb{B}}$-adic valuation as small as possible, i.e. such that $\Phi(x, x) \in \mathfrak{m}_{\mathbb{B}}^r = \mathcal{O}_{\mathbb{B}} \Phi(x_0, x_0)$ for all $x \in \tilde{N}$. By the usual polarization process it follows that $\Phi(x, y) - \Phi(x, y)^{\iota} \in \mathfrak{m}_{\mathbb{B}}^r$, and also $\Phi(x, y) + \Phi(x, y)^{\iota} \in \mathfrak{m}_{\mathbb{B}}^r$ by replacing tx for x. Consequently $\Phi(\tilde{N}, \tilde{N}) \subset \mathfrak{m}_{\mathbb{B}}^r$. Therefore we obtain an orthogonal direct sum $\tilde{N} = (\tilde{N} \cap (\mathbb{B}x_0)^{\perp}) \oplus \mathcal{O}_{\mathbb{B}} x_0$, as any $x \in \tilde{N}$ has $\Phi(x, x_0)\Phi(x_0, x_0)^{-1} = \alpha \in \mathcal{O}_{\mathbb{B}}$ which allows to write x as a sum of $\alpha x_0 \in \tilde{N}$ and $x - \alpha x_0 \in \tilde{N} \cap (\mathbb{B}x_0)^{\perp}$.

Having obtained a decomposition $\tilde{N} = \bigoplus_{i=1}^{n} \tilde{N}_i$ we search for basis elements $\tilde{x}_i \in \tilde{N}_i$ with $\Phi(\tilde{x}_i, \tilde{x}_i)$ manageable: In $\tilde{N}_i \otimes \mathbb{Q}$ one can certainly find elements $\tilde{x}_i$ with $\Phi(\tilde{x}_i, \tilde{x}_i) \in W(\mathbb{F}_{p^2})^{\times} \cup FW(\mathbb{F}_{p^2})^{\times}$ for example by [16, Chapter 10, Theorem(3.6.(i))]. Observe that the $\mathfrak{m}_{\mathbb{B}}$-adic valuation of $\Phi(\tilde{x}_i, \tilde{x}_i)$ must be congruent modulo 2 to $r_i = \operatorname{length}_{\mathcal{O}_{\mathbb{B}}} \tilde{N}_i^t / \tilde{N}_i$. Hence after adjusting the $\tilde{x}_i$'s by multiplying

them by $F^{r_i/2}$, if r_i is even, and by $F^{(r_i-1)/2}$, if r_i is odd, one gets generators of the $\mathcal{O}_{\mathbb{B}}$-modules $\tilde{N}_i$ on which the sesquilinear form takes values in $F^{r_i} W(\mathbb{F}_{p^2})^\times$.

It is clear how to obtain the desired basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ from these generators. If r_i is even N_i will be of type (i) with $n_i = r_i/2$, and if r_i is odd then N_i will be of the type (ii) with $n_i = (r_i - 1)/2$. □

Remark 2.2. Suppose N is a superspecial Dieudonné module of rank 2 with a symmetric form ϕ. Then one checks from the above classification that (N, ϕ) is isometric to $(N, -\phi)$. It follows that $N^{\oplus 2}$, the orthogonal direct sum of two copies of N, is hyperbolic. One checks this by using the sesquilinear form (2) as $\Phi((u_1 + u_2, u_1 - u_2), (v_1 + v_2, v_1 - v_2)) = (u_1 + u_2)w(v_1 + v_2)^t - (u_1 - u_2)w(v_1 - v_2)^t = (2u_1w)v_2^t - u_2(2v_1w)^t$. (cf. [9, Remark(6.1)] for the analog in the anti-symmetric setting)

For later use we note an immediate corollary:

Corollary 2.3. *Let (N_1, ϕ_1) and (N_2, ϕ_2) be supersingular Dieudonné modules of rank two, equipped with symmetric pairings. There exists an isometry between them if and only if the following holds:*

$$\begin{aligned} \text{length}_{W(k)} \, N_1^t/N_1 &= \text{length}_{W(k)} \, N_2^t/N_2 \\ \text{crisdisc}(N_1, \phi_1) &= \text{crisdisc}(N_2, \phi_2). \end{aligned}$$

Consequently for any non negative integer n, there is only one isometry class of rank two supersingular Dieudonné modules with pairing (N, ϕ) where $\text{length}_{W(k)} \, N^t/N = 2n$. There are two such classes of modules with pairing where $\text{length}_{W(k)} \, N^t/N = 2n + 1$.

2.3 Classification of symmetric Dieudonné modules

This section is the core of the work, we give a classification of Dieudonné modules with the additional structure of interest.

Theorem 2.4. *Let M be a supersingular Dieudonné module over $W(k)$ with perfect symmetric pairing ϕ. Assume that:*

$$\begin{aligned} \text{rank}_{W(k)} \, M &= 8 \\ \text{crisdisc}(M, \phi) &= 1. \end{aligned}$$

Consider $S^0(M) = N$, the smallest superspecial Dieudonné lattice in $M \otimes \mathbb{Q}$, which contains M. Choose a decomposition $N = \bigoplus_{i=1}^{4} N_i$ with properties as granted by Theorem 2.1, and with $S_0(M) = N^t = \bigoplus_{i=1}^{4} F^{r_i} N_i$ for integers $r_1 \leq r_2 \leq r_3 \leq r_4$. Then (r_1, r_2, r_3, r_4) is one of

(i) $(0, 0, 0, 0)$

(ii) $(1, 1, 1, 1)$

(iii) $(0, 2, 2, 2)$

(iv) $(2, 2, 2, 2)$,

moreover, there exists a superspecial Dieudonné lattice Q, *which contains* FM *and satisfies*

(a) $Q^t = Q$

(b) $\dim_k(M/M \cap Q) = \dim_k(Q/M \cap Q) = 1$.

If M *is of the form (iii) or (iv), then the superspecial Dieudonné lattice* Q, *satisfying (a) and (b) is unique.*

Proof. For the proof we need two auxiliary lemmas:

Lemma 2.5. *Let the assumptions on* M *be as in the above theorem, then there exist two different indices* i_1 *and* i_2 *such that* N_{i_1} *and* N_{i_2} *are isometric.*

Proof. If an even integer r occurs twice amongst the various r_i's one is done, and if an odd integer r occurs three times one is done as well, use the pigeon hole principle and Corollary 2.3. The condition on the discriminant forces the number of indices i with r_i odd to be even. This means that one is left with checking the lemma for the r_i-quadruples $(0, 1, 2, 3)$, $(0, 2, 3, 3)$, $(0, 1, 1, 2)$, and $(1, 1, 3, 3)$.

The three quadruples with $r_1 = 0$ do not arise, because otherwise M would be an orthogonal direct sum of N_1 and some supersingular Dieudonné module M' of rank 6 and equipped with a perfect symmetric form ϕ'. Applying Fact 2 to M' would give that M' has Oort invariant 1 or 3, as $\mathrm{crisdisc}(M') = \mathrm{crisdisc}(N_1) = -t^2$. Fact 3 applied to M' would further imply that the elementary divisors of $S^0(M')/S_0(M')$ are either all 0 or all equal to 2. Hence the elementary divisors of $S^0(M)/S_0(M)$ would be $(0, 0, 0, 0)$ or $(0, 2, 2, 2)$.

It remains to do the $(r_1, r_2, r_3, r_4) = (1, 1, 3, 3)$-case. Assume that no two of the N_i's were isometric. This would lead to a basis $x_i, Fx_i = Vx_i = y_i$ with

$$\phi(x_1, x_1) = p^{-1}, \ \phi(y_1, y_1) = 1$$
$$\phi(x_2, x_2) = \epsilon p^{-1}, \ \phi(y_2, y_2) = \epsilon^\sigma$$
$$\phi(x_3, x_3) = p^{-2}, \ \phi(y_3, y_3) = p^{-1}$$
$$\phi(x_4, x_4) = \epsilon p^{-2}, \ \phi(y_4, y_4) = \epsilon^\sigma p^{-1},$$
$$\text{other products} = 0,$$

and with ϵ some non-square in $W(\mathbb{F}_{p^2})^\times$. The module M has to contain an element of the form $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \beta_3 y_3 + \beta_4 y_4$ such that $\beta_i, \alpha_i \in W(k)$ but not both of α_3 and α_4 in $pW(k)$. As

$$\phi(x, x) = p^{-1}(\alpha_1^2 + \epsilon \alpha_2^2 + \beta_3^2 + \epsilon^\sigma \beta_4^2) + p^{-2}(\alpha_3^2 + \epsilon \alpha_4^2)$$

one has $\alpha_3^2 + \epsilon\alpha_4^2 \equiv 0 \pmod{p}$, but as

$$\phi(x, F^2 x) = \alpha_1 \alpha_1^{\sigma^2} + \epsilon\alpha_2 \alpha_2^{\sigma^2} + \beta_3 \beta_3^{\sigma^2} + \epsilon^\sigma \beta_4 \beta_4^{\sigma^2} + p^{-1}(\alpha_3 \alpha_3^{\sigma^2} + \epsilon\alpha_4 \alpha_4^{\sigma^2})$$

one has $\alpha_3^{p^2+1} + \epsilon\alpha_4^{p^2+1} \equiv 0 \pmod{p}$ as well. As ϵ is a non-square in $W(\mathbb{F}_{p^2})^\times$, one has $\epsilon^{\frac{p^2-1}{2}} \equiv -1 \pmod{p}$, so that we derive the contradiction

$$\alpha_3^{p^2+1} \equiv (-\epsilon\alpha_4^2)^{\frac{p^2+1}{2}} \equiv \epsilon\alpha_4^{p^2+1} \pmod{p}.$$

$\square$

Lemma 2.6. *With the same notation as in the theorem $r_i \leq 2$ for all indices i.*

Proof. Observe that the lemma would be immediate if one of the r_i was zero. So we can assume $0 < r_i$ for all indices i. Pick two indices $i \neq j$ with $r_i = r_j = r$ and $\mathrm{crisdisc}(N_i) = \mathrm{crisdisc}(N_j)$, according to the previous lemma such indices will exist. Say $(i,j) = (1,2)$ after relabeling, and write according to Remark 2.2 $N_1 \oplus N_2 = A \oplus B$, with $\phi(A, A) = \phi(B, B) = 0$ and $A \times F^r B \to W(k)$ a perfect pairing. Consider along the lines of [9, Proposition(6.3)] a $W(k)$-module M' which is the image of $(B \oplus N_3 \oplus N_4) \cap M$ under the projection map $B \oplus N_3 \oplus N_4 \to N_3 \oplus N_4$. M' inherits a perfect form and is indeed canonically isomorphic to the sub-quotient $(B^\perp \cap M)/(B \cap M)$ of M. One has $\mathrm{crisdisc}(M') = 1$ because M' is isogenous to $N_3 \oplus N_4$. By Fact 2 it follows that M' is superspecial. Furthermore the proof of [9, Proposition(6.3)] shows that $FN_3 \oplus FN_4 \subset M' \subset N_3 \oplus N_4$. For convenience of the reader we reproduce the argument in loc.cit.: Pick an element in M of the form $x = e + f + n_3 + n_4$ with $e \in \tilde{A}$, $f \in B$, $n_3 \in N_3$, $n_4 \in N_4$ and

$$S^0(M) = S^0(W(k)[F, V]x),$$

it exists due to Fact 4. The elements $F^3 x$, $F^2 Vx$, $FV^2 x$, $V^3 x$ will then form a basis of the k-vector space $F^3 N/F^4 N$ so that $F^3 x - F^2 Vx$, $F^2 Vx - FV^2 x$, $FV^2 x - V^3 x$ is a basis of $F^3(B \oplus N_3 \oplus N_4)/F^4(B \oplus N_3 \oplus N_4)$. It follows that

$$S^0(W(k)[F, V](F - V)x) = F(B \oplus N_3 \oplus N_4),$$

but $(F - V)x \in M \cap (B \oplus N_3 \oplus N_4)$ which projects surjectively onto M'. As S^0 is a functor in supersingular Dieudonné modules $FN_3 \oplus FN_4$ will be contained in $S^0(M') = M'$, and consequently

$$FN_3 \oplus FN_4 \subset M' = M'^t \subset F^{-1}N_3^t \oplus F^{-1}N_4^t = F^{r_3-1}N_3 \oplus F^{r_4-1}N_4$$

i.e. $r_3, r_4 \leq 2$. However, $r_3 \equiv r_4 \pmod 2$, as $\mathrm{crisdisc}(N_3) = \mathrm{crisdisc}(N_4)$. Therefore $r_3 = r_4$, as $r_3, r_4 \in \{1, 2\}$. Now, note that this does indeed imply that N_3 is isometric to N_4.
In order to find that $r_1, r_2 \leq 2$ also, we redo the whole argument, with the roles of N_1 and N_2 being replaced by N_3 and N_4. $\square$

Return to proof of theorem

We move on to investigate the set of possible quadruples (r_1, r_2, r_3, r_4). If one of the numbers in that sequence is 0, then Fact 2 shows that we must have either $(0, 0, 0, 0)$ or $(0, 2, 2, 2)$. For the remaining cases $(2, 2, 2, 2)$, $(1, 1, 1, 1)$ and $(1, 1, 2, 2)$ are conceivable. We show that $(1, 1, 2, 2)$ can not arise: Assume we had a Dieudonné module M with $(r_1, r_2, r_3, r_4) = (1, 1, 2, 2)$. It would follow that one had $\mathrm{crisdisc}(N_3) = \mathrm{crisdisc}(N_4)$ by Corollary 2.3, and so would $\mathrm{crisdisc}(N_1) = \mathrm{crisdisc}(N_2)$. By applying Remark 2.2 to both $N_1 \oplus N_2$ and $N_3 \oplus N_4$ one obtains a basis of N consisting of say e_1, e_2, f_1, f_2, $Fe_1 = Ve_1$, $Fe_2 = Ve_2$, $Ff_1 = Vf_1$, $Ff_2 = Vf_2$ and with the only non-zero products being given by

$$\phi(Fe_1, Ff_1) = 1$$
$$\phi(e_1, f_1) = \phi(e_2, Ff_2) = \phi(f_2, Fe_2) = p^{-1}.$$

As $F^{-1}N^t$ is superspecial one has $M \not\subset F^{-1}N^t$, so that M contains an element of the form $x = \alpha_1 e_1 + \beta_1 f_1 + \alpha_2 e_2 + \beta_2 f_2 + \alpha_3 Fe_2 + \beta_3 Ff_2$, with all $\alpha_1, \ldots, \beta_3 \in W(k)$ and at least one of α_2 and β_2 a unit. From $Fx \in \alpha_2^\sigma Fe_2 + \beta_2^\sigma Ff_2 + N^t$ and $\phi(M, M) \subset W(k)$ one infers $\phi(x, Fx) \in p^{-1}(\alpha_2^\sigma \beta_2 + \beta_2^\sigma \alpha_2) + W(k)$, which means that $\alpha_2^\sigma \beta_2 + \beta_2^\sigma \alpha_2 \equiv 0 \pmod{p}$. As we may alter the elements $\alpha_1, \ldots, \beta_3$ by any element in $pW(k)$ we can actually assume that $\alpha_2^\sigma \beta_2 + \beta_2^\sigma \alpha_2 = 0$, but then the Dieudonné module

$$W(k)Fx + N^t = W(k)(\alpha_2^\sigma Fe_2 + \beta_2^\sigma Ff_2) + N^t$$

is superspecial contradicting $S_0(M) = N^t$.

Having done the first assertion of the theorem we now focus on the existence of Q. If M is of the form (i), then use Remark 2.2 to write $N_1 \oplus N_2$ as direct sum of two isotropic Dieudonné modules A and B, between which there is the duality that is induced from the pairing on N. Then one finds that $Q = F^{-1}A \oplus FB \oplus N_3 \oplus N_4$ is a superspecial Dieudonné lattice that does the job. Similarly for the (ii)-case: Write $N = A_1 \oplus A_2 \oplus B_1 \oplus B_2$ with isotropic A_i and B_i, this time equipped with a canonical isomorphism $A_i^t \cong FB_i$. The superspecial lattices

$$FA_1 \oplus FA_2 \oplus B_1 \oplus B_2$$
$$A_1 \oplus FA_2 \oplus FB_1 \oplus B_2$$

both satisfy $Q^t = Q$, and one of them satisfies property (b) as well.

In the (iii)-case property (a) forces to look at $Q = N_1 \oplus \bigoplus_{i=2}^4 FN_i$, whereas $Q = FN$ in the (iv)-case. We have to show that this module does indeed satisfy (b), to this end observe that the numbers $\dim_k M/M \cap Q$ and $\dim_k M \cap Q/M \cap FQ$ are nonzero and sum up to 4, it thus suffices to see that the first of them is strictly smaller than the second. In the (iv)-case this is the content of Fact 5. In the (iii)-case apply Fact 5 to the orthogonal complement of N_1 in M, which is a Dieudonné module of rank 6 with perfect symmetric form. □

2.4 Moduli of symmetric Dieudonné modules

We consider the graded $\mathbb{F}_p$-algebra $R := \mathbb{F}_p[A_1, A_2, B_1, B_2]/(\sum_{i=1}^2 A_i B_i^p + B_i A_i^p)$, and its associated projective variety $\overline{X}_1 := \mathrm{Proj}\, R$, which is smooth of relative dimension 2. Let $\overline{Y}_1$ denote the affine

chart determined by $A_1 \neq 0$, it is the spectrum of $R_{(A_1)} \cong \mathbb{F}_p[a_2, b_1, b_2]/(b_1 + b_1^p + a_2 b_2^p + b_2 a_2^p)$, where $a_2 := \frac{A_2}{A_1}$, $b_1 := \frac{B_1}{A_1}$, and $b_2 := \frac{B_2}{A_1}$. Let $\alpha_2, \beta_1, \beta_2 \in W(R_{(A_1)})$ be lifts of a_2, b_1, b_2 with $\beta_1 + {}^F\beta_1 + \alpha_2 {}^F\beta_2 + \beta_2 {}^F\alpha_2 = 0$. Let $T_{(A_1)}$ be the $W(R_{(A_1)})$-module $\bigoplus_{i=1}^4 W(R_{(A_1)})t_i$, $L_{(A_1)}$ be the $W(R_{(A_1)})$-module $\bigoplus_{i=1}^4 W(R_{(A_1)})l_i$, and $M_{(A_1)}$ be $L_{(A_1)} \oplus T_{(A_1)}$. Putting:

$$F(t_1) = l_1$$
$$F(t_2) = l_2 + (\beta_2 - {}^{F^2}\beta_2)t_1$$
$$F(t_3) = l_3 + ({}^{F^2}\alpha_2 - \alpha_2)t_2 + ({}^{F^2}\beta_2 - \beta_2)t_4$$
$$F(t_4) = l_4 + (\alpha_2 - {}^{F^2}\alpha_2)t_1$$
$$V^{-1}(l_1) = t_1$$
$$V^{-1}(l_2) = t_2$$
$$V^{-1}(l_3) = t_3$$
$$V^{-1}(l_4) = t_4$$

and using the formula $V^{-1}({}^V\alpha x) = \alpha F(x)$ defines the structure of a display ([17]) on $M_{(A_1)}$, which moreover has the normal decomposition $L_{(A_1)} \oplus T_{(A_1)}$. One checks that a pairing is given on $M_{(A_1)}$ by $\phi(l_i, l_j) = \phi(t_i, t_j) = 0$, $\phi(l_i, t_j) = \delta_{|i-j|,2}$. Let also $N = L_N \oplus T_N$ be the display obtained from the formulas $F(t_i) = l_i$, $V^{-1}(l_i) = t_i$ and with pairing defined analogously. Putting:

$$\epsilon(t_1) = pt_3$$
$$\epsilon(t_2) = l_2 - {}^F\beta_2 l_3$$
$$\epsilon(t_3) = t_1 + {}^F\alpha_2 t_2 + {}^F\beta_1 t_3 + {}^F\beta_2 t_4$$
$$\epsilon(t_4) = -{}^F\alpha_2 l_3 + l_4$$
$$\epsilon(l_1) = pl_3$$
$$\epsilon(l_2) = pt_2 - p\beta_2 t_3$$
$$\epsilon(l_3) = l_1 + \alpha_2 l_2 + \beta_1 l_3 + \beta_2 l_4$$
$$\epsilon(l_4) = -p\alpha_2 t_3 + pt_4$$

defines an embedding of displays $\epsilon_{(A_1)} : M_{(A_1)} \hookrightarrow N \times_{\mathbb{F}_p} \overline{Y}_1$, satisfying $p\phi(x, y) = \phi(\epsilon(x), \epsilon(y))$. Neither $M_{(A_1)}$ nor $\epsilon_{(A_1)}$ depend on the choice of the lifts $\alpha_2, \beta_1, \beta_2$, which can be checked upon passage to the perfection $R_{(A_1)}^{\text{perf}}$ (here notice that $R_{(A_1)} \to R_{(A_1)}^{\text{perf}}$ is flat, because $R_{(A_1)}$ is regular). Moreover, the natural action of the Kleinian group on $\overline{X}_1$ gives rise to analogous subdisplays of the constant display N regarded over each of the translates $\{A_2 \neq 0\}$, $\{B_1, \neq 0\}$, and $\{B_2 \neq 0\}$, which in turn gives rise to an inclusion

$$\epsilon : M \hookrightarrow N \times_{\mathbb{F}_p} \overline{X}_1,$$

of sheaves of displays with respect to the Zariski topology of $\overline{X}_1$. This is because the closed points can be used to check the cocycle condition. However, notice that there does not exist a global normal decomposition for M.

2.5 Miscellaneous

The study of families of Dieudonné modules with our additional structure within a given isogeny class is meaningful not just for the supersingular one. Recall that every isogeny class of Dieudonné modules can be written as a direct sum of certain simple ones. These are parameterized by pairs of coprime non-negative integers a and b and denoted by $G_{a,b}$, see [10] for details. The isogeny class $G_{a,b}$ contains usually more than one Dieudonné module except if a or b is equal to 1, in which case we are allowed to speak of "the" Dieudonné module of type $G_{a,b}$. We have the following result:

Corollary 2.7. *Let* M *be a non-supersingular Dieudonné module over* $W(k)$ *that is equipped with a perfect symmetric pairing* ϕ. *Assume that:*

$$\operatorname{rank}_{W(k)} M = 8$$
$$\operatorname{crisdisc}(M, \phi) = 1.$$

Then M *is an orthogonal direct sum* $\bigoplus_i M_i$ *where for each of the* (M_i, ϕ) *one of the following alternatives hold:*

(i.n) (M_i, ϕ) *can be written as* $A \oplus B$ *with mutually dual isotropic Dieudonné modules* A *and* B, *which lie in the isogeny classes* $G_{1,n}$ *and* $G_{n,1}$ *for some* $n \in \{0, 1, 2, 3\}$.

(ii) (M_i, ϕ) *is supersingular of rank* 2 *and the perfect pairing thereon is the one described by part* *(i)* *of theorem 2.1.*

(iii) (M_i, ϕ) *is supersingular of rank* 4, *and the pairing is such that* $S_0(M_i)$ *decomposes into the two Dieudonné modules with pairings described by part* *(ii)* *of theorem 2.1.*

Moreover, the only combinations which occur are:

- $4 \times (\mathrm{i}.\,0)$

- $2 \times (\mathrm{i}.\,0) \oplus (\mathrm{i}.\,1)$

- $(\mathrm{i}.\,0) \oplus (\mathrm{ii}) \oplus (\mathrm{iii})$

- $(\mathrm{i}.\,0) \oplus (\mathrm{i}.\,2)$

- $(\mathrm{i}.\,3)$

Proof. We consider the canonical decomposition of $M = M_0 \oplus M' \oplus M_1$ into the étale-local, local-local, local-étale parts. The assertion of the corollary has solely something to do with M' which is of some even rank equal to $8 - 2f$ and has $\operatorname{crisdisc}(M') = (-1)^f$, here f is the p-rank of M. As M' is also self-dual it can have only one of the following isogeny types:

(1) $3 \times G_{1,1}$

(2) $2 \times G_{1,1}$

(3) $G_{1,2} \oplus G_{2,1}$

(4) $G_{1,3} \oplus G_{3,1}$

If M' has the above isogeny types 3., or 4. we deduce from [7, Paragraph(16), Satz(3)] and $a(M') = 2$ that M' is a direct sum of two Dieudonné modules A and B, each with Oort invariant equal to one. The assertion on the pairing is then immediate as neither A nor B is selfdual.

If M' has isogeny type $2G_{1,1}$ it must be superspecial. Then use theorem 2.1 in conjunction with remark 2.2 to check that M' has the shape $A \oplus B$ with isotropic A and B.

In the case in which the isogeny type of M' is $3G_{1,1}$, we have to work a bit harder: First consider a diagonalization of $S^0(M') = N = \bigoplus_{i=1}^{3} N_i$ with $S_0(M') = N^t = F^{r_i}N_i$. An analysis as in the proof of lemma 2.5 yields that $(r_1, r_2, r_3) = (0, 1, 1)$, therefore the orthogonal direct summand (N_1, φ) has a complement with perfect form, say M'', its Oort invariant is 1. Therefore crisdisc$(M'') = t^2$. As $r_2 = r_3 = 1$ this implies that crisdisc(N_2, φ), and crisdisc(N_3, φ), are the two numbers p, and pt^2, which is what we wanted. $\qquad \square$

3 The Shimura variety S_{K^p}

3.1 Further Notation

Before we proceed we want to introduce the input data for our PEL-moduli problem: Fix once and for all a quaternion algebra B over $\mathbb{Q}$ and write R for the set of places at which B is non-split. Assume that $\infty \in R$, i.e. that $B_{\mathbb{R}}$ is definite. Let p be a prime which is not in $\{2\} \cup R$ and choose a maximal $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B \subset B$, together with an isomorphism $\kappa_p : \mathbb{Z}_p \otimes \mathcal{O}_B \cong \mathrm{Mat}_2(\mathbb{Z}_p)$. The standard involution $b \mapsto b^\iota = \mathrm{tr}(b) - b$ preserves $\mathcal{O}_B$ and is positive.

Let V be a left B-module of rank 4 with non-degenerate alternating pairing satisfying $(bv, w) = (v, b^\iota w)$. For simplicity we require that the skew-Hermitian B-module V is hyperbolic in the following sense: We want it to have a B-basis e_1, e_2, f_1, f_2 such that $(\sum_{i=1}^{2} a_i e_i + b_i f_i, \sum_{i=1}^{2} a_i' e_i + b_i' f_i) = \mathrm{tr}_{B/\mathbb{Q}}(\sum_{i=1}^{2} a_i b_i'^\iota - b_i a_i'^\iota)$ for all $a_i, b_i, a_i', b_i' \in B$. Set further $\Lambda_0 = \bigoplus_{i=1}^{2} \mathcal{O}_B e_i \oplus \mathcal{O}_B f_i$, it is a self-dual $\mathcal{O}_B$-invariant $\mathbb{Z}_{(p)}$-lattice in V.

Let $G/\mathbb{Q}$ be the reductive group of all B-linear symplectic similitudes of V. This group is a form of $\mathrm{GO}(8)$. Write $K_p \subset G(\mathbb{Q}_p)$ for the hyperspecial subgroup consisting of group elements that preserve Λ_0 and let $K^p \subset G(\mathbb{A}^{\infty, p})$ be an arbitrary compact open subgroup.

Finally we specify a particular $*$-homomorphism $h_0 : \mathbb{C} \to \mathrm{End}_B(V_{\mathbb{R}})$ by the rule $h_0(i)(\sum_{i=1}^{2} a_i e_i + b_i f_i) = \sum_{i=1}^{2} b_i e_i - a_i f_i$, and $\mathbb{R}$-linear extension. The reflex field of (G, h_0) is equal to $\mathbb{Q}$.

Now, for every connected scheme $S_{K^p}/\mathbb{Z}_{(p)}$ with a geometric base point s we consider the set of $\mathbb{Z}_{(p)}$-isogeny classes of quadruples $(A, \lambda, \imath, \overline{\eta})$ with:

(M1) A is a 8-dimensional abelian scheme over S up to prime-to-p isogeny

(M2) $\lambda : A \to A^t$ is a $\mathbb{Z}_{(p)}^\times$-class of prime-to-p polarizations of A

(M3) $\imath : \mathcal{O}_B \to \operatorname{End}(A) \otimes \mathbb{Z}_{(p)}$ is a homomorphism satisfying $\imath(b^t) = \imath(b)^*$, here $*$ is the Rosati involution associated to λ

(M4) $\overline{\eta}$ is a $\pi_1(S, s)$-invariant K^p-orbit of $\mathcal{O}_B$-linear isomorphisms $\eta : V \otimes \mathbb{A}^{\infty, p} \cong H_1(A_s, \mathbb{A}^{\infty, p})$ which are compatible with the alternating form up to scalars.

By geometric invariant theory this functor is representable by a quasi-projective $\mathbb{Z}_{(p)}$-scheme S_{K^p}. Moreover, the deformation theory of Grothendieck-Messing shows that S is smooth of relative dimension 6 over $\mathbb{Z}_{(p)}$, cf. [6, Chapter 5]. See also [6, Chapter 8] for the complex uniformizations of $S_{K^p}(\mathbb{C})$.

Finally, let us write $S_{K^p}^{si}$ (resp. $S_{K^p}^{sp}$) for the subsets $S_{K^p} \times \mathbb{F}_p^{ac}$ whose sets of geometric points consist of those quadruples $(A, \lambda, \imath, \overline{\eta})$ where $\mathbb{D}(A[p^\infty])$ is supersingular (resp. superspecial), here $\mathbb{D}(G)$ denotes the (covariant) Dieudonné module of a p-divisible group G over a perfect field. Notice that we always have $\operatorname{crisdisc}(\mathbb{D}(G), \phi) = 1$, by [1].

3.2   Morita equivalence

Let us write G^* for the Serre-dual of a p-divisible group $G = \bigcup_l G[p^l]$ over some base scheme S. We will say that G is polarized (resp. anti-polarized) if it is endowed with an isomorphism ϕ to its dual which satisfies $\phi = -\phi^*$ (resp. $\phi = \phi^*$). In particular, consider the anti-polarized p-divisible groups $G_1 := \mathcal{BT}(M)$ and $G_0 := \mathcal{BT}(N)$, where M and N are as in section 2.4. The emdedding $\epsilon : M \hookrightarrow N \times_{\mathbb{F}_p} \overline{X}_1$ gives rise to a canonical isogeny $\epsilon : G_1 \to G_0 \times_{\mathbb{F}_p} \overline{X}_1$ satisfying $\epsilon^* \circ \epsilon = p\,\mathrm{id}_{G_1}$ and $\epsilon \circ \epsilon^* = p\,\mathrm{id}_{G_0 \times_{\mathbb{F}_p} \overline{X}_1}$, notice also that $\ker(\epsilon) \subset G_1[p]$ and $\ker(\epsilon^*) \subset G_0[p] \times_{\mathbb{F}_p} \overline{X}_1$ are finite, flat, maximal isotropic subgroup schemes of order p^4.
If an isomorphism $\mathbb{Z}_p \otimes \mathcal{O}_B \overset{\kappa_p}{\to} \operatorname{Mat}_2(\mathbb{Z}_p)$ is fixed once and for all, one obtains a Morita-equivalence

$$(G, \phi) \mapsto \left(G^{\oplus 2}, \begin{pmatrix} 0 & \phi \\ -\phi & 0 \end{pmatrix}\right)$$

from the category of anti-polarized p-divisible groups to the category of polarized p-divisible groups with Rosati-invariant $\mathcal{O}_B$-action. In this manner one obtains an anti-polarized p-divisible group (G, ϕ) from every S-valued point on S_{K^p}, say represented by $(A, \lambda, \imath, \overline{\eta})$, by the requirement

$$(A[p^\infty], \psi_\lambda) \cong \left(G^{\oplus 2}, \begin{pmatrix} 0 & \phi \\ -\phi & 0 \end{pmatrix}\right),$$

where $\psi_\lambda : A[p^\infty] \to A[p^\infty]^*$ is the p-adic Weil-pairing, which is induced from the polarization $\lambda : A \to A^t$. If S is the spectrum of a perfect field of characteric p, we always have $\operatorname{crisdisc}(\mathbb{D}(G), \phi) =$

1, by [1]. We next want to define a family of morphisms

$$c_{x, \eta_p} : \overline{X}_1 \times \mathbb{F}_p^{ac} \to S_{K^p} \times \mathbb{F}_p^{ac} \tag{4}$$

which are indexed by superspecial $\mathbb{F}_p^{ac}$-points $x = (A, \lambda, \iota, \overline{\eta})$, equipped with the following additional datum: By a frame for x we mean an isomorphism $\eta_p : G_0 \times_{\mathbb{F}_p} \mathbb{F}_p^{ac} \to G$, where (G, φ) corresponds to $x \in S_{K^p}(\mathbb{F}_p^{ac})$ by the above Morita-equivalence while (G_0, φ_0) is the previously exhibited anti-polarized p-divisible group. Let us consider the abelian variety which is defined by the exact sequence:

$$0 \to \eta_p(\ker(\epsilon^*))^{\oplus 2} \to A \xrightarrow{e^t} A_1 \to 0,$$

the isotropicity and the $\mathcal{O}_B$-invariance of $\eta_p(\ker(\epsilon^*))^{\oplus 2}$ give rise to a canonical $\mathbb{Z}_{(p)}^\times$-class of prime-to-p polarizations $\lambda_1 : A_1 \to A_1^t$, together with a Rosati-invariant operation $\iota_1 : \mathcal{O}_B \to \operatorname{End}(A_1) \otimes \mathbb{Z}_{(p)}$ and level structure $\overline{\eta}_1$, each gotten by transport of structure. Finally one sees that the quadruple $x_1 = (A_1, \lambda_1, \iota_1, \overline{\eta}_1)$ thus obtained constitutes a $\overline{X}_1 \times \mathbb{F}_p^{ac}$-valued point, whose classifying morphism we define to be (4). It is easy to see that the image of c_{x, η_p} is a closed subset, whose geometric points consist of exactly those quadruples $(A_1, \lambda_1, \iota_1, \overline{\eta}_1)$ which allow an $\mathcal{O}_B$-linear isogeny $e : A_1 \to A$, wich is compatible with the level structure and satisfies $p\lambda_1 = e^t \circ \lambda \circ e$.

Remark 3.1. Fix $(A, \lambda, \iota, \overline{\eta}) = x \in S_{K^p}^{sp}(\mathbb{F}_p^{ac})$. Notice, that we have just shown, that the Zariski-closed subset $c_{x, \eta_p}(\overline{X}_1 \times \mathbb{F}_p^{ac}) := S_{x, K^p}^{sp}$ does not dependent on the choice of frame.

3.3   Description of $S_{K^p}^{si}$

Now, we would like to investigate whether or not c_{x, η_p} is a closed immersion, the next lemma is a step towards this direction:

Lemma 3.2. *Let* x *and* η_p *be as above, then* c_{x, η_p} *induces an injection on the tangentspaces to each geometric point* $u \in \overline{X}_1(k)$*, where* k *is an arbitrary algebraically closed field of characteristic* p*.*

Proof. Recall that every k-display P of dimension d and codimension c allows structural equations:

$$F(t_j) = \sum_{i=1}^{d} u_{i,j} t_i + \sum_{i=1}^{c} u_{i+d,j} l_i$$

$$V^{-1}(l_j) = \sum_{i=1}^{d} u_{i,j+d} t_i + \sum_{i=1}^{c} u_{i+d,j+d} l_i$$

for some display-matrix

$$\begin{pmatrix} u_{1,1} & \cdots & u_{1,c+d} \\ \vdots & \ddots & \vdots \\ u_{c+d,1} & \cdots & u_{c+d,c+d} \end{pmatrix} = U \in \operatorname{GL}(c+d, W(k)),$$

where $t_1, \ldots, t_d, l_1, \ldots, l_c \in P$, and $t_1 + Q, \ldots, t_d + Q \in P/Q$ are bases. Let L and T be the $W(k)$-submodules of P that are generated by $l_1, \ldots, l_c$ and $t_1, \ldots, t_d$, and write $J := \mathrm{Hom}_{W(k)}(L, T)$. Due to the technique of Norman-Oort the isomorphism classes of infinitesimal deformations of P over the ring of dual numbers $k_D := k[s]/(s^2)$ are parameterized by the elements in $J \otimes_{W(k)} k = \mathrm{Hom}_k(Q/pP, P/Q)$, in fact each deformation may be described explicitly as follows: Pick a tangent direction $N \in J \otimes_{W(k)} k$, say with $d \times c$-matrix representation

$$\begin{pmatrix} n_{1,1} & \ldots & n_{1,c} \\ \vdots & \ddots & \vdots \\ n_{d,1} & \ldots & n_{d,c} \end{pmatrix}$$

(with respect to the two bases above). Write $W(sk_D)$ for the kernel of the natural map from $W(k_D)$ to $W(k)$, and choose elements $\tilde{n}_{i,j} \in W(sk_D)$ whose 0-th Witt coordinate is equal to the dual number $sn_{i,j}$. Then

$$\tilde{U} := \begin{pmatrix} 1 & \ldots & 0 & \tilde{n}_{1,1} & \ldots & \tilde{n}_{1,c} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & \tilde{n}_{d,1} & \ldots & \tilde{n}_{d,c} \\ 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 1 \end{pmatrix} \quad U \in \mathrm{GL}(c + d, W(k_D))$$

displays an infinitesimal deformation of P, that corresponds to the tangent direction N, in particular it is the trivial deformation if and only of $N = 0$.

Now let $(X_1 : X_2 : Y_1 : Y_2)$ be the homogeneous coordinates of $u \in \overline{X}_1(k)$, and fix one of its non-zero tangent directions $u' \in \overline{X}_1(k_D)$. To finish the proof of the lemma we only have to show that the associated k_D-display $M_{u'}$ is a non-trivial infinitesimal deformation (of M_u, i.e. the special fiber of $M_{u'}$). Of course we can assume $(X_1 : X_2 : Y_1 : Y_2) = (1 : x_2 : y_1 : y_2)$ from the start, so let $(1 : x_2 + sa : y_1 - s(ay_2^p - bx_2^p) : y_2 + sb)$ be the homogeneous coordinates of u', where $(a, b) \in k^2 - \{(0,0)\}$. Now recall from section 2.4 that the restriction of M to the affine chart $\mathrm{Spec}\,\mathbb{F}_p[a_2, b_1, b_2]/(b_1 + b_1^p + a_2 b_2^p + b_2 a_2^p) \subset \overline{X}_1$ has already a normal decomposition and is explicitly displayed in an extremely convenient way, namely by means of the matrix $U = \begin{pmatrix} H & E \\ E & 0 \end{pmatrix}$, where E denotes the identity matrix, and where the (so-called 'Hasse-Witt') matrix H is given by:

$$\begin{pmatrix} 0 & \beta_2 - {}^{F^2}\beta_2 & 0 & \alpha_2 - {}^{F^2}\alpha_2 \\ 0 & 0 & {}^{F^2}\alpha_2 - \alpha_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & {}^{F^2}\beta_2 - \beta_2 & 0 \end{pmatrix},$$

for certain $\alpha_2, \beta_1, \beta_2 \in W(\mathbb{F}_p[a_2, b_1, b_2]/(b_1 + b_1^p + a_2 b_2^p + b_2 a_2^p))$. Now consider the sk_D-valued Witt-vectors $\alpha := u'(\alpha_2) - u(\alpha_2)$ and $\beta := u'(\beta_2) - u(\beta_2)$, in fact it is easy to see that

$u'(\beta_1) - u(\beta_1) = -(\alpha u(\beta_2)^\sigma + \beta u(\alpha_2)^\sigma)$, because α and β are killed by F. Moreover, the 0th Witt-coordinates of α and β are just sa and sb. It follows immediately that $u'(U) = \begin{pmatrix} E & \tilde{N} \\ 0 & E \end{pmatrix} u(U)$, with $\tilde{N}$ being the deformation matrix:

$$\begin{pmatrix} 0 & \beta & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & 0 \end{pmatrix},$$

whose matrix of 0th Witt-components is clearly nonvanishing. □

As a consequence of theorem 2.4 we have:

$$S^{si}_{K^p} = \bigcup_{x \in S^{sp}_{K^p}} S^{si}_{x, K^p},$$

and $S^{sp}_{K^p}$ is a finite set of closed points. It follows from this (or from Grothendieck's specialization theorem [3, p.149]), that $S^{si}_{K^p}$ is Zariski closed. Our aim is to describe $S^{si}_{K^p}$ together with its induced reduced subscheme structure. Let us fix $x \in S^{sp}_{K^p}$, which classifies some quadruple $(A, \lambda, \iota, \overline{\eta})$, and let $*$ denote the Rosati-involution on the $\mathbb{Q}$-algebra $\mathrm{End}^0_B(A)$. Let us write $I_x/\mathbb{Q}$ for the group scheme which represents the functor

$$C \mapsto \{g \in (\mathrm{End}^0_B(A) \otimes C)^\times | gg^t \in C^\times\}. \tag{5}$$

Every full level structure $\eta : V \otimes \mathbb{A}^{\infty, p} \cong H_1(A, \mathbb{A}^{\infty, p})$ yields an isomorphism

$$I \times \mathbb{A}^{\infty, p} \xrightarrow{\cong} G \times \mathbb{A}^{\infty, p}; \gamma \mapsto \eta^{-1} \gamma \eta.$$

Notice that the preimage of K^p under the above isomorphism depends only on the K^p-orbit of η, and hence we can define $K^p_x := \eta K^p \eta^{-1}$ for any $\eta \in \overline{\eta}$, this is again a compact open subgroup of $I_x(\mathbb{A}^{\infty, p})$. Consider the compact set $\tilde{K}_p := \{\gamma \in I(\mathbb{Q}_p) | \gamma, \gamma^{-1} \in p^{-1}\mathbb{Z}_p \otimes \mathrm{End}_B(A)\}$, and let us say that K^p is superneat for x if and only if $I_x(\mathbb{Q}) \cap \tilde{K}_p \times K^p_x = \{1\}$. The left-hand side is always a finite group, because I_x is anisotropic. In particular K^p will always contain some a compact open subgroup which is superneat for every $x \in S^{si}_{K^p}$

Lemma 3.3. *If K^p is superneat for x, then (4) is a closed immersion.*

Proof. A morphism from a proper $\mathbb{F}^{ac}_p$-variety to a separated one is a closed immersion if and only if it radical and injective on the tangent spaces to all $\mathbb{F}^{ac}_p$-valued points, this is elementary and can be proved along the lines of [4, Lemma 7.4.]. In view of lemma 3.2 it suffices to check that (4) is indeed injective on geometric points. Suppose it wasn't. Then there existed $S_{K^p}(k) \ni x_1 = (A_1, \lambda_1, \iota_1, \overline{\eta}_1)$ which lies in the image of (4) in two different ways. According to the thoughts at the end of subsection 3.2, this means that there existed two degree-p^8-isogenies $e, e' : A_1 \to A$ each of which induce the additional structures $\lambda_1, \iota_1, \overline{\eta}_1$ from the additional structures $\lambda, \iota, \overline{\eta}$ on A. It follows immediately that $\mathrm{id}_A \neq e' \circ e^{-1}$ is in contradiction to K^p being superneat for x. □

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