

Computing the resolvent of composite operators

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ABSTRACT

Based in a very recent paper by Micchelli et al. [8], we present an algorithmic approach for computing the resolvent of composite operators: the composition of a monotone operator and a continuous linear mapping. The proposed algorithm can be used, for example, for solving problems arising in image processing and traffic equilibrium. Furthermore, our algorithm gives an alternative to Dykstra-like method for evaluating the resolvent of the sum of two maximal monotone operators.

RESUMEN

Basados en un artículo reciente de Micchelli et al. [8], presentamos una manera algorítmica para calcular la resolvente de operadores compuestos: la composición de un operador monótono y una aplicación lineal continua. El algoritmo propuesto puede usarse, por ejemplo, para resolver problemas que aparecen en procesamiento de imágenes y equilibrio de tránsito. Además, nuestro algoritmo entrega una alternativa a métodos tipo Dykstra para evaluar al resolvente de la suma de dos operadores monótonos maximales.

Keywords and Phrases: 49J53, 65K10, 49M37, 90C25.

2010 AMS Mathematics Subject Classification: maximal monotone operators, KM-algorithm, Yosida regularization, Douglas-Rachford algorithm, Dykstra-like method.

1 Introduction and preliminaries

To begin with, let us recall the following concepts which are of common use in the context of convex and nonlinear analysis, see for example Takahashi [13] or [1]. Throughout, H is a real Hilbert space, $\langle \cdot, \cdot \rangle$ denotes the associated scalar product and $\| \cdot \|$ stands for the corresponding norm. An operator with domain $\text{Dom}(T)$ and range $\text{R}(T)$ is said to be monotone if

$$\langle u - v, x - y \rangle \geq 0 \quad \text{whenever} \quad u \in T(x), v \in T(y).$$

It is said to be maximal monotone if, in addition, its graph, $\text{gph}T := \{(x, y) \in H \times H : y \in T(x)\}$, is not properly contained in the graph of any other monotone operator. It is well-known that for each $x \in H$ and $\lambda > 0$ there is a unique $z \in H$ such that

$$x \in (I + \lambda T)z. \tag{1}$$

The single-valued operator $J_\lambda^\Gamma := (I + \lambda T)^{-1}$ is called the resolvent of T of parameter λ . It is a nonexpansive mapping which is everywhere defined and is related to its Yosida approximate, namely $T_\lambda(x) := \frac{x - J_\lambda^\Gamma(x)}{\lambda}$, by the relation

$$T_\lambda(x) \in T(J_\lambda^\Gamma(x)). \tag{2}$$

The latter is Lipschitz continuous with constant $\frac{1}{\lambda}$. Recall also that the inverse T^{-1} of T is the operator defined by $x \in T^{-1}(y)$ if and only if $T(x) = y$.

Now, let $A : H_1 \rightarrow H_2$ be a continuous linear operator with adjoint A^* , $T : H_2 \rightarrow H_2$ a maximal monotone operator, H_1 and H_2 being two Hilbert spaces. It is easily checked that the composite operator A^*TA is monotone. This kind of operator appears, for example, in partial differential equations in divergence form and in signal and image processing. Without further conditions, however, it may fail to be maximal monotone, see for sufficient conditions [12] or [13].

Now, let us state the two following key facts:

Fact 1: A^*TA is maximal monotone, see for instance [12]-Corollary 4.4, if $0 \in \text{ri}(\text{R}(A) - \text{Dom}T)$, where ri stands for the relative interior of a set.

The Krasnoselski-Mann algorithm is a widely used method for solving fixed-point problems. This algorithm generates from an arbitrary initial guess $v_0 \in C$ (a closed convex set), a sequence (y_k) by the recursive formula

$$y_{k+1} = (1 - \alpha_k)y_k + \alpha_k Q(y_k), \quad k \in \mathbb{N},$$

where (α_k) is a sequence in $[0, 1]$.

Fact 2: It is known that for a nonexpansive mapping Q , the Krasnoselski-Mann's algorithm converges weakly to a fixed point of Q provided that the sequence (α_k) is such that $\sum_{k=0}^{\infty} \alpha_k(1 - \alpha_k) = +\infty$ and the underlying space is a Hilbert space, see for example [2].

Given $x \in H_1$, we will focus our attention in this paper on the following problem

$$\text{Compute} \quad y := J_1^{A^*TA}(x). \tag{3}$$

The problem (1.3) was studied in the case where $T = \partial\phi$ because it arises in many applications in signal processing and image denoising, see for example, [8], [9] and references therein. More precisely, If $\phi : H_2 \rightarrow \mathbb{R} \cup \{+\infty\}$ is a convex and lower semicontinuous, and $A : H_1 \rightarrow H_2$ is a continuous and linear mapping, then the composition $\phi \circ A$ is also convex and lower semicontinuous. Furthermore, by the chain rule of convex analysis,

$$A^* \partial\phi A \subset \partial(\phi \circ A),$$

where equality holds whenever the constraint qualification $0 \in \text{ri}(\text{R}(A) - \text{Dom}\phi)$ is satisfied, see for instance [12], (here $\text{Dom}\phi = \{x \in H_2; \phi(x) < +\infty\}$ and $\partial\phi(x)$ means the subdifferential of ϕ at x and is equal to the set $\{u \in H_2 : \phi(y) \geq \phi(x) + \langle u, y - x \rangle$ for all $y \in H_2\}$ as usual). In this context (1.3) reduces to the problem of evaluating the proximity operator,

$$\text{prox}_{\phi \circ A} x := \text{argmin}_u \{ \phi \circ A(u) + \frac{1}{2} \|u - x\|^2 \}, \tag{4}$$

of $\phi \circ A$ at x . This arises, for instance, in studying the L_1 /TV image denoising model that involves the total-variation regularization term which is non-differentiable. This causes algorithmic difficulties for its numerical treatment. To overcome the difficulties, the authors formulate in [8] the total-variation as a composition of a convex function (the l_1 -norm or the l_2 - norm) and the first order difference operator, and then express the solution of the model in terms of the proximity operator of the composition, by identifying the solution as fixed point of a nonlinear mapping expressed in terms of the proximity operator of the l_1 -norm or the l_2 -norm, each of which is explicitly given.

Our aim here is to extend their analysis to evaluate the resolvent of the main operator A^*TA . To begin with, let us recall that M. Fukushima [6] proved that if $A \circ A^*$ is an isomorphism, then the operator A^*TA is maximal monotone. Whereas, in finite dimensional setting, S. Robinson [13] observed that we may avoid this condition provided that $\text{R}(A) \cap \text{ri}(\text{Dom}T) \neq \emptyset$. Moreover,

$$J_{\lambda}^{A^*TA}(x) = x - \lambda u \text{ with } u = (T^{-1} + \lambda AA^*)^{-1}(Ax).$$

However, the above formula is difficult to evaluate in practice since it involves T^{-1} . To overcome this difficulty, M. Fukushima [6] proposed an alternative computation of $y = J_{\lambda}^{A^*TA}(x)$: Given $x \in H_1$

(i) find the unique solution z of

$$0 \in \frac{1}{\lambda} (AA^*)^{-1}(z - Ax) + Tz,$$

(ii) Compute u , by

$$u = \frac{1}{\lambda} (AA^*)^{-1}(Ax - z)$$

(iii) Compute $y = x - \lambda A^*u$.

Let us remark the potential difficulties in implementing the above algorithm lie in the fact that it involves the inverse operator AA^* . The evaluation of such a mapping is in general expensive.

In the sequel we will follow another approach developed in [8] in a convex optimization context.

2 Fixed-point approach

Under the assumption that we can explicitly compute the resolvent operator of T and by assuming that A^*TA is a maximal monotone operator, our aim is to develop an algorithm for evaluating the resolvent of A^*TA . Remember that from (1.1), for each $x \in H_1$ there is a unique $z := J_1^{A^*TA}x \in H_1$ such that $x \in (I + A^*TA)z$. Our main is to provide a constructive method to compute it. From (1.2), we clearly have

$$J_1^{A^*TA}(x) \in x - A^*T(A(J_1^{A^*TA}x)). \quad (5)$$

This combined with the fact that

$$y \in A^*TA(x) \Leftrightarrow x = J_1^{A^*TA}(x + y)$$

enable us to establish a relationship between the resolvent of A^*TA and that of T . To that end, we define the following affine transformation for a fixed $x \in H_1$ at $y \in H_2$ by

$$Fy := Ax + (I - \lambda AA^*)y$$

and the operator

$$Q := (I - J_{1/\lambda}^T) \circ F. \quad (6)$$

Theorem 2.1. *If T is a maximal monotone operator, A a continuous linear operator and $\lambda > 0$ then*

$$J_1^{A^*TA}x = x - \lambda A^*y$$

if and only if y is a fixed point of Q .

Proof. According to (2.1), we can write

$$J_1^{A^*TA}x = x - \lambda A^*y,$$

where $y \in \frac{1}{\lambda}T(AJ_1^{A^*TA}x)$. Thus $y \in \frac{1}{\lambda}T(A(x - \lambda A^*y))$.

Using, for instance, the fact that $y \in T(x) \Leftrightarrow x = J_1^T(x + y)$, we deduce

$$Ax - \lambda AA^*y = J_{1/\lambda}^T(Ax + (I - \lambda AA^*)y),$$

that is y is a fixed-point of Q .

Conversely, if y is a fixed-point of Q then

$$Ax - \lambda AA^*y = J_{1/\lambda}^T(Ax + (I - \lambda AA^*)y),$$

Definition of the resolvent yields

$$\lambda y \in TA(x - \lambda A^*y),$$

thus $\lambda A^*y \in A^*TA(x - \lambda A^*y)$, and by (2.1) we obtain $J_1^{A^*TA}x = x - \lambda A^*y$. This completes the proof. \square

Hence, it suffices to find a fixed-point \mathbf{y} of \mathcal{Q} and the resolvent of T is then equal to $\mathbf{x} - \lambda A^* \mathbf{y}$.

Now, by assuming $\|I - \lambda A A^*\| \leq 1$ and having in mind that $I - J_{1/\lambda}^T$ is nonexpansive, we easily derive:

Lemma 2.2. *If T is a maximal monotone operator and A a continuous linear mapping and $\lambda > 0$ such that $\|I - \lambda A A^*\| \leq 1$, then the operator \mathcal{Q} is nonexpansive.*

To find a fixed point \mathbf{y}^∞ of the operator \mathcal{Q} , we can use, for example, the KM-algorithm, namely

$$\mathbf{y}_{k+1} = \alpha_k \mathbf{y}_k + (1 - \alpha_k) \mathcal{Q}(\mathbf{y}_k), \tag{7}$$

and then set $J_1^{A^* T A} \mathbf{x} = \mathbf{x} - \lambda A^* \mathbf{y}^\infty$.

Using fact 2, we derive the following result.

Corollary 2.3. *Under assumptions of Lemma 1.2, for any $\alpha_k \in [0, 1]$ satisfying $\sum_k \alpha_k (1 - \alpha_k) = +\infty$, the sequence (\mathbf{x}_k) generated by (2.3) weakly converges to a fixed-point of \mathcal{Q} .*

3 Applications

3.1 Image denoising

L_1/TV models can be cast into the following general form

$$0 \in \lambda S(\mathbf{x}) + A^* T A \mathbf{x}, \tag{8}$$

where S and T are two maximal monotone operator and A a linear continuous mapping.

For example, given a noisy image \mathbf{x} which was contaminated by impulsive noise, we consider a denoised image of \mathbf{x} as a minimizer of the following L_1/TV model

$$\min\{\lambda \|\mathbf{u} - \mathbf{x}\|_1 + \|\mathbf{u}\|_{TV}\}, \tag{9}$$

where $\|\cdot\|_1$ represents the l_1 -norm, $\|\cdot\|_{TV}$ denotes the total-variation, and λ is the regularization parameter balancing the fidelity term $\|\mathbf{u} - \mathbf{x}\|_1$ and the regularization term $\|\cdot\|_{TV}$. By rewriting $\|\mathbf{u}\|_{TV} = \phi \circ A$, with ϕ a convex lower-semicontinuous function and A a real matrix and by using the chain rule, we obtain the following optimality condition of (3.2)

$$0 \in \lambda \partial \|\mathbf{u} - \mathbf{x}\|_1 + A^t \partial \phi A(\mathbf{u}),$$

which is nothing else than (3.1) with $S = \partial \|\cdot - \mathbf{x}\|_1$ and $T = \partial \phi$.

It is well accepted that the l_1 -norm fidelity term can effectively suppress the effect of outliers that may contaminate a given image, and is therefore particularly suitable for handling non-Gaussian additive noise. The L_1/TV model (3.1) has many distinctive and desirable features. For the

anisotropic total variation, ϕ is expressed with $\|\cdot\|_1$ while for the isotropic total variation, ϕ is expressed with $\|\cdot\|_2$. Thus, we can obtain their proximity mappings in closed-forms. Indeed, for $x \in \mathbb{R}^m$ and $\lambda > 0$

$$\text{prox}_{\frac{1}{\lambda}\|\cdot\|_1}(x) = (\text{prox}_{\frac{1}{\lambda}|\cdot|}(x), \dots, \text{prox}_{\frac{1}{\lambda}|\cdot|}(x)) \text{ with } \text{prox}_{\frac{1}{\lambda}|\cdot|}(x_i) = \max(|x_i| - \frac{1}{\lambda}, 0)\text{sign}(x_i),$$

while

$$\text{prox}_{\frac{1}{\lambda}\|\cdot\|_2}(x) = \max(\|x\|_2 - \frac{1}{\lambda}, 0) \frac{x}{\|x\|_2}.$$

Since in section 2, we developed a method for computing the resolvent of the operator A^*TA , to solve (3.1), we can make use of any splitting algorithm for finding the zero of the sum of two maximal monotone operators, for instance that of Passty which consists in the Picard iteration for the composition of the resolvents of λS and A^*TA . But, it is well known that we only have ergodic convergence. The algorithm which is of common use in this type of context is the so-called Douglas-Rachford (DR) proposed in its initial form by Lions and Mercier [7] and that takes here the following form

$$\begin{cases} x_k = J_Y^{A^*TA}(y_k) \text{ (by a loop which uses the method introduced in section 2);} \\ y_{k+1} = y_k + \alpha_k(J_{\lambda Y}^S(2x_k - y_k) - y_k); \end{cases} \tag{10}$$

Thanks to the well-known convergence result for DR-algorithm, we derive that the sequence (y_k) converges weakly to a fixed-point y^∞ of $(2J_Y^{A^*TA} - I) \circ (2J_{\lambda Y}^S - I)$ and $J_{\lambda Y}^S y^\infty$ solves (1.3) provided that $\sum_k \alpha_k(2 - \alpha_k) = +\infty$ and $\alpha_k < 2$.

Moreover, if $\dim H < +\infty$, then the sequence (x_k) converges to a solution of (3.1).

3.2 Resolvent of a sum of two operators

Let $x \in H$, let T_1 and T_2 be two maximal monotone operators from H to H . To compute the resolvent of the sum of T_1 and T_2 at x , Bauschke and Combettes [4] proposed the following Dykstra-like method: and set

$$\begin{cases} x_0 = x, \\ p_0 = 0 \\ q_0 = 0 \end{cases} \text{ and } \begin{cases} y_k = J_1^{T_2}(x_k + p_k); \\ p_{k+1} = x_k + p_k - y_k, \end{cases} \text{ and } \begin{cases} x_{k+1} = J_1^{T_1}(y_k + q_k); \\ q_{k+1} = y_k + q_k - x_{k+1}. \end{cases} \tag{11}$$

They proved that if $x \in R(\text{Id} + T_1 + T_2)$, then both the sequences (x_k) and (y_k) strongly converges to $J_1^{T_1+T_2}x$.

Our aim is to propose an alternative approach for computing the resolvent of a monotone operator which can be decomposed as a sum of two maximal monotone operators, such that their individual resolvents can be implemented easily. Both methods we present will proceed by splitting in the sense that, at each iteration, they employ these resolvents separately.

First, note that if T_1 and T_2 are set-valued mappings from H to H , their pointwise sum can be expressed in the composite form A^*TA ; by defining

$$\mathcal{H} = H \times H; Ax = (x, x) \text{ and } T(x_1, x_2) = T_1(x_1) \times T_2(x_2).$$

Indeed, then

$$A^*(y_1, y_2) = y_1 + y_2 \text{ and so } A^*TA(x) = T_1(x) + T_2(x).$$

This fact will allow us to give alternative algorithms to compute the resolvent operator of the sum of two maximal monotone operators relying on the fixed-point approach developed in section 2.

It is easily seen that finding a fixed point $y = (y_1, y_2)$ of the operator \mathcal{Q} , defined by (2.2), amounts to solving the following system

$$\begin{cases} y_1 = \frac{1}{\lambda}(I - J_1^{T_1})(x - \lambda y_2); \\ y_2 = \frac{1}{\lambda}(I - J_1^{T_2})(x - \lambda y_1), \end{cases} \tag{12}$$

Note that the operator $\tilde{\mathcal{Q}}(y_1, y_2) := \begin{cases} \frac{1}{\lambda}(I - J_1^{T_1})(x - \lambda y_2); \\ \frac{1}{\lambda}(I - J_1^{T_2})(x - \lambda y_1), \end{cases}$ is nonexpansive for all $\lambda > 0$.

Thus, we can use the algorithm

$$(y_1^{k+1}, y_2^{k+1}) = \alpha_k(y_1^k, y_2^k) + (1 - \alpha_k)\tilde{\mathcal{Q}}(y_1^k, y_2^k), \tag{13}$$

to find a fixed-point (y_1^∞, y_2^∞) and then we deduce the resolvent of the sum of T_1 and T_2 . Indeed, we have the following result:

Proposition 3.1. *Let T_1, T_2 be two maximal monotone operators, then for any $\alpha_k \in [0, 1]$ satisfying $\sum_k \alpha_k(1 - \alpha_k) = +\infty$, the sequence (y_1^k, y_2^k) generated by (3.6) weakly converges to a fixed-point (y_1^∞, y_2^∞) . Furthermore, we have $J_1^{T_1+T_2}(x) = x - \lambda A^*(y_1^\infty, y_2^\infty) = x - \lambda(y_1^\infty + y_2^\infty)$.*

We would like to emphasize that we can also use a Von Neumann-like alternating algorithm for solving system (3.5). Indeed, the latter is equivalent to

$$\begin{cases} \lambda y_1 = (I - J_1^{T_1})(x - (I - J_1^{T_2})(x - \lambda y_1)); \\ \lambda y_2 = (I - J_1^{T_2})(x - \lambda y_1). \end{cases} \tag{14}$$

By defining $\tilde{A}(y) = -A(-y)$ for a given operator A , a simple computation gives

$$J_1^{\tilde{A}}(y) = -J_1^A(-y) \text{ and } J_1^{\tilde{A}(\cdot-x)}(y) = x + J_1^A(y - x).$$

Hence, by setting $A := T_1^{-1}$, $B := T_2^{-1}$, $v_1 := \lambda y_1$ and $v_2 := \lambda y_2$, and according to the fact that $I - J_1^A = J_1^{A^{-1}}$, we finally obtain

$$\begin{cases} v_1 = J_1^A \circ J_1^{\tilde{B}(\cdot-x)}(v_1); \\ v_2 = J_1^B \circ J_1^{\tilde{A}(\cdot-x)}(v_2). \end{cases} \tag{15}$$

This suggests to consider the following alternating resolvent method:

$$\begin{cases} v_1^0 \in H \text{ and } \tilde{v}_1^k = J_1^{\tilde{B}(-x)}(v_1^k), v_1^{k+1} = J_1^A(\tilde{v}_1^k); \\ v_2^0 \in H \text{ and } \tilde{v}_2^k = J_1^{\tilde{A}(-x)}(v_2^k), v_2^{k+1} = J_1^A(\tilde{v}_2^k). \end{cases} \quad (16)$$

From [3]-Theorem 3.3, we deduce:

Proposition 3.2. *Let T_1, T_2 be two maximal monotone operators, then the sequence (v_1^k, v_2^k) generated by (3.9) weakly converges to a solution (v_1^∞, v_2^∞) of (3.8) provided that the latter exists. Moreover, the resolvent of the sum of T_1 and T_2 at x is then given by $J_1^{T_1+T_2}(x) = x - \lambda A^*(y_1^\infty, y_2^\infty) = x - \lambda(y_1^\infty + y_2^\infty)$.*

To conclude, it is worth mentioning that in the case of convex optimization, the problem under consideration in this section amounts to evaluating the *proximity operator of the sum of two proper, lower semicontinuous convex functions* $\phi, \psi : H \rightarrow \mathbf{R} \cup \{+\infty\}$, namely: given x ,

$$\text{compute } y := \text{prox}_{\phi+\psi}(x). \quad (17)$$

In this context, the resolvent operator is nothing else than the proximity operator and according to the fact that $(I - \text{prox}_\phi) = \text{prox}_{\phi^*}$, system (3.5) reduces to

$$\begin{cases} y_1 = \frac{1}{\lambda}(I - \text{prox}_\phi)(x - \lambda y_2) = \frac{1}{\lambda}\text{prox}_{\phi^*}(x - \lambda y_2); \\ y_2 = \frac{1}{\lambda}(I - \text{prox}_\psi)(x - \lambda y_1) = \frac{1}{\lambda}\text{prox}_{\psi^*}(x - \lambda y_1), \end{cases} \quad (18)$$

where ϕ^*, ψ^* stand for the Fenchel conjugate of the functions ϕ and ψ . Remember that the Fenchel conjugate of a given function f is defined as $f^*(x) = \sup_u \{\langle x, u \rangle - f(u)\}$.

Finally, we would like to emphasize that we can also consider as an application the traffic equilibrium problem consider in [6]. The advantage of our approach is that it does not require the inversion of the operator AA^* . Roughly speaking, see [6], the traffic equilibrium problem can be written as (3.1) with $\lambda = 1$, $S = \partial\delta_C$, namely

$$0 \in A^*TAx + \partial\delta_C(x), \quad (19)$$

$S = \partial\delta_C$ the partial differential of the indicator function of a polyhedral convex set, T a cost function (not assumed to be single-valued as in [6], but we suppose that its resolvent operator can be computed efficiently) and A satisfying

$$AA^* = \nu I \text{ for } \nu \in \mathbf{N}. \quad (20)$$

The latter is also satisfied in image restoration problems by the so-called tight frames operators. It is well known that in this case the resolvent operator of $\partial\delta_C$ is exactly the projection onto C (algorithms for computing projection which do not require any particular hypothesis on the input

polyhedral sets are available) and a simple calculation shows that the method developed in section 2 gives

$$J_1^{A^*T A}(x) = x - A^*T_\nu(Ax).$$

The latter formula is in fact valid for any positive real number ν and generalizes a formula by Combettes and Pesquet [5] obtained in the case where T is the subdifferential of a proper lower-semicontinuous function and was used in many applications in image restoration. More precisely, relying on a qualification condition and by taking into account (1.4), we clearly have

$$\text{prox}_{\phi \circ A} x = x - A^*(\partial\phi)_\nu(Ax) = x - \nu^{-1}A^*(I - \text{prox}_{\nu\phi})Ax.$$

Now, an application of Passty’s algorithm gives

$$x_{k+1} = P_C(x_k - \frac{1}{\nu}A^*(I - J_\nu^T)Ax_k), \tag{21}$$

which is nothing else than an extension of the celebrated CQ-algorithm of Byrne, see [10]. [10]-Theorem 3.1 assures:

Proposition 3.3. *Let T be a given maximal monotone operator and C a nonempty closed convex set, the sequence $f(x_k)$ generated by (3.14) converges to a solution of the traffic equilibrium problem (3.12) if $\nu > L$, with L being the spectral radius of the operator A^*A .*

While applying the Douglas-Rachford’s algorithm yields to

$$\begin{cases} x_k = y_k - \frac{1}{\nu}A^*(I - J_\nu^T)Ay_k; \\ y_{k+1} = (1 - \alpha_k)y_k + \alpha_k P_C(y_k - \frac{2}{\nu}A^*(I - J_\nu^T)Ay_k). \end{cases} \tag{22}$$

Which looks like a relaxed version of algorithm (3.13). Since, we are in a finite dimensional setting, we obtain

Proposition 3.4. *Let T be a given maximal monotone operator and C a nonempty closed convex set, , then for any $\alpha_k \in]0, 2[$ satisfying $\sum_k \alpha_k(2 - \alpha_k) = +\infty$, the sequence (x_k) generated by (3.15) converges to a solution of the equilibrium problem (3.12).*

Received: April 2014. Accepted: August 2014.

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