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Pseudo-Almost Periodic and Pseudo-Almost Automorphic Solutions to Some Evolution Equations Involving Theoretical Measure Theory

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ABSTRACT

Motivated by the recent works by the first and the second named authors, in this paper we introduce the notion of doubly-weighted pseudo-almost periodicity (respectively, doubly-weighted pseudo-almost automorphy) using theoretical measure theory. Basic properties of these new spaces are studied. To illustrate our work, we study, under Acquistapace–Terreni conditions and exponential dichotomy, the existence of (μ, ν)-pseudo almost periodic (respectively, (μ, ν)-pseudo almost automorphic) solutions to some nonautonomous partial evolution equations in Banach spaces. A few illustrative examples will be discussed at the end of the paper.

RESUMEN

Motivado por los trabajos recientes del primer y segundo autor, en este artículo introducimos la noción de seudo-casi periodicidad con doble peso (seudo-casi automorfía con doble peso respectivamente) usando Teoría de la Medida. Se estudian las propiedades básicas de estos espacios nuevos. Para ilustrar nuestro trabajo, bajo las condiciones de Acquistapace-Terreni y dicotomía exponencial estudiamos la existencia de soluciones (respectivamente, (μ, ν) seudo-casi periódicas (μ, ν) seudo-casi automórficas) para algunas ecuaciones parciales de evolución autónomas en espacios de Banach. Algunos ejemplos ilustrativos se discutirán al final del artículo.

Keywords and Phrases: Evolution family; exponential dichotomy; Acquistapace–Terreni conditions; pseudo-almost periodic; pseudo-almost automorphic; evolution equation; nonautonomous equation; doubly-weighted pseudo-almost periodic; doubly-weighted pseudo-almost automorphy; (μ, ν)-pseudo-almost periodicity; (μ, ν)-pseudo-almost automorphy; neutral systems; positive measure.

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1 Introduction

Motivated by the recent works by Ezzinbi *et al.* [12, 13] and Diagana [30], in this paper we make extensive use of theoretical measure theory to introduce and study the concept of doubly-weighted pseudo almost periodicity (respectively, doubly-weighted pseudo almost automorphy). Obviously, these new notions generalize all the different notions of weighted pseudo-almost periodicity (respectively, weighted pseudo-almost automorphy) recently introduced in the literature. In contrast with [12, 13], here the idea consists of using two positive measures instead of one. Doing so will provide us a larger and richer class of weighted ergodic spaces. Basic properties of these new functions will be studied including their translation invariance and compositions etc.

To illustrate our study, we study the existence of (μ, ν)-pseudo-almost periodic (respectively, (μ, ν)-pseudo-almost automorphic) solutions to the following nonautonomous differential equations,

$$\frac{d}{dt}u(t) = A(t)u(t) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

and

$$\frac{d}{dt}\Big(u(t) - G(t, u(t))\Big) = A(t)\Big(u(t) - G(t, u(t))\Big) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

where $A(t) : D(A(t)) \subset X \mapsto X$ for $t \in \mathbb{R}$ is a family of closed linear operators on a Banach space X, satisfying the well-known Acquistapace–Terreni conditions, and $F, G : \mathbb{R} \times X \mapsto X$ are jointly continuous functions satisfying some additional conditions. One should indicate that the autonomous case, i.e., $A(t) = A$ for all $t \in \mathbb{R}$, and the periodic case, that is, $A(t + \theta) = A(t)$ for some $\theta > 0$, have been extensively studied, see [8, 10, 40, 41, 53, 56] for the almost periodic case and [18, 22, 39, 42, 50, 51] for the almost automorphic case. Recently, Diagana [24, 25, 26, 32] studied the existence and uniqueness of weighted pseudo-almost periodic and weighted pseudo-almost automorphic solutions to some classes of nonautonomous partial evolution equations of type Eq. (1.1). Similarly, in Diagana [33], the existence of pseudo-almost periodic solutions to Eq. (1.2) has been studied in the particular case when $G = 0$. In this paper it goes back to studying the existence of doubly-weighted pseudo-almost periodic (respectively, doubly-weighted pseudo-almost automorphic) solutions in the general case as outlined above using theoretical measure theory.

The existence and uniqueness of almost periodic, almost automorphic, pseudo-almost periodic and pseudo-almost automorphic solutions is one of the most attractive topics in the qualitative theory of ordinary or functional differential equations due to applications in the physical sciences, mathematical biology, and control theory. The concept of almost automorphy, which was introduced by Bochner [15], is an important generalization of the classical almost periodicity in the sense of Bohr. For basic results on almost periodic and almost automorphic functions we refer the reader to [7, 59, 61], where the authors give an important overview about their applications to differential equations. In recent years, the existence of almost periodic, pseudo-almost periodic, almost automorphic, and pseudo-almost automorphic solutions to different kinds of differential equations have been extensively investigated by many people, see, e.g., [3, 4, 5, 16, 17, 19, 20, 30, 23, 24, 32, 33, 34, 35, 36, 37, 39, 43, 44, 45, 46, 48, 57, 58, 60] and the references therein.

The concept of weighted pseudo-almost periodicity, which was introduced by Diagana [25, 26, 27, 29] is a natural generalization of the classical pseudo-almost periodicity due to Zhang [59, 60, 61]. A few years later, Blot et al. [11], introduced the concept of weighted pseudo-almost automorphy as a generalization of weighted pseudo-almost periodicity. More recently, Ezzinbi *et al.* [12, 13] presented a new approach to study weighted pseudo-almost periodic and weighted pseudo-almost automorphic functions using theoretical measure theory, which turns out to be more general than Diagana's approach.

Let us explain the meaning of this notion as introduced by Ezzinbi et al.*et al.* [12, 13]. Let μ be a positive measure on $\mathbb{R}$. We say that a continuous function $f : \mathbb{R} \mapsto X$ is μ-pseudo-almost periodic (respectively, μ-pseudo almost automorphic) if $f = g + \varphi$, where g is almost periodic (respectively, almost automorphic) and φ is ergodic with respect to the measure μ in the sense that

$$\lim_{r\to\infty} \frac{1}{\mu(Q_r)} \int_{Q_r} \|\varphi(s)\| d\mu(s) = 0,$$

where $Q_r := [-r, r]$ and $\mu(Q_r) := \int_{Q_r} d\mu(t)$.

One can observe that a ρ-weighted pseudo almost automorphic function is μ-pseudo almost automorphic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its RadonNikodym derivative is ρ,

$$\frac{d\mu(t)}{dt} = \rho(t).$$

Here we generalize the above-mentioned notion of μ-pseudo-almost periodicity. Fix two positive measures μ, ν in $\mathbb{R}$. We say that a function $f : \mathbb{R} \mapsto X$ is (μ, ν)-pseudo-almost periodic (respectively, (μ, ν)-pseudo-almost automorphic) if

$$f = g + \varphi,$$

where g is almost periodic (respectively, almost automorphic) and φ is (μ, ν)-ergodic in the sense that

$$\lim_{r\to\infty} \frac{1}{\nu(Q_r)} \int_{Q_r} \|\varphi(s)\| d\mu(s) = 0,$$

Clearly, the (μ, μ)-pseudo-almost periodicity coincides with the μ-pseudo-almost periodicity. More generally, the (μ, ν)-pseudo-almost periodicity coincides with the μ-pseudo-almost periodicity when the measures μ and ν are equivalent.

In this paper, we introduce and study properties of (μ, ν)-pseudo-almost periodic functions and make use of these new functions to study the existence and uniqueness of (μ, ν)-pseudo-almost periodic (respectively, (μ, ν)- pseudo-almost automorphic) solutions of the nonautonomous partial evolution equations Eq. (1.1) and Eq. (1.2) in a Banach space.

The organization of this paper is as follows. In Section 2, we recall some definitions and lemmas of (μ,ν)-pseudo almost periodic functions, (μ,ν)-pseudo-almost automorphic functions, and the basic notations of evolution family and exponential dichotomy. In Section 3, we study the existence and uniqueness of (μ,ν)-pseudo almost periodic (respectively, (μ,ν)-pseudo almost automorphic) solutions to both Eq. (1.1) and Eq. (1.2). In Section 4, we give some examples to illustrate our abstract results.

2 Preliminaries

2.1 (μ,ν)-Pseudo-Almost Periodic and (μ,ν)-Pseudo-Almost Automorphic Functions

Let $(X,\|\cdot\|)$, $(Y,\|\cdot\|)$ be two Banach spaces and let $BC(\mathbb{R},X)$ (respectively, $BC(\mathbb{R}\times Y,X)$) be the space of bounded continuous functions $f:\mathbb{R}\longrightarrow X$ (respectively, jointly bounded continuous functions $f:\mathbb{R}\times Y\longrightarrow X$). Obviously, the space $BC(\mathbb{R},X)$ equipped with the super norm

$$\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|$$

is a Banach space. Let $B(X,Y)$ denote the Banach spaces of all bounded linear operator from X into Y equipped with natural topology with $B(X,X)=B(X)$.

Definition 2.1. *[21] A continuous function $f:\mathbb{R}\mapsto X$ is said to be almost periodic if for every $\varepsilon>0$ there exists a positive number $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a number τ such that*

$$\|f(t+\tau)-f(t)\| < \varepsilon \quad \textit{for} \quad t\in\mathbb{R}.$$

Let $AP(\mathbb{R},X)$ denote the collection of almost periodic functions from $\mathbb{R}$ to X. It can be easily shown that $(AP(\mathbb{R},X),\|\cdot\|_\infty)$ is a Banach space.

Definition 2.2. *[38] A jointly continuous function $f:\mathbb{R}\times Y\mapsto X$ is said to be almost periodic in t uniformly for $y\in Y$, if for every $\varepsilon>0$, and any compact subset K of Y, there exists a positive number $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a number τ such that*

$$\|f(t+\tau,y)-f(t,y)\| < \varepsilon \quad \textit{for} \quad (t,y)\in\mathbb{R}\times K.$$

We denote the set of such functions as $APU(\mathbb{R}\times Y,X)$.

Let $\mu,\nu\in\mathcal{M}$. If $f:\mathbb{R}\mapsto X$ is a bounded continuous function, we define its *doubly-weighted mean*, if the limit exists, by

$$\mathcal{M}(f,\mu,\nu) := \lim_{r\to\infty}\frac{1}{\nu(Q_r)}\int_{Q_r} f(t)d\mu(t).$$

It is well-known that if $f \in AP(\mathbb{R}, X)$, then its mean defined by

$$\mathcal{M}(f) := \lim_{r\to\infty} \frac{1}{2r} \int_{Q_r} f(t)dt$$

exists [15]. Consequently, for every $\lambda \in \mathbb{R}$, the following limit

$$a(f, \lambda) := \lim_{r\to\infty} \frac{1}{2r} \int_{Q_r} f(t)e^{-i\lambda t}dt$$

exists and is called the Bohr transform of f.

It is well-known that $a(f, \lambda)$ is nonzero at most at countably many points [15]. The set defined by

$$\sigma_b(f) := \Big\{\lambda \in \mathbb{R} : a(f, \lambda) \neq 0\Big\}$$

is called the Bohr spectrum of f [47].

Theorem 2.3. *[47] Let $f \in AP(\mathbb{R}, X)$. Then for every $\varepsilon > 0$ there exists a trigonometric polynomial*

$$P_\varepsilon(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}$$

where $a_k \in X$ and $\lambda_k \in \sigma_b(f)$ such that $\|f(t) - P_\varepsilon(t)\| < \varepsilon$ for all $t \in \mathbb{R}$.

Theorem 2.4. *Let $\mu, \nu \in \mathcal{M}$ and suppose that $\displaystyle\lim_{r\to\infty} \frac{\mu(Q_r)}{\nu(Q_r)} = \theta_{\mu\nu}$. If $f : \mathbb{R} \mapsto X$ is an almost periodic function such that*

$$\lim_{r\to\infty} \left| \frac{1}{\nu(Q_r)} \int_{Q_r} e^{i\lambda t} d\mu(t) \right| = 0 \tag{2.1}$$

for all $0 \neq \lambda \in \sigma_b(f)$, then the doubly-weighted mean of f,

$$\mathcal{M}(f, \mu, \nu) = \lim_{T\to\infty} \frac{1}{\nu(Q_T)} \int_{Q_T} f(t)d\mu(t)$$

exists. Furthermore, $\mathcal{M}(f, \mu, \nu) = \theta_{\mu\nu}\mathcal{M}(f)$.

Proof. The proof of this theorem was given in [30] in the case of measures of the form $\rho(t)dt$. For the sake of completeness we reproduce it here for positive measures. If f is a trigonometric polynomial, say, $f(t) = \sum_{k=0}^{n} a_k e^{i\lambda_k t}$ where $a_k \in X - \{0\}$ and $\lambda_k \in \mathbb{R}$ for $k = 1, 2, ..., n$, then $\sigma_b(f) = \{\lambda_k : \ k = 1, 2, ..., n\}$. Moreover,

$$\begin{aligned}
\frac{1}{\nu(Q_r)} \int_{Q_r} f(t)d\mu(t) &= a_0 \frac{\mu(Q_r)}{\nu(Q_r)} + \frac{1}{\nu(Q_r)} \int_{Q_r} \Big[\sum_{k=1}^{n} a_k e^{i\lambda_k t}\Big] d\mu(t) \\
&= a_0 \frac{\mu(Q_r)}{\nu(Q_r)} + \sum_{k=1}^{n} a_k \Big[\frac{1}{\nu(Q_r)} \int_{Q_r} e^{i\lambda_k t} d\mu(t)\Big]
\end{aligned}$$

and hence

$$\left\|\frac{1}{\nu(Q_r)}\int_{Q_r} f(t)d\mu(t) - a_0\frac{\mu(Q_r)}{\nu(Q_r)}\right\| \leq \sum_{k=1}^{n}\|a_k\|\left|\frac{1}{\nu(Q_r)}\int_{Q_r} e^{i\lambda_k t}d\mu(t)\right|$$

which by Eq. (2.1) yields

$$\left\|\frac{1}{\nu(Q_r)}\int_{Q_r} f(t)d\mu(t) - a_0\theta_{\mu\nu}\right\| \to 0 \text{ as } r \to \infty$$

and therefore $\mathcal{M}(f,\mu,\nu) = a_0\theta_{\mu\nu} = \theta_{\mu\nu}\mathcal{M}(f)$.

If in the finite sequence of λ_k there exist $\lambda_{n_k} = 0$ for $k = 1, 2, ...l$ with $a_m \in X - \{0\}$ for all $m \neq n_k$ $(k = 1, 2, ..., l)$, it can be easily shown that

$$\mathcal{M}(f,\mu,\nu) = \theta_{\mu\nu}\sum_{k=1}^{l} a_{n_k} = \theta_{\mu\nu}\mathcal{M}(f).$$

Now if $f : \mathbb{R} \mapsto X$ is an arbitrary almost periodic function, then for every $\varepsilon > 0$ there exists a trigonometric polynomial (Theorem 2.3) P_ε defined by

$$P_\varepsilon(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}$$

where $a_k \in X$ and $\lambda_k \in \sigma_b(f)$ such that

$$\|f(t) - P_\varepsilon(t)\| < \varepsilon \tag{2.2}$$

for all $t \in \mathbb{R}$.

Proceeding as in Bohr [15] it follows that there exists r_0 such that for all $r_1, r_2 > r_0$,

$$\left\|\frac{1}{\nu(Q_{r_1})}\int_{Q_{r_1}} P_\varepsilon(t)d\mu(t) - \frac{1}{\nu(Q_{r_2})}\int_{Q_{r_2}} P_\varepsilon(t)d\mu(t)\right\| = \theta_{\mu\nu}\left\|\mathcal{M}(P_\varepsilon) - \mathcal{M}(P_\varepsilon)\right\| = 0 < \varepsilon.$$

In view of the above it follows that for all $r_1, r_2 > r_0$,

$$\begin{aligned}\left\|\frac{1}{\nu(Q_{r_1})}\int_{Q_{r_1}} f(t)d\mu(t) - \frac{1}{\nu(Q_{r_2})}\int_{Q_{r_2}} P_\varepsilon(t)d\mu(t)\right\| &\leq \frac{1}{\nu(Q_{r_1})}\int_{Q_{r_1}} \|f(t) - P_\varepsilon(t)\|d\mu(t)\\ &+ \left\|\frac{1}{\nu(Q_{r_1})}\int_{Q_{r_1}} P_\varepsilon(t)d\mu(t) - \frac{1}{\nu(Q_{r_2})}\int_{Q_{r_2}} P_\varepsilon(t)d\mu(t)\right\| < \varepsilon.\end{aligned}$$

Now for all $r > r_0$,

$$\left\|\frac{1}{\nu(Q_r)}\int_{Q_r} f(t)d\mu(t) - \frac{1}{\nu(Q_r)}\int_{Q_r} P_\varepsilon(t)d\mu(t)\right\| < \varepsilon$$

and hence $\mathcal{M}(f,\mu,\nu) = \mathcal{M}(P_\varepsilon,\mu,\nu) = \theta_{\mu\nu}\mathcal{M}(P_\varepsilon) = \theta_{\mu\nu}\mathcal{M}(f)$. The proof is complete.

□

Definition 2.5. *[51] A continuous function* $f : \mathbb{R} \to X$ *is called almost automorphic if for every sequence* $(\sigma_n)_{n\in N}$ *there exists a subsequence* $(s_n)_{n\in N} \subset (\sigma_n)_{n\in\mathbb{N}}$ *such that*

$$\lim_{n,m\to\infty} f(t + s_n - s_m) = f(t) \quad \text{for each } t \in \mathbb{R}.$$

Equivalently,

$$g(t) := \lim_{n\to\infty} f(t + s_n) \quad \text{and} \quad f(t) = \lim_{n\to\infty} g(t - s_n)$$

are well defined for each $t \in \mathbb{R}$.

Let $AA(\mathbb{R}, X)$ denote the collection of all almost automorphic functions from $\mathbb{R}$ to X. It can be easily shown that $(AA(\mathbb{R}, X), \|.\|_\infty)$ is a Banach space.

Definition 2.6. *[13] A function* $f : \mathbb{R} \times X \to Y$ *is said to be almost automorphic in* t *uniformly with respect to* x *in* X *if the following two conditions hold:*
(i) for all $x \in X$, $f(., x) \in AA(\mathbb{R}, Y)$,
(ii) f *is uniformly continuous on each compact set* K *in* X *with respect to the second variable* x, *namely, for each compact set* K *in* X, *for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x_1,\ x_2\ \in K$, *one has*

$$\|x_1 - x_2\| \le \delta \Rightarrow \sup_{t\in\mathbb{R}} \|f(t, x_1) - f(t, x_2)\| \le \varepsilon.$$

Denote by $AAU(\mathbb{R} \times X, Y)$ *the set of all such functions.*

Remark 2.7. *[13] Note that in the above limit the function* g *is just measurable. If the convergence in both limits is uniform in* $t \in \mathbb{R}$, *then* f *is almost periodic. The concept of almost automorphy is then larger than almost periodicity. If* f *is almost automorphic, then its range is relatively compact, thus bounded in norm.*

Example 2.8. *[49] Let* $k :\ \mathbb{R} \to \mathbb{R}$ *be such that*

$$k(t) = \sin\Big(\frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)}\Big),\ \ t \in \mathbb{R}.$$

Then k *is almost automorphic, but it is not uniformly continuous on* $\mathbb{R}$. *Then, it is not almost periodic.*

In what follows, we introduce a new concept of ergodicity, which will generalize those given in [12] and [29, 31].

Let $\mathcal{B}$ denote the Lebesque σ-field of $\mathbb{R}$ and let $\mathcal{M}$ be the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < \infty$, for all $a, b \in \mathbb{R}$ $(a \le b)$.

Definition 2.9. *[12] Let* $\mu, \nu \in \mathcal{M}$. *The measures* μ *and* ν *are said to be equivalent there exist constants* $c_0, c_1 > 0$ *and a bounded interval* $\Omega \subset \mathbb{R}$ *(eventually* $\emptyset$*) such that*

$$c_0\nu(A) \le \mu(A) \le c_1\nu(A)$$

for all $A \in \mathcal{B}$ *satisfying* $A \cap \Omega = \emptyset$.

We introduce the following new space.

Definition 2.10. *Let* $\mu, \nu \in \mathcal{M}$. *A bounded continuous function* $f : \mathbb{R} \to X$ *is said to be* (μ, ν)-*ergodic if*

$$\lim_{r\to\infty} \frac{1}{\nu(Q_r)} \int_{Q_r} \|f(s)\| d\mu(s) = 0.$$

We then denote the collection of all such functions by $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$.

We are now ready to introduce the notion of (μ, ν)-pseudo-almost periodicity (respectively, (μ, ν)-pseudo-almost automorphy) for two positive measures $\mu, \nu \in \mathcal{M}$.

Definition 2.11. *Let* $\mu, \nu \in \mathcal{M}$. *A continuous function* $f : \mathbb{R} \to X$ *is said to be* (μ, ν)-*pseudo almost periodic if it can be written in the form*

$$f = g + h,$$

where $g \in AP(\mathbb{R}, X)$ *and* $h \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. *The collection of such functions is denoted by* $PAP(\mathbb{R}, X, \mu, \nu)$.

Definition 2.12. *Let* $\mu, \nu \in \mathcal{M}$. *A continuous function* $f : \mathbb{R} \to X$ *is said to be* (μ, ν)-*pseudo almost automorphic if it can be written in the form*

$$f = g + h,$$

where $g \in AA(\mathbb{R}, X)$ *and* $h \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. *The collection of such functions will be denoted by* $PAA(\mathbb{R}, X, \mu, \nu)$.

We formulate the following hypotheses.

(M.1) Let $\mu, \nu \in \mathcal{M}$ such that

$$\limsup_{r\to\infty} \frac{\mu(Q_r)}{\nu(Q_r)} < \infty. \tag{2.3}$$

(M.2) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a + \tau : \quad a \in A\}) \leq \beta\mu(A) \quad \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

Theorem 2.13. *Let* $\mu, \nu \in \mathcal{M}$ *satisfy* ***(M.2)***. *Then the spaces* $PAP(\mathbb{R}, X, \mu, \nu)$ *and* $PAA(\mathbb{R}, X, \mu, \nu)$ *are translation invariants.*

Proof. We show that $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ is translation invariant. Let $f \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$, we will show that $t \mapsto f(t + s)$ belongs to $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ for each $s \in \mathbb{R}$.

Indeed, letting $\mu_s = \mu(\{t + s : t \in A\})$ for $A \in \mathcal{B}$ it follows from **(M.2)** that μ and μ_s are equivalent (see [12]).

Now

$$\begin{aligned}
\frac{1}{\nu(Q_r)}\int_{Q_r} \|f(t+s)\|d\mu(t) &= \frac{\nu(Q_{r+|s|})}{\nu(Q_r)}\cdot\frac{1}{\nu(Q_{r+|s|})}\int_{Q_r}\|f(t+s)\|d\mu(t)\\
&= \frac{\nu(Q_{r+|s|})}{\nu(Q_r)}\cdot\frac{1}{\nu(Q_{r+|s|})}\int_{Q_{r+|s|}}\|f(t)\|d\mu_{-s}(t)\\
&\leq \frac{\nu(Q_{r+|s|})}{\nu(Q_r)}\cdot\frac{cst.}{\nu(Q_{r+|s|})}\int_{Q_{r+|s|}}\|f(t)\|d\mu(t).
\end{aligned}$$

Since ν satisfies **(M.2)** and $f \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$, we have

$$\lim_{r\to\infty}\frac{1}{\nu(Q_r)}\int_{Q_r}\|f(t+s)\|d\mu(t) = 0.$$

Therefore, $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ is translation invariant. Since $AP(\mathbb{R}, X)$ and $AA(\mathbb{R}, X)$ are translation invariants, then $PAP(\mathbb{R}, X, \mu, \nu)$ and $PAA(\mathbb{R}, X, \mu, \nu)$ are translation invariants.

$\square$

Theorem 2.14. *Let $\mu, \nu \in \mathcal{M}$ satisfy **(M.1)**, then $(\mathcal{E}(\mathbb{R}, X, \mu, \nu), \|.\|_\infty)$ is a Banach space.*

Proof. It is clear that $(\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ is a vector subspace of $BC(\mathbb{R}, X)$. To complete the proof, it is enough to prove that $(\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ is closed in $BC(\mathbb{R}, X)$. If $(f_n)_n$ be a sequence in $(\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ such that

$$\lim_{n\to\infty} f_n = f$$

uniformly in $\mathbb{R}$.

From $\nu(\mathbb{R}) = \infty$, it follows $\nu(Q_r) > 0$ for r sufficiently large. Using the inequality

$$\int_{Q_r}\|f(t)\|d\mu(t) \leq \int_{Q_r}\|f(t) - f_n(t)\|d\mu(t) + \int_{Q_r}\|f_n(t)\|d\mu(t)$$

we deduce that

$$\frac{1}{\nu(Q_r)}\int_{Q_r}\|f(t)\|d\mu(t) \leq \frac{\mu(Q_r)}{\nu(Q_r)}\|f - f_n\|_\infty + \frac{1}{\nu(Q_r)}\int_{Q_r}\|f_n(t)\|d\mu(t),$$

then from **(M.1)** we have

$$\limsup_{r\to\infty}\frac{1}{\nu(Q_r)}\int_{Q_r}\|f(t)\|d\mu(t) \leq cst.\|f - f_n\|_\infty$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty}\|f - f_n\|_\infty = 0$, we deduce that

$$\lim_{r\to\infty}\frac{1}{\nu(Q_r)}\int_{Q_r}\|f(t)\|d\mu(t) = 0.$$

$\square$

Lemma 2.15. *[13] Let $g \in AA(\mathbb{R}, X)$ and $\varepsilon > 0$ be given. Then there exist $s_1, ..., s_m \in \mathbb{R}$ such that*

$$\mathbb{R} = \bigcup_{m}^{i=1} (s_i + C_\varepsilon), \text{ where } C_\varepsilon := \{t \in \mathbb{R} : \|g(t) - g(0)\| < \varepsilon\}.$$

Theorem 2.16. *Let $\mu, \nu \in \mathcal{M}$ and $f \in PAA(\mathbb{R}, X, \mu, \nu)$ be such that*

$$f = g + \phi,$$

where $g \in AA(\mathbb{R}, X)$ and $\phi \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. If $PAA(\mathbb{R}, X, \mu, \nu)$ is translation invariant, then

$$\{g(t);\ t \in \mathbb{R}\} \subset \overline{\{f(t);\ t \in \mathbb{R}\}}, (\text{the closure of the range of } f). \tag{2.4}$$

Proof. The proof is similar to the one given in [13]. Indeed, if we assume that (2.4) does not hold, then there exists $t_0 \in \mathbb{R}$ such that $g(t_0)$ is not in $\{f(t);\ t \in \mathbb{R}\}$. Since the spaces $AA(\mathbb{R}, X)$ and $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ are translation invariants, we can assume that $t_0 = 0$, then there exists $\varepsilon > 0$ such that $\|f(t) - g(0)\| > 2\varepsilon$ for all $t \in \mathbb{R}$. Then we have

$$\|\phi(t)\| = \|f(t) - g(t)\| \geq \|f(t) - g(0)\| - \|g(t) - g(0)\| \geq \varepsilon$$

for all $t \in C_\varepsilon$. Therefore,

$$\|\phi(t - s_i)\| \geq \varepsilon, \quad \text{for all } i \in \{1, ..., m\}, \text{ and } \ t \in s_i + C_\varepsilon.$$

Let ϕ be the function defined by

$$\phi(t) := \sum_{i=1}^{i=m} \|\phi(t - s_i)\|.$$

From Lemma 2.15, we deduce that

$$\|\phi(t)\| \geq \varepsilon \ \text{ for all } t \in \mathbb{R}. \tag{2.5}$$

Since $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ is translation invariant, then $[t \to \phi(t - s_i)] \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$ for all $i \in \{1, ..., m\}$, then $\phi \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$ which is a contradiction. Consequently (2.4) holds. □

Theorem 2.17. *Let $\mu, \nu \in \mathcal{M}$ satisfy **(M.2)**, then the decomposition of a (μ, ν)-pseudo almost automorphic function in the form $f = g + h$, where $g \in AA(\mathbb{R}, X)$ and $h \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$, is unique.*

Proof. Suppose that $f = g_1 + \phi_1 = g_2 + \phi_2$, where $g_1, g_2 \in AA(\mathbb{R}, X)$ and $\phi_1, \phi_2 \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. Then $0 = (g_1 - g_2) + (\phi_1 - \phi_2) \in PAA(\mathbb{R}, X, \mu, \nu)$ where $g_1 - g_2 \in AA(\mathbb{R}, X)$ and $\phi_1 - \phi_2 \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. From Theorem 2.16, we obtain $(g_1 - g_2)(\mathbb{R}) \subset \{0\}$, therefore we have $g_1 = g_2$ and $\phi_1 = \phi_2$. □

From Theorem 2.17, we deduce

Theorem 2.18. *Let $\mu, \nu \in \mathcal{M}$ satisfy **(M.2)**, then the decomposition of a (μ, ν)-pseudo almost periodic function in the form $f = g + h$, where $g \in AP(\mathbb{R}, X)$ and $h \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$, is unique.*

Theorem 2.19. *Let $\mu, \nu \in \mathcal{M}$ satisfy **(M.1)** and **(M.2)**. Then, the spaces $(PAP(\mathbb{R}, X, \mu, \nu), \|.\|_\infty)$ and $(PAA(\mathbb{R}, X, \mu, \nu), \|.\|_\infty)$ are Banach spaces.*

Proof. The proof is similar to the one given in [13], in fact we assume that $(f_n)_n$ is a Cauchy sequence in $PAA(\mathbb{R}, X, \mu, \nu)$. We have $f_n = g_n + \phi_n$ where $g_n \in AA(\mathbb{R}, X)$ and $\phi_n \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. From Theorem 2.16 we see that

$$\|g_n - g_m\|_\infty \le \|f_n - f_m\|_\infty,$$

therefore $(g_n)_n$ is a Cauchy sequence in the Banach space $(AA(\mathbb{R}, X), \|.\|_\infty)$. So, $\phi_n = f_n - g_n$ is also a Cauchy sequence in the Banach space $\mathcal{E}((\mathbb{R}, X, \mu, \nu), \|.\|_\infty)$. Then we have $\lim_{n\to\infty} g_n = g \in AA(\mathbb{R}, X)$ and $\lim_{n\to\infty} \phi_n = \phi \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. Finally we have

$$\lim_{n\to\infty} f_n = g + \phi \in PAA(\mathbb{R}, X, \mu, \nu).$$

The proof for $PAP(\mathbb{R}, X, \mu, \nu)$ is similar to that of $PAA(\mathbb{R}, X, \mu, \nu)$. □

Definition 2.20. *Let $\mu, \nu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times Y \to X$ is said to be (μ, ν)-ergodic in t uniformly with respect to $y \in Y$ if the following conditions are true*
(i) For all $y \in Y$, $f(., y) \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$.
(ii) f is uniformly continuous on each compact set K in Y with respect to the second variable y.

The collection of such function is denoted by $\mathcal{E}U(\mathbb{R} \times Y, X, \mu, \nu)$.

Definition 2.21. *Let $\mu, \nu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times Y \to X$ is said to be (μ, ν)-pseudo almost periodic if is written in the form*

$$f = g + h,$$

where $g \in APU(\mathbb{R} \times Y, X)$ and $h \in \mathcal{E}U(\mathbb{R} \times Y, X, \mu, \nu)$. The collection of such functions is denoted by $PAPU(\mathbb{R} \times Y, X, \mu, \nu)$.

Theorem 2.22. *Let μ, $\nu \in \mathcal{M}$ and I be a bounded interval (eventually $I = \emptyset$). Assume that **(M1)** and $f \in BC(\mathbb{R}, X)$. Then the following assertions are equivalent:*
(i) $f \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$.
(ii) $\displaystyle\lim_{r\to\infty} \frac{1}{\nu(Q_r \setminus I)} \int_{Q_r \setminus I} \|f(t)\| d\mu(t) = 0.$
(iii) *For any* $\varepsilon > 0$, $\displaystyle\lim_{r\to\infty} \frac{\mu(\{t \in Q_r \setminus I : \|f(t)\| > \varepsilon\})}{\nu(\{Q_r \setminus I)} = 0.$

Proof. The proof is similar to the one given in [13], in fact we have
(i) $\Leftrightarrow$ (ii) : Denote by $A = \nu(I)$, $B = \int_I \|f(t)\| d\mu(t)$ and $C = \mu(I)$. Since the interval I is bounded

and the function f is bounded and continuous, then A, B and C are finite. For $r > 0$ such that $I \subset Q_r$ and $\nu(Q_r \setminus I) > 0$, we have

$$\begin{aligned}\frac{1}{\nu(Q_r \setminus I)} \int_{Q_r \setminus I} \|f(t)\| d\mu(t) &= \frac{1}{\nu(Q_r) - A}\Big(\int_{Q_r} \|f(t)\| d\mu(t) - B\Big) \\ &= \frac{\nu(Q_r)}{\nu(Q_r) - A}\Big(\frac{1}{\nu(Q_r)}\int_{Q_r} \|f(t)\| d\mu(t) - \frac{B}{\nu(Q_r)}\Big).\end{aligned}$$

Since $\nu(\mathbb{R}) = \infty$, we deduce that (ii) is equivalent to (i).
$(iii) \Rightarrow (ii)$ Denote by A_r^ε and B_r^ε the following sets

$$A_r^\varepsilon = \{t \in Q_r \setminus I : \|f(t)\| > \varepsilon\} \text{ and } B_r^\varepsilon = \{t \in Q_r \setminus I : \|f(t)\| \le \varepsilon\}.$$

Assume that (iii) holds, that is

$$\lim_{r\to\infty} \frac{\mu(A_r^\varepsilon)}{\nu(Q_r \setminus I)} = 0.$$

From the following equality

$$\int_{Q_r \setminus I} \|f(t)\| d\mu(t) = \int_{A_r^\varepsilon} \|f(t)\| d\mu(t) + \int_{B_r^\varepsilon} \|f(t)\| d\mu(t),$$

and **(M.1)**, we deduce for r large enough that

,

$$\begin{aligned}\frac{1}{\nu(Q_r \setminus I)} \int_{Q_r \setminus I} \|f(t)\| d\mu(t) &\le \|f\|_\infty \frac{\mu(A_r^\varepsilon)}{\nu(Q_r \setminus I)} + \frac{\mu(B_r^\varepsilon)}{\nu(Q_r \setminus I)}\varepsilon \\ &\le \|f\|_\infty \frac{\mu(A_r^\varepsilon)}{\nu(Q_r \setminus I)} + \frac{\mu(Q_r \setminus I)}{\nu(Q_r \setminus I)}\varepsilon \\ &= \|f\|_\infty \frac{\mu(A_r^\varepsilon)}{\nu(Q_r \setminus I)} + \frac{\mu(Q_r) - C}{\nu(Q_r) - A}\varepsilon \\ &= \|f\|_\infty \frac{\mu(A_r^\varepsilon)}{\nu(Q_r \setminus I)} + \frac{\mu(Q_r)}{\nu(Q_r)} \frac{1 - \frac{C}{\mu(Q_r)}}{1 - \frac{A}{\nu(Q_r)}}\varepsilon \\ &\le \|f\|_\infty \frac{\mu(A_r^\varepsilon)}{\nu(Q_r \setminus I)} + cst. \frac{1 - \frac{C}{\mu(Q_r)}}{1 - \frac{A}{\nu(Q_r)}}\varepsilon.\end{aligned}$$

Since $\mu(\mathbb{R}) = \nu(\mathbb{R}) = \infty$, then for all $\varepsilon > 0$ we have

$$\limsup_{r\to\infty} \frac{1}{\nu(Q_r \setminus I)} \int_{Q_r \setminus I} \|f(t)\| d\mu(t) \le cst.\varepsilon,$$

consequently (ii) holds.
$(ii) \Rightarrow (iii)$ Assume that (ii) holds. From the following inequality:

$$\begin{aligned}\frac{1}{\nu(Q_r \setminus I)} \int_{Q_r \setminus I} \|f(t)\| d\mu(t) &\geq \frac{1}{\nu(Q_r \setminus I)} \int_{A_r^\varepsilon} \|f(t)\| d\mu(t) \\ &\geq \varepsilon \frac{\mu(A_r^\varepsilon)}{\nu(Q_r \setminus I)},\end{aligned}$$

for r sufficiently large, we obtain (iii). □

Theorem 2.23. *[12] Let $F \in APU(\mathbb{R} \times X, Y)$ and $h \in AP(\mathbb{R}, X)$. Then $[t \longmapsto F(t, h(t))] \in AP(\mathbb{R}, Y)$.*

Proposition 2.24. *[12] Let $f : \mathbb{R} \times X \to Y$ be a continuous function. Then $f \in APU(\mathbb{R} \times X, Y)$ if and only if the two following conditions hold:*
(i) for all $x \in X$, $f(., x) \in AP(\mathbb{R}, Y)$,
(ii) f is uniformly continuous on each compact set K in X with respect to the second variable x, namely, for each compact set K in X, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1,\ x_2\ \in K$, one has

$$\|x_1 - x_2\| \leq \delta \Rightarrow \sup_{t \in \mathbb{R}} \|f(t, x_1) - f(t, x_2)\| \leq \varepsilon.$$

The proof of our result of composition of (μ, ν)-pseudo-almost periodic functions is based on the following lemma due to Schwartz [54].

Lemma 2.25. *If $\Psi \in C(X, Y)$, then for each compact set K in X and all $\varepsilon > 0$, there exists $\delta > 0$, such that for any $x_1, x_2 \in X$, one has*

$$x_1 \in K \text{ and } \|x_1 - x_2\| \leq \delta \ \Rightarrow\ \|\Psi(x_1) - \Psi(x_2)\| \leq \varepsilon.$$

Theorem 2.26. *Let $\mu,\ \nu \in \mathcal{M}$, $F \in PAPU(\mathbb{R} \times X, Y, \mu, \nu)$ and $h \in PAP(\mathbb{R}, Y, \mu, \nu)$. Assume that **(M.1)** and the following hypothesis holds:*

(C) *For all bounded subset B of Y, F is bounded on $\mathbb{R} \times B$.*

Then $t \longmapsto F(t, h(t)) \in PAP(\mathbb{R}, Y, \mu, \nu)$.

Proof. The function $[t \mapsto F(t, h(t))]$ is continuous and by Hypothesis (C), it is bounded. Since $h \in PAP(\mathbb{R}, X, \mu, \nu)$, we can write

$$h = h_1 + h_2,$$

where $h_1 \in AP(\mathbb{R}, X)$ and h_2 is (μ, ν)-ergodic. Since $F \in PAPU(\mathbb{R} \times X, Y, \mu, \nu)$, we have

$$F = F_1 + F_2$$

where $F_1 \in APU(\mathbb{R} \times X, Y)$ and $F_2 \in \mathcal{E}(\mathbb{R} \times X, Y, \mu, \nu)$. The function F can be written in the form

$$\begin{aligned} F(t, h(t)) &= F_1(t, h_1(t)) + [F(t, h(t)) - F(t, h_1(t))] + [F(t, h_1(t)) - F_1(t, h_1(t))] \\ &= F_1(t, h_1(t)) + [F(t, h(t)) - F(t, h_1(t))] + F_2(t, h_1(t)). \end{aligned}$$

From Theorem 2.23, we have $[t \longmapsto F_1(t, h_1(t))] \in AP(\mathbb{R}, Y)$. Denote by K the closure of the range of h_1: $K = \overline{\{h_1(t) : t \in \mathbb{R}\}}$. Since h_1 is almost periodic, K is a compact subset of X. Denote by Φ the function defined by

$$\Phi : X \to PAP(\mathbb{R}, Y, \mu, \nu)$$

$$x \mapsto F(., x)$$

Since $F \in PAPU(\mathbb{R} \times X, Y, \mu, \nu)$), by using Proposition 2.24, we deduce that the restriction of Φ on all compact K of X, is uniformly continuous, which is equivalent to saying that the function Φ is continuous on X. From Lemma 2.25 applied to Φ, we deduce that for given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $t \in \mathbb{R}$, ξ_1 and $\xi_2 \in X$, one has

$$\xi_1 \in K \text{ and } \|\xi_1 - \xi_2\| \leq \delta \Rightarrow \|F(t, \xi_1) - F(t, \xi_2)\| \leq \varepsilon.$$

Since $h(t) = h_1(t) + h_2(t)$ and $h_1(t) \in K$, we have

$$t \in \mathbb{R} \text{ and } \|h_2(t)\| \leq \delta \Rightarrow \|F(t, h(t)) - F(t, h_2(t))\| \leq \varepsilon,$$

therefore, we have

$$\frac{\mu\{t \in Q_r : \|F(t, h(t)) - F(t, h_1(t))\| > \varepsilon\}}{\nu(Q_r)} \leq \frac{\mu\{t \in Q_r : \|h_2(t)\| > \delta\}}{\nu(Q_r)}.$$

Since h_2 is (μ, ν)-ergodic, Theorem 2.22 yields that for the above-mentioned δ we have

$$\lim_{r \to \infty} \frac{\mu\{t \in Q_r : \|h_2(t)\| > \delta\}}{\nu(Q_r)} = 0,$$

then we obtain

$$\lim_{r \to \infty} \frac{\mu\{t \in Q_r : \|F(t, h(t)) - F(t, h_1(t))\| > \varepsilon\}}{\nu(Q_r)} = 0.$$

By Theorem 2.22 we have $t \mapsto F(t, h(t)) - F(t, h_1(t))$ is (μ, ν)-ergodic. Now to complete the proof, it is enough to prove that $t \mapsto F_2(t, h_1(t))$ is (μ, ν)-ergodic. Since F_2 is uniformly continuous on the compact set $K = \overline{\{h_1(t) : t \in \mathbb{R}\}}$ with respect to the second variable x, we deduce that for given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $t \in \mathbb{R}$, ξ_1 and $\xi_2 \in K$, one has

$$\|\xi_1 - \xi_2\| \leq \delta \Rightarrow \|F_2(t, \xi_1) - F_2(t, \xi_2)\| \leq \varepsilon.$$

then, there exist $n(\varepsilon)$ and $\{x_i\}_{i=1}^{n(\varepsilon)} \subset K$, such that

$$K \subset \bigcup_{i=1}^{n(\varepsilon)} B(x_i, \delta),$$

and then

$$\|F_2(t,h_1(t))\| \leq \varepsilon + \sum_{i=1}^{n(\varepsilon)} \|F_2(t,x_i)\|.$$

Since

$$\forall i \in \{1,...,n(\varepsilon)\}, \lim_{r\to+\infty} \frac{1}{\nu(Q_r)} \int_{Q_r} \|F_2(t,x_i)\| d\mu(t) = 0,$$

we deduce that

$$\forall \varepsilon > 0, \quad \limsup_{r\to\infty} \frac{1}{\nu(Q_r)} \int_{Q_r} \|F_2(t,h_1(t))\| d\mu(t) \leq \varepsilon,$$

that implies

$$\lim_{r\to\infty} \frac{1}{\nu(Q_r)} \int_{Q_r} \|F_2(t,h_1(t))\| d\mu(t) = 0,$$

then $t \mapsto F_2(t,h_1(t))$ is (μ,ν)-ergodic and the theorem is proved.

□

Definition 2.27. *Let $\mu,\nu \in \mathcal{M}$. A continuous function $f : \mathbb{R}\times Y \to X$ is said to be (μ,ν)-pseudo almost automorphic if is written in the form*

$$f = g + h,$$

where $g \in AAU(\mathbb{R}\times Y, X)$ and $h \in \mathcal{E}U(\mathbb{R}\times Y, X, \mu, \nu)$. The collection of such functions is denoted by $PAAU(\mathbb{R}\times Y, X, \mu, \nu)$.

Theorem 2.28. *Let $\mu,\nu \in \mathcal{M}$, $F \in PAAU(\mathbb{R}\times X, Y, \mu, \nu)$ and $h \in PAA(\mathbb{R}, Y, \mu, \nu)$. Assume that, for all bounded subset B of Y, F is bounded on $\mathbb{R}\times B$. Then $t \longmapsto F(t,h(t)) \in PAA(\mathbb{R}, X, \mu, \nu)$.*

Proof. The proof for $PAA(\mathbb{R}, Y, \mu, \nu)$ is similar to that of $PAP(\mathbb{R}, Y, \mu, \nu)$.

□

2.2 Evolution Families and Exponential Dichotomy

(H0) A family of closed linear operators $A(t)$ for $t \in \mathbb{R}$ on X with domain $D(A(t))$ (possibly not densely defined) is said to satisfy the so-called Acquistapace–Terreni conditions, if there exist constants $\omega \in \mathbb{R}$, $\theta \in (\frac{\pi}{2},\pi)$, $K, L \geq 0$ and $\mu_0, \nu_0 \in (0,1]$ with $\mu_0 + \nu_0 > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t)-\omega) \ni \lambda, \quad \|R(\lambda, A(t)-\omega)\| \leq \frac{K}{1+|\lambda|} \tag{2.6}$$

and

$$\|(A(t)-\omega)R(\lambda, A(t)-\omega)\,[R(\omega, A(t)) - R(\omega, A(s))]\| \leq L\,\frac{|t-s|^{\mu_0}}{|\lambda|^{\nu_0}} \tag{2.7}$$

for $t,s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C}\setminus\{0\} : |\arg\lambda| \leq \theta\}$.

Note that in the particular case when $A(t)$ has a constant domain $D = D(A(t))$, it is well-known on [6] that condition (2.7) can be replaced with the following one: There exist constants L and $0 < \gamma \leq 1$ such that

$$\|(A(t) - A(s))R(\omega, A(r))\| \leq L|t - s|^{\gamma}, \text{ for all}, \ s, t, r \in \mathbb{R}. \tag{2.8}$$

For a given family of linear operators $A(t)$, the existence of an evolution family associated with it is not always guaranteed. However, if $A(t)$ satisfies Acquistapace–Terreni, then there exists a unique evolution family(see[1, 2, 52])

$$\mathcal{U} = \{U(t, s) : t, s \in \mathbb{R}, \ t \geq s\}$$

on X associated with $A(t)$ such that $U(t, s)X \subseteq D(A(t))$ for all $t, s \in \mathbb{R}$ with $t \geq s$, and,

(1) $U(t, r)U(r, s) = U(t, s)$ and $U(s, s) = I$ for all $t \geq r \geq s$ and $t, r, s \in \mathbb{R}$;

(2) the map $(t, s) \to U(t, s)x$ is continuous for all $x \in X$, $t \geq s$ and $t, s \in \mathbb{R}$;

(3) $U(\cdot, s) \in C^1((s, \infty), B(X))$, $\dfrac{\partial U}{\partial t}(t, s) = A(t)U(t, s)$ and

$$\left\|A(t)^k U(t, s)\right\| \leq K\,(t - s)^{-k}$$

for $0 < t - s \leq 1$, $k = 0, 1$.

Definition 2.29. *An evolution family* $(U(t, s))_{t \geq s}$ *on a Banach space* X *is* X *is called hyperbolic (or has exponential dichotomy) if there exist projections* $P(t), t \in \mathbb{R}$*, uniformly bounded and strongly continuous in* t*, and constant* $N \geq 1$*,* $\delta > 0$ *such that*

(1) $U(t, s)P(s) = P(t)U(t, s)$ *for* $t \geq s$*;*

(2) the restriction $U_Q(t, s) : Q(s)X \to Q(t)X$ *of* $U(t, s)$ *is invertible for* $t, s \in \mathbb{R}$ *and we set* $U_Q(t, s) = U(s, t)^{-1}$*;*

(3)

$$\|U(t, s)P(s)\| \leq Ne^{-\delta(t-s)} \tag{2.9}$$

and

$$\|U_Q(s, t)Q(t)\| \leq Ne^{-\delta(t-s)} \tag{2.10}$$

for $t \geq s$ *and* $t, s \in \mathbb{R}$*, were* $Q(t) := I - P(t)$.

3 Existence Results

To study the existence and uniqueness of (μ, ν)-pseudo almost periodic (respectively, (μ, ν)-pseudo almost automorphic) solutions to equation (1.1), in addition to above, we also assume that the next assumption holds:

(H1) The evolution family U generated by $A(.)$ has an exponential dichotomy with constants $N \geq 1$, $\delta > 0$ and dichotomy projections $P(t)$.

We recall from [48, 55], the following sufficient conditions to fulfill the assumption **(H1)**.

(H1.1) Let $(A(t), D(A(t)))_{t\in\mathbb{R}}$ be generators of analytic semigroups on X of the same type. Suppose that $D(A(t)) = D(A(0))$, $A(t)$ is invertible, $\sup_{t,s\in\mathbb{R}} \|A(t)A(s)^{-1}\|$ is finite, and

$$\|A(t)A(s)^{-1} - I\| \leq L_0|t-s|^{\mu_1}$$

for $t, s \in \mathbb{R}$ and constants $L_0 \geq 0$ and $0 < \mu_1 \leq 1$.

(H1.2) The semigroups $(e^{\tau A(t)})_{\tau\geq 0}$, $t \in \mathbb{R}$, are hyperbolic with projection P_t and constants $N, \delta > 0$. Moreover, let

$$\|A(t)e^{\tau A(t)}P_t\| \leq \Psi(\tau) \text{ and } \|A(t)e^{\tau A_Q(t)}Q_t\| \leq \Psi(-\tau)$$

for $\tau > 0$ and a function Ψ such that $\mathbb{R} \ni s \to \varphi(s) := |s|^{\mu}\Psi(s)$ is integrable with $L_0\|\varphi\|_{L^1(\mathbb{R})} < 1$.

Now, we first introduce the definition of the mild solution to Eq. (1.1).

Definition 3.1. *A continuous function* $u : \mathbb{R} \mapsto X$ *is called a bounded mild solution of equation* (1.1) *if:*

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)F(\tau, u(\tau))d\tau, \quad t \geq s, \quad t, s \in \mathbb{R}. \tag{3.1}$$

Theorem 3.2. *Assume that* **(H0)** *and* **(H1)** *hold. If there exists* $0 < K_F < \frac{\delta}{2N}$ *such that*

$$\|F(t,u) - F(t,v)\| \leq K_F\|u - v\|,$$

for all $u, v \in X$ *and* $t \in \mathbb{R}$*, then the equation* (1.1) *has a unique bounded mild solution* $u : \mathbb{R} \mapsto X$ *given by*

$$u(t) = \int_{\mathbb{R}} \Gamma(t,s)F(s,u(s))ds, \quad t \in \mathbb{R},$$

where the operator family $\Gamma(t,s)$*, called Green's function corresponding to* U *and* $P(.)$*, is given by*

$$\begin{cases} \Gamma(t,s) = U(t,s)P(s), & t \geq s,\ t,s \in \mathbb{R} \\ \Gamma(t,s) = -U_Q(t,s)Q(s), & t < s,\ t,s \in \mathbb{R}. \end{cases}$$

Proof. If one supposes

$$u(t) = \int_{\mathbb{R}} \Gamma(t,\tau)F(\tau, u(\tau))d\tau, \quad t \in \mathbb{R},$$

Thus we have

$$u(t) = \int_{-\infty}^{t} U(t,\tau)P(\tau)F(\tau,u(\tau))d\tau - \int_{t}^{+\infty} U_Q(t,\tau)Q(\tau)F(\tau,u(\tau))d\tau, \quad \text{for all } t \in \mathbb{R}.$$

For $t = s$, one obtains

$$u(s) = \int_{-\infty}^{s} U(s,\tau)P(\tau)F(\tau,u(\tau))d\tau - \int_{s}^{+\infty} U_Q(s,\tau)Q(\tau)F(\tau,u(\tau))d\tau,$$

and

$$U(t,s)u(s) = \int_{-\infty}^{s} U(t,\tau)P(\tau)F(\tau,u(\tau))d\tau - \int_{s}^{+\infty} U_Q(t,\tau)Q(\tau)F(\tau,u(\tau))d\tau.$$

Now, we have

$$u(t) - U(t,s)u(s) = \int_{s}^{t} U(t,\tau)f(\tau,u(\tau))d\tau.$$

Then, $u(t) = \int_{\mathbb{R}} \Gamma(t,\tau)F(\tau,u(\tau))d\tau$ checks equation (3.1).
Consider the nonlinear operator $\mathbb{K}$ defined on X by

$$\mathbb{K}u(t) = \int_{\mathbb{R}} \Gamma(t,\tau)F(\tau,u(\tau))d\tau, \quad t \in \mathbb{R}.$$

To complete the proof, one has to show that $\mathbb{K}$ is a contraction map on X.
From assumption **(H1)**, there exist two constant $N \geq 1$ and $\delta > 0$ such that

$$\|\Gamma(t,s)\| \leq Ne^{-\delta|t-s|} \quad \text{for all } t,s \in \mathbb{R}$$

If $u, v \in X$, then one has

$$\|\mathbb{K}v - \mathbb{K}u\|_{\infty} < \frac{2NK_F}{\delta}\|v - u\|_{\infty},$$

and $\mathbb{K}$ is a contraction map on X. Therefore, $\mathbb{K}$ has unique fixed point in X, that is, there exists unique $u \in X$ such that $\mathbb{K}u = u$. Therefore, Eq.(1.1) has unique mild solution.

□

Denote by Γ_1 and Γ_2 the nonlinear integral operators defined by,

$$(\Gamma_1 u)(t) := \int_{-\infty}^{t} U(t,s)P(s)F(s,u(s))ds,$$

and

$$(\Gamma_2 u)(t) := \int_{t}^{+\infty} U_Q(t,s)Q(s)F(s,u(s))ds.$$

In the rest of the paper, we fix $\mu, \nu \in \mathcal{M}$ satisfy **(M1)** and **(M2)**.

3.1 Existence of (μ, ν)-Pseudo-Almost Periodic Solutions

In addition to the previous assumptions, we require the following additional ones:

(H2) $R(\omega, A(\cdot)) \in AP(\mathbb{R}, \mathcal{L}(X))$.

(H3) We suppose $F : \mathbb{R} \times X \mapsto X$ belongs to $PAP(\mathbb{R} \times X, X, \mu, \nu)$ and there exists $K_F > 0$ such that

$$\|F(t, u) - F(t, v)\| \leq K_F \|u - v\|,$$

for all $u, v \in X$ and $t \in \mathbb{R}$.

The following Lemma plays an important role to prove main results of this work.

Lemma 3.3. *[48] Assume that* **(H0)**, **(H1)** *and* **(H2)** *hold. Then* $r \to \Gamma(t + r, s + r)$ *belongs to* $AP(\mathbb{R}, \mathcal{L}(X))$ *for* $t, s \in \mathbb{R}$, *where we may take the same pseudo periods for* t, s *with* $|t - s| \geq h > 0$. *If* $f \in AP(\mathbb{R}, \mathcal{L}(X))$, *then the unique bounded mild solution* $u(t) = \int_{\mathbb{R}} \Gamma(t, s) f(s) ds$ *of the following equation*

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R},$$

is almost periodic.

Lemma 3.4. *Under assumptions* **(H0)**–**(H3)**, *then the integral operators* Γ_1 *and* Γ_2 *defined above map* $PAP(\mathbb{R}, X, \mu, \nu)$ *into itself.*

Proof. Let $u \in PAP(\mathbb{R}, X, \mu, \nu)$. Setting $h(t) = F(t, u(t))$, using assumption **(H3)** and Theorem 2.26 it follows that $h \in PAP(\mathbb{R}, X, \mu)$. Now write $h = \psi_1 + \psi_2$ where $\psi_1 \in AP(\mathbb{R}, X)$ and $\psi_2 \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. That is, $\Gamma_1 h = \Xi(\psi_1) + \Xi(\psi_2)$ where

$$\Xi\psi_1(t) := \int_{-\infty}^{t} U(t, s)P(s)\psi_1(s)ds, \quad \text{and} \quad \Xi\psi_2(t) := \int_{-\infty}^{t} U(t, s)P(s)\psi_2(s)ds.$$

From Lemma 3.3, we have $\Xi(\psi_1) \in AP(\mathbb{R}, X)$. To complete the proof, we will prove that $\Xi(\psi_2) \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. Now, let $r > 0$. Again from Eq. (2.9), we have

$$\begin{aligned}
\frac{1}{\nu(Q_r)} \int_{Q_r} \|(\Xi\psi_2)(t)\| d\mu(t) &\leq \frac{1}{\nu(Q_r)} \int_{Q_r} \int_{0}^{+\infty} \|U(t, t - s)P(t - s)\psi_2(t - s)\| ds d\mu(t) \\
&\leq \frac{N}{\nu(Q_r)} \int_{Q_r} \int_{0}^{+\infty} e^{-\delta s} \|\psi_2(t - s)\| ds d\mu(t) \\
&\leq N \int_{0}^{+\infty} e^{-\delta s} \left(\frac{1}{\nu(Q_r)} \int_{Q_r} \|\psi_2(t - s)\| d\mu(t) \right) ds.
\end{aligned}$$

Since μ and ν satisfy **(M2)**, then from Theorem 2.13, we have $t \mapsto \psi_2(t-s) \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$ for every $s \in \mathbb{R}$. By the Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{r\to\infty} \frac{1}{\nu(Q_r)} \int_{Q_r} \|(\Xi\psi_2)(t)\| d\mu(t) = 0.$$

The proof for $\Gamma_2 u(\cdot)$ is similar to that of $\Gamma_1 u(\cdot)$ except that one makes use of equation (2.10) instead of equation (2.9). ☐

Theorem 3.5. *Under assumptions* **(H0)—(H3)**, *then Eq.* (1.1) *has a unique* (μ, ν)*-pseudo almost periodic mild solution whenever* K_F *is small enough.*

Proof. Consider the nonlinear operator $\mathbb{M}$ defined on $PAP(\mathbb{R}, X, \mu, \nu)$ by

$$\mathbb{M}u(t) = \int_{-\infty}^{t} U(t,s)P(s)F(s,u(s))ds - \int_{t}^{+\infty} U_Q(t,s)Q(s)F(s,u(s))ds, \quad \text{for all } \ t \in \mathbb{R}.$$

In view of Lemma 3.4, it follows that $\mathbb{M}$ maps $PAP(\mathbb{R}, X, \mu, \nu)$ into itself. To complete the proof one has to show that $\mathbb{M}$ is a contraction map on $PAP(\mathbb{R}, X, \mu, \nu)$. Let $u, v \in PAP(\mathbb{R}, X, \mu, \nu)$. Firstly, we have

$$\begin{aligned}
\|\Gamma_1(v)(t) - \Gamma_1(u)(t)\| &\leq \int_{-\infty}^{t} \|U(t,s)P(s)\left[F(s,v(s)) - F(s,u(s))\right]\|ds \\
&\leq NK_F \int_{-\infty}^{t} e^{-\delta(t-s)} \|v(s) - u(s)\| ds \\
&\leq NK_F \delta^{-1} \|v - u\|_\infty.
\end{aligned}$$

Next, we have

$$\begin{aligned}
\|\Gamma_2(v)(t) - \Gamma_2(u)(t)\| &\leq \int_{t}^{+\infty} \|U_Q(t,s)Q(s)\left[F(s,v(s)) - F(s,u(s))\right]\|ds \\
&\leq N \int_{t}^{+\infty} e^{\delta(t-s)} \|F(s,v(s)) - F(s,u(s))\| ds \\
&\leq NK_F \int_{t}^{+\infty} e^{\delta(t-s)} \|v(s) - u(s)\| ds \\
&\leq NK_F \|v - u\|_\infty \int_{t}^{+\infty} e^{\delta(t-s)} ds \\
&= NK_F \delta^{-1} \|v - u\|_\infty.
\end{aligned}$$

Finally, combining previous approximations it follows that

$$\|\mathbb{M}v - \mathbb{M}u\|_\infty < 2NK_F\delta^{-1}\|v - u\|_\infty.$$

Thus if K_F is small enough, that is, $K_F < \delta(2N)^{-1}$, then $\mathbb{M}$ is a contraction map on $PAP(\mathbb{R}, X, \mu, \nu)$. Therefore, $\mathbb{M}$ has unique fixed point in $PAP(\mathbb{R}, X, \mu, \nu)$, that is, there exists unique function u satisfying $\mathbb{M}u = u$, which is the unique (μ, ν)-pseudo almost periodic mild solution to Eq. (1.1). ☐

3.2 Existence of (μ,ν)-Pseudo-Almost Automorphic Solutions

In this section we consider the following conditions:

(H'2) $R(\omega, A(\cdot)) \in AA(\mathbb{R}, \mathcal{L}(X))$.

(H'3) We suppose $F : \mathbb{R} \times X \mapsto X$ belongs to $PAA(\mathbb{R} \times X, X, \mu, \nu)$ and there exists $K_F > 0$ such that

$$\|F(t,u) - F(t,\nu)\| \leq K_F \|u - \nu\|_\infty,$$

for all $u, \nu \in X$ and $t \in \mathbb{R}$.

Lemma 3.6. *[9] Assume that* **(H0)**, **(H1)** *and* **(H'2)** *hold. Let a sequence $(s'_l)_{l\in\mathbb{N}} \in \mathbb{R}$ there is a subsequence $(s_l)_{l\in\mathbb{N}}$ such that for every $h > 0$*

$$\|\Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t,s)\| \to 0, \quad k, l \to \infty$$

for $|t - s| \geq h$.

Lemma 3.7. *Under assumptions* **(H0)**, **(H1)** , **(H'2)** *and* **(H'3)**, *then the integral operators Γ_1 and Γ_2 defined above map $PAA(\mathbb{R}, X, \mu, \nu)$ into itself.*

Proof. Let $u \in PAA(\mathbb{R}, X, \mu, \nu)$. Setting $g(t) = F(t, u(t))$, using assumption **(H'3)** and Theorem 2.28 it follows that $g \in PAA(\mathbb{R}, X, \mu, \nu)$. Now write $g = u_1 + u_2$ where $u_1 \in AA(\mathbb{R}, X)$ and $u_2 \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. That is, $\Gamma_1 g = Su_1 + Su_2$, where

$$Su_1(t) := \int_{-\infty}^{t} U(t,s)P(s)u_1(s)ds, \quad \text{and} \quad Su_2(t) := \int_{-\infty}^{t} U(t,s)P(s)u_2(s)ds.$$

From Eq. (2.9), we obtain

$$\|Su_1(t)\| \leq N\delta^{-1}\|u_1\|_\infty \ \text{ and } \ \|Su_2(t)\| \leq N\delta^{-1}\|u_2\|_\infty \ \text{ for all } \ t \in \mathbb{R}.$$

Then $Su_1, Su_2 \in BC(\mathbb{R}, X)$. Now, we prove that $Su_1 \in AA(\mathbb{R}, X)$. Since $u_1 \in AA(\mathbb{R}, X)$, then for every sequence $(\tau'_n)_{n\in\mathbb{N}} \in \mathbb{R}$ there exists a subsequence $(\tau_n)_{n\in\mathbb{N}}$ such that

$$\nu_1(t) := \lim_{n\to\infty} u_1(t + \tau_n), \tag{3.2}$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} \nu_1(t - \tau_n) = u_1(t), \tag{3.3}$$

for each $t \in \mathbb{R}$. Set

$$M(t) = \int_{-\infty}^{t} U(t,s)P(s)u_1(s)ds \ \text{ and } \ N(t) = \int_{-\infty}^{t} U(t,s)P(s)\nu_1(s)ds, \quad t \in \mathbb{R}.$$

Now, we have

$$\begin{aligned} M(t+\tau_n)-N(t) &= \int_{-\infty}^{t+\tau_n} U(t+\tau_n,s)P(s)u_1(s)ds - \int_{-\infty}^{t} U(t,s)P(s)v_1(s)ds \\ &= \int_{-\infty}^{t} U(t+\tau_n,s+\tau_n)P(s+\tau_n)u_1(s+\tau_n)ds - \int_{-\infty}^{t} U(t,s)P(s)v_1(s)ds \\ &= \int_{-\infty}^{t} U(t+\tau_n,s+\tau_n)P(s+\tau_n)\Big[u_1(s+\tau_n)-v_1(s)\Big]ds \\ &+ \int_{-\infty}^{t} \Big[U(t+\tau_n,s+\tau_n)P(s+\tau_n)-U(t,s)P(s)\Big]v_1(s)ds. \end{aligned}$$

Using Eq. (2.9), Eq. (3.2) and the Lebesgue's Dominated Convergence Theorem, it follows that

$$\|\int_{-\infty}^{t} U(t+\tau_n,s+\tau_n)P(s+\tau_n)\Big[u_1(s+\tau_n)-v_1(s)\Big]ds\| \to 0 \ \text{ as } \ n\to\infty,\ t\in\mathbb{R}. \tag{3.4}$$

Similarly, using Lemma 3.6 it follows that

$$\|\int_{-\infty}^{t} \Big[U(t+\tau_n,s+\tau_n)P(s+\tau_n)-U(t,s)P(s)\Big]v_1(s)ds\| \to 0 \text{ as } n\to\infty,\ t\in\mathbb{R}. \tag{3.5}$$

Therefore, we have

$$N(t) := \lim_{n\to\infty} M(t+\tau_n),\ t\in\mathbb{R}. \tag{3.6}$$

Using similar ideas as the previous ones, then

$$M(t) := \lim_{n\to\infty} N(t-\tau_n),\ t\in\mathbb{R}. \tag{3.7}$$

Therefore, $Su_1 \in AA(\mathbb{R},X)$. Finally, by using the same stages that the proof of Lemma 3.4 one obtains $Su_2 \in \mathcal{E}(\mathbb{R},X,\mu,\nu)$. The proof for $\Gamma_2 u(\cdot)$ is similar to that of $\Gamma_1 u(\cdot)$ except that one makes use of equation (2.10) instead of equation 2.9. □

Theorem 3.8. *Under assumptions* **(H0)**, **(H1)** , **(H'2)** *and* **(H'3)**, *then Eq.* (1.1) *has a unique* (μ,ν)*-pseudo almost automorphic mild solution whenever* K_F *is small enough.*

Proof. The proof for Theorem 3.8 is similar to that of Theorem 3.5 except that one makes use of Lemma 3.7 instead of Lemma 3.4. □

3.3 Neutral Systems

In this subsection, we establish the existence and uniqueness of (μ,ν)-pseudo-almost periodic (respectively, (μ,ν)-pseudo-almost automorphic) solutions for the nonautonomous neutral partial evolution equation (1.2). For that, we need the following assumptions:

(H4) We suppose $G : \mathbb{R}\times X \mapsto X$ belongs to $PAP(\mathbb{R}\times X, X, \mu, \nu)$ and there exists $K_G > 0$ such that

$$\|G(t,u) - G(t,v)\| \le K_G\|u - v\|,$$

for all $u, v \in X$ and $t \in \mathbb{R}$.

(H'4) $G : \mathbb{R}\times X \mapsto X$ belongs to $PAA(\mathbb{R}\times X, X, \mu, \nu)$ and there exists $K_G > 0$ such that

$$\|G(t,u) - G(t,v)\| \le K_G\|u - v\|,$$

for all $u, v \in X$ and $t \in \mathbb{R}$.

Definition 3.9. *A function $v : \mathbb{R} \mapsto X$ is said to be a bounded mild solution to equation* (1.2) *and we have:*

$$v(t) = G(t, v(t)) + \int_{-\infty}^{t} U(t,s)P(s)F(s, v(s))ds - \int_{t}^{+\infty} U_Q(t,s)Q(s)F(s, v(s))ds$$

for all $t \in \mathbb{R}$.

Theorem 3.10. *If assumptions* **(H0)**, **(H1)**, **(H2)**, **(H3)** *and* **(H4)** *hold and* $\Big(K_G + 2NK_F\delta^{-1}\Big) < 1$, *then Eq.* (1.2) *has a unique* (μ,ν)*-pseudo almost periodic mild solution.*

Proof. We consider the nonlinear operator $\mathbb{W}$ defined on $PAP(\mathbb{R}, X, \mu)$ by

$$\mathbb{W}v(t) = G(t, v(t)) + \int_{-\infty}^{t} U(t,s)P(s)F(s, v(s))ds - \int_{t}^{+\infty} U_Q(t,s)Q(s)F(s, v(s))ds$$

for all $t \in \mathbb{R}$.

From **(H4)**, Theorem (2.26), and Lemma 3.4 it follows that $\mathbb{W}$ maps $PAP(\mathbb{R}, X, \mu, \nu)$ into itself. To complete the proof we need to show that $\mathbb{W}$ is a contraction map on $PAP(\mathbb{R}, X, \mu, \nu)$. For that, letting $u, v \in PAP(\mathbb{R}, X, \mu, \nu)$, we obtain

$$\|\mathbb{W}v - \mathbb{W}u\|_\infty \le \Big(K_G + 2NK_F\delta^{-1}\Big)\|v - u\|_\infty,$$

which yields $\mathbb{W}$ is a contraction map on $PAP(\mathbb{R}, X, \mu, \nu)$. Therefore, $\mathbb{W}$ has unique fixed point in $PAP(\mathbb{R}, X, \mu, \nu)$. Therefore, Eq.(1.2), has unique (μ,ν)-pseudo-almost periodic mild solution.

□

Theorem 3.11. *If assumptions* **(H0)**, **(H1)**, **(H'2)**, **(H'3)** *and* **(H'4)** *hold and* $\Big(K_G + 2NK_F\delta^{-1}\Big) < 1$, *then Eq.* (1.2) *has a unique* (μ,ν)*-pseudo almost automorphic mild solution.*

Proof. Similarly the proof of Theorem 3.10, we can show, by using the assumption **(H'4)**, Theorem 2.28 and Lemma 3.7, that the Eq. (1.2) has a unique (μ,ν)-pseudo almost automorphic mild solution.

□

4 Examples

Example 4.1. Let $X = L^2([0,1])$ be equipped with its natural topology. In order to illustrate Theorem 3.5, we consider the following one-dimensional heat equation with Dirichlet conditions,

$$\begin{aligned} \frac{\partial}{\partial t}\Big[v(t,x)\Big] &= \Big[\frac{\partial^2}{\partial x^2} + \xi\Big(\sin(at)+\sin(bt)\Big)\Big]v(t,x) + f(t,v(t,x)), \quad \text{on } \mathbb{R}\times(0,1) \\ v(t,0) &= v(t,1) = 0, \quad t\in\mathbb{R}, \end{aligned} \tag{4.1}$$

where the coefficient $\xi \in]0,\frac{1}{2}[$, the constants $a,b\in\mathbb{R}$ with $\frac{a}{b}\notin\mathbb{Q}$, and the forcing term $f:\mathbb{R}\times X\mapsto X$ is continuous function.

In order to write Eq.(4.1) in the abstract form Eq.(1.1), we consider the linear operator $A: D(A)\subset X\longrightarrow X$, given by

$$D(A) = H^2(0,1)\cap H_0^1(0,1) \text{ and } Au = u'' \text{ for } u\in D(A).$$

It is well known that A is the infinitesimal generator of an exponentially stable C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ such that $\|T(t)\|\leq e^{-\pi^2 t}$ for $t\geq 0$.

Define a family of linear operator $A(t)$ as follows:

$$\begin{cases} D(A(t)) = D(A) = H^2[0,1]\cap H_0^1[0,1] \\ A(t)v = \Big[A+\xi\Big(\sin(at)+\sin(bt)\Big)\Big]v, \; v\in D(A). \end{cases}$$

Obviously, $D(A(t)) = D(A)$. Furthermore,

$$\|A(t)-A(s)\| = \|\xi\Big(\sin(at)-\sin(as)+\sin(bt)-\sin(bs)\Big)\| \leq \xi\Big(|a|+|b|\Big)|t-s|,$$

for all $s,t\in\mathbb{R}$ and hence **(H0)** holds. Consequently, $A(t)$ generates an evolution family, which we denote by $U(t,s)_{t\geq s}$ and which satisfies

$$U(t,s)v = T(t-s)\exp\Big[\int_s^t \xi\Big(\sin(a\tau)+\sin(b\tau)\Big)d\tau\Big]v.$$

Since $\|U(t,s)\|\leq e^{-(\pi^2-1)(t-s)}$ for $t\geq s$ and $t,s\in\mathbb{R}$, it follows that **(H1)** holds with $N=1$, $\delta=\pi^2-1>0$. And since $t\mapsto\sin(at)+\sin(bt)$ is almost periodic, then $R(\omega,A(\cdot))\in AP(\mathbb{R},\mathcal{L}(X))$ and so **(H2)** holds.

Let $F : \mathbb{R} \times X \mapsto X$ be the mapping defined by $F(t, \varphi)(x) = f(t, \varphi(x))$ for $x \in [0, 1]$, and let $y : \mathbb{R} \to X$ be the function defined by $y(t) = v(t, .)$, for $t \in \mathbb{R}$. Then Eq. (4.1) takes the abstract form,

$$\frac{d}{dt}\big[y(t)\big] = A(t)y(t) + F(t, y(t)), \quad t \in \mathbb{R}. \tag{4.2}$$

Let $\mu = \nu$ and suppose that its Radon-Nikodym derivative is given by

$$\rho(t) = \begin{cases} e^t & \text{if } \ t \le 0, \\ 1 & \text{if } \ t > 0. \end{cases}$$

Then from [12], $\mu \in \mathcal{M}$ satisfies **(M1)** and **(M2)**.

If we assume that f is μ-pseudo almost periodic in $t \in \mathbb{R}$ uniformly in $u \in X$ and is globally Lipschitz with respect to the second argument in the following sense: there exists $K_f > 0$ such that

$$\Big\|f(t, u) - f(t, v)\Big\| \le K_f \Big\|u - v\Big\| \text{ for all } t \in \mathbb{R} \text{ and } u, v \in X,$$

then F satisfies **(H3)**.

We deduce all assumptions **(H0)**,**(H1)**,**(H2)**,**(H3)**, **(M.1)** and **(M.2)** of Theorem 3.5 are satisfied and thus equation (4.1) has a unique (μ, μ)-pseudo almost periodic solution whenever K_f is small enough ($K_f < \frac{\pi^2-1}{2}$).

To illustrate the result in Theorem 3.11, we consider the following equation

$$\begin{aligned} \frac{\partial}{\partial t}\Big[v(t,x) - g_1(t, v(t,x))\Big] = \Big(\frac{\partial^2}{\partial x^2} + \xi(\sin(at) + \sin(bt))\Big)\Big[v(t,x) - g_1(t, v(t,x))\Big] \\ +f_1(t, v(t,x)), \quad \text{on } \mathbb{R} \times (0,1) \\ v(t,0) = v(t,1) = 0, \quad t \in \mathbb{R}, \end{aligned} \tag{4.3}$$

where the coefficient $\xi \in (0, \frac{1}{2})$, $a, b \in \mathbb{R}$ and $\frac{a}{b} \notin \mathbb{Q}$, $f_1, g_1 : \mathbb{R} \times X \mapsto X$ are given by

$$f_1(t, x) = x \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t} + \max_{k \in \mathbb{Z}}\{e^{-(t \pm k^2)^2}\} \cos x, \ t \in \mathbb{R}\ , \ x \in X,$$

$$g_1(t, x) = \frac{x}{4} \sin \frac{1}{2 + \sin t + \sin \sqrt{2}t} + \frac{1}{4} \max_{k \in \mathbb{Z}}\{e^{-(t \pm k^2)^2}\} \cos x, \ t \in \mathbb{R}\ , \ x \in X.$$

Clearly, $f_1, g_1 \ \in PAA(\mathbb{R} \times X, X, \mu, \mu)$ and satisfies the Lipschitz condition in Theorem 3.11 with $N = 1$, $\delta = \pi^2 - 1$, $K_{f_1} = 2$ and $K_{g_1} = \frac{1}{2}$. By Theorem 3.11, the evolution equation (4.3) has a unique (μ, μ)-pseudo almost automorphic solution, with μ being the measure defined in the example above.

Example 4.2. Let $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ be an open bounded subset with regular boundary $\Gamma = \partial\Omega$ and let $X = L^2(\Omega)$ equipped with its natural topology $\|\cdot\|_2$.

We study the existence of (μ,ν)-pseudo-almost automorphic solutions to the N-dimensional heat equation

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} = a(t,x)\Delta\varphi + g(t,\varphi), & \text{in } \mathbb{R}\times\Omega \\ \\ \varphi = 0, & \text{on } \mathbb{R}\times\Gamma \end{cases} \tag{4.4}$$

where $a:\mathbb{R}\times\Omega\mapsto\mathbb{R}$ is almost automorphic, and $g:\mathbb{R}\times L^2(\Omega)\mapsto L^2(\Omega)$ is (μ,ν)-pseudo almost automorphic function.

Define the linear operator appearing in Eq. (4.4) as follows:

$$A(t)u = a(t,x)\Delta u \quad \text{for all} \quad u\in D(A(t)) = H_0^1(\Omega)\cap H^2(\Omega),$$

where $a:\mathbb{R}\times\Omega\mapsto\mathbb{R}$, in addition of being almost automorphic satisfies the following assumptions:

(h.1) $\inf\limits_{t\in\mathbb{R},x\in\Omega} a(t,x) = m_0 > 0$, and

(h.2) there exists $L>0$ and $0<\mu\le 1$ such that

$$|a(t,x)-a(s,x)| \le L|s-t|^{\mu}$$

for all $t,s\in\mathbb{R}$ uniformly in $x\in\Omega$.

(h.3) $\sup\limits_{t\in\mathbb{R},x\in\Omega} a(t,x) < \infty$.

(h.4) g is μ-pseudo-almost periodic in $t\in\mathbb{R}$ uniformly in $u\in L^2(\Omega)$ and satisfying globally Lipschitz with respect to the second argument in the following sense: there exists $K_g>0$ such that

$$\|g(t,u)-g(t,v)\|_2 \le K_g\|u-v\|_2 \quad \text{for all } t\in\mathbb{R} \text{ and } u,v\in L^2(\Omega),$$

A classical example of a function a satisfying the above-mentioned assumptions is for instance

$$a(t,x) = 3+\sin(t)+\sin(\sqrt{2}t)+l(x), \text{ for } x\in\Omega \text{ and } t\in\mathbb{R},$$

where $l:\Omega\longmapsto\mathbb{R}^+$, continuous and bounded on Ω.

Under previous assumptions, it is clear that the operators $A(t)$ defined above are invertible and satisfy Acquistapace–Terreni conditions. Moreover, it can be easily shown that

$$R\Big(\omega, a(\cdot,x)\Delta\Big)\varphi = \frac{1}{a(\cdot,x)}R\Big(\frac{\omega}{a(\cdot,x)},\Delta\Big)\varphi \in AA(\mathbb{R},L^2(\Omega))$$

for each $\varphi\in L^2(\Omega)$ with

$$\Big\|R\Big(\omega, a\Delta\Big)\Big\|_{B(L^2(\Omega))} \le \frac{\text{const.}}{|\omega|}.$$

Let $F : \mathbb{R} \times L^2(\Omega) \mapsto L^2(\Omega)$ be the mapping defined by

$$F(t, \phi)(x) = g(t, \phi(x)) \text{ for } x \in \Omega,$$

and $z : \mathbb{R} \to L^2(\Omega)$ be the function defined by $z(t) = \varphi(t, .)$, for $t \in \mathbb{R}$. Then the Eq.(4.4) takes the abstract form

$$\frac{d}{dt}\big[z(t)\big] = A(t)z(t) + F(t, z(t)), \quad t \in \mathbb{R}, \tag{4.5}$$

Furthermore, all assumptions **(H0)**, **(H1)**, **(H'2)**, **(H'3)**, **(M1)** and **(M2)** of Theorem 3.8 are fulfilled. Then, the evolution equation (4.4) has a unique (μ, ν)-pseudo almost automorphic solution whenever K_g is small enough, with μ, ν being arbitrary positive measures of $\mathcal{M}$ satisfying **(M.1)** and **(M.2)**.

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